MODERN MATHEMATICS

an Introduction

A series of mathematics texts under the general editorship of

CARL B. ALLENDOERFER

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MODERN MATHEMATICS an Introduction

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PREFACE

This book is an attempt to provide an introduction to modern mathematics for those college students and others whose appreciation and understanding will be enhanced by a comprehensive presentation. Consequently some topics bear names that are familiar from earlier study. However, the resemblance often ends there, for it has been the author's aim to develop elementary as well as advanced topics in a rich and rewarding manner and to seek to impart mature insight and understanding throughout this text.

The author begins by developing all the essentials of mathematics from the earliest concept of number through secondary school mathematics. The reader will find that the presentation is thoroughly suffused with relevant, meaningful, and enlightening contributions from modern mathematics. There are, for example, Dedekind's definition of the irrational number and a portion of Cantor's theory of infinite sets to round out the study of the number system in the very first chapter. It will be found that the important set concept and language are employed generally throughout the book.

The chapter on the number system is followed first by a development of the essentials of geometry and then by those of trigonometry. The deductive method, used constantly in the book, is brought into sharp focus by a study of the non-Euclidean geometries of Riemann and Lobachevsky. Each of these geometries is carried far enough deductively to achieve significant results, including some of those that are contrary to familiar theorems of Euclid. Thus the existence of other valid postulate systems is made manifest. However, we go one step further. The chapter on symbolic logic, which follows the discussions of geometry and trigonometry, carries the axiomatic method to an even higher level of abstraction and permits symbolic analyses of the deductive procedures that were employed in earlier chapters.

In the chapter on functions there is developed, concurrently with some finishing touches on secondary mathematics from the modern point of view, other basic concepts such as sequence and limit. In addition to the usual graphs, attention is given to graphs resulting from the greatest integer symbol, the absolute-value symbol, and the inequality symbol. All these contribute to a comprehensive understanding of the continuity and discontinuity of functions.

The stage is set for the wonderful method of analytic geometry. There is a thorough development of concepts relating to the straight line and to straight line segments. The determinant is introduced in order to function

as a method for the solution of various problems. The locus approach leads us easily to the analytic treatment of conic sections. Translation and rotation of axes are utilized to provide a broader view of the notion of frames of reference.

The correspondence of points and ordered triples creates the basis for the study of three-dimensional geometry. Planes, points, and distances are considered. The set concept leads us to the quadric surfaces. The fourth dimension is considered briefly in the light of the above developments.

The scope of the analytic method is broadened further through the description of relations by means of parametric equations. Polar coordinates provide an effective introduction to a different coordinate system. These notions are extended to include different space coordinate methods.

All is in readiness now for the development of the essential concepts of the calculus. Further discussion of limit, continuity, and indeterminate forms leads to the crucial definition of the derivative. The basic theorems are developed, and the applications include extreme problems, related rate problems, and Newton's method of determining irrational roots. The inverse approach is applied again, this time in the form of the antiderivative.

The problem of finding an area under a curve serves to introduce the subject of integration. The relation to the antiderivative is carefully developed. Some of the applications that are discussed are those of work, length of arc, volumes, and surfaces.

The transcendental functions are utilized generously and appropriately. Indeed they play something of a culminating role at the end where, in a discussion of infinite series, Maclaurin expansions are obtained for a number of them.

With few exceptions, sections of every chapter are followed by a list of exercises. Some of these provide for a direct application of a learning experience, and some provide for and encourage creative participation and discovery. In addition there occur at frequent intervals lists of review exercises that strengthen the learning bonds and help to integrate concepts and techniques. Answers to selected odd-numbered exercises may be found at the end of the text.

I should like to close with an expression of sincere gratitude to Dr. Carl Allendoerfer for his painstaking reading of the manuscript and for his many constructive suggestions. I am very appreciative too of the general assistance, guidance, and courtesy extended to me by many people at Macmillan.

Samuel I. Altwerger

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INDEX OF SYMBOLS

$\aleph_0$	aleph-null, 63
$\wedge$	and, 171
$\mid \ \mid$	absolute value, 34
D^{-1}	antiderivative, 356
$\approx$	approximate, 47
$C_{n,r}$	binomial coefficient, 211
$\cong$	congruent, 72
$\wedge$	conjunction, 171
C	continuum, 64
$\backsim$	curl, 171
δ	delta, 232
$D_xy, \frac{dy}{dx}, y', f'(x), Df(x), \frac{df(x)}{dx}$	derivative, 323
Df	derivative (of function), 327
$D(f+g)$	derivative (of sum of functions), 330
$Df[g]$	derivative (of function of function), 333
$D(f \cdot g)$	derivative of product of functions, 345
$D\left(\frac{f}{g}\right)$	derivative of quotient of functions, 345
$D^2y, \frac{d^2y}{dx^2}$	derivative of second order, 393
$\mid \ \mid$	determinant, 248
δ	distance, 225
Δ	distance, 322
ϵ	epsilon, 232
$\longleftrightarrow$	equivalent proposition, 176
$!$	factorial, 210
$f:\{\ \}$	function, 187
$f[g]$	function of a function, 332
$f(x)$	function value, 187
$>$	greater than, 33
$[\ ]$	greatest integer, 223
$\rightarrow$	implication, 174
$\frac{ds}{dt}$	instantaneous speed, 299

$\int$	integral, 366
$\cap$	intersection, 65
f^{-1}	inverse function, 189
$<$	less than, 33
lim	limit, 122
log	logarithm, 214
ln	logarithm natural, 214
$\sim$	negation, 171
$\neq$	not equal, 29
$\veebar$	or, exclusive, 172
$\vee$	or, inclusive, 172
$\perp$	perpendicular, 68
π	pi, 135
$\cap$	product of sets, 65
$\sqrt{}$	radical, 47
$\{\ \}$	set, 187
$\sum$	summation, 365
$\sim$	similar, 72
$\triangle$	triangle, 74
$\cup$	union of sets, 65

MODERN MATHEMATICS
an Introduction

I

THE NUMBER SYSTEM OF MATHEMATICS

1. INTRODUCTION

It is possible through an intermingling of the old and the new in mathematics to achieve an understanding of the mathematical needs of modern times. This goal is attainable by those who are curious about mathematics and who are ready to support their curiosity with a fair amount of application. The demand for mathematical knowledge is indeed great.

The fabulous computers are reminders of the fact that mathematics is grounded in arithmetic. The machines, it may be known, are but ingenious arithmetical robots. In earliest times the invention of arithmetic, as well as other portions of mathematics, resulted from the demands of everyday living. The evolution was slow and the variety of the processes developed was great. Who has heard today of numbering by means of the letters of the alphabet, of multiplication by means of a doubling and halving process or of computing by means of finger positions? There remains little trace today of these early mathematical tools.

Often the logic of a mathematical process came later than its invention and use. The logic of arithmetic was developed thousands of years after numbers were first used. In this book we shall reverse the historical procedure substantially by developing significant portions of the theory of mathematics as the preliminary step so that we can gain the insight necessary to establish the plausibility of mathematical technic.

EXERCISES (I-1)

1. The ancient process of multiplication by doubling and halving may be illustrated with 8×6:

A	B	
8	6	
16	3	
32	1	$16 + 32 = 48$

We start with 8 in column A and write 6 in column B. At each step we double the previous entry in column A and take one-half of the previous entry in column B, dropping any remainder which arises from the division. The process continues until a 1 is reached in column B. From column A we now select those numbers which lie opposite odd numbers in column B. The sum of these numbers is the desired product.

Try the foregoing method on the following:

a. 14×8 b. 15×7 c. 28×21 d. 104×17

2. The construction of addition and multiplication tables was for each of us an experience in our youth. To appreciate better the scope of the problems involved in such an effort, construct
   a. An addition table with Roman numerals through V + V.
   b. A multiplication table through V $\times$ V.
3. Use the tables you constructed in the preceding exercise to obtain the results for

a. VI + II b. VI · II c. VIII + II d. VIII · II

Note: The center dot (·) is another symbol for multiplication.

4. The following example, 14×18, illustrates a method of *doubling* which was used for a long time:

$$\begin{array}{ccc} & A & B \\ & 1 & 18 \\ 14 & \left\{\begin{array}{l} 2 \\ 4 \\ 8 \end{array}\right. & \left.\begin{array}{r} 36 \\ 72 \\ 144 \end{array}\right\} 252 \end{array}$$

$$14 \times 18 = 252$$

   a. How are the numbers in the column A evolved?
   b. How are the numbers in column B evolved?
   c. What is the relationship between any pair of numbers in the same row?
   d. Justify the fact that the sum of the last three numbers in column B gives the desired product.
5. Try the doubling method on each of the following:

a. $8 \cdot 6$ b. $13 \cdot 15$ c. $23 \cdot 26$.

6. The numbers IV and VI indicate that the Romans employed *subtraction* and *addition* conventions in the representation of numbers.
   a. What do you think is meant by these conventions?
   b. Illustrate these conventions with other numbers in the Roman numeral system.
   c. The symbol "XIX" would be ambiguous were it not for an additional convention. Explain.
7. The Greeks and the Hebrews, who used their alphabets for numbers, used the additive principle. In the Hebrew system, where ח = 8 and י = 10, 18 is written as י ח, where the number is read from right to left.

If כ = 20, write 28 in Hebrew.

8. In Egyptian hieroglyphics, $|$ = 1, $|||$ = 3, $\begin{smallmatrix}|||\\||\end{smallmatrix}$ = 5, and $\cap$ = 10. The Egyptians also used the additive principle and read the numbers from right to left. Represent the numbers 12 and 14 in Egyptian hieroglyphics.

2. EVOLVING SYMBOLS

The beginnings of counting, by means of pebbles and like objects, are hard to detect today. The gears in a car speedometer, however, mechanically ticking off one mile at a time, may suggest primitive man's groping toward a counting medium as he put aside one pebble for each of his sheep. Perhaps some of us would hesitate to call this counting, for no other symbols, written or spoken, accompanied the procedure. Yet the shepherd's collection of pebbles gave him notions of number. How far is this really removed from our monthly bank statements?

Eventually pebbles gave way to higher levels of abstraction, to markings in sand and to notches on sticks. The resemblance to our tally marks, at inventory time, is striking. History is partially imprinted in our language. *Tally* comes from *telia*, which means cutting, and *calculate* comes from *calculus*, which means pebbles.

These means of counting soon gave way to more abstract symbolism. The Syriacs, the Greeks, and the Sumerians had different systems; for example:

Syrian:	$\mu\mu = 4$
Greek:	$\kappa\alpha = 21$
Sumerian:	$\gamma\!-\!\!- = 60$

New symbols and conjunctions of symbols had to be invented to meet the relentless demand for even moderately large numbers. The need for the invention of a system became increasingly imperative. It was recognized that such a system would have to have a *fixed base* to avoid the need for new symbols as greater numbers became used.

A number system with a fixed base and fortified with certain conventions makes it possible to cope conveniently with numbers of any size. This realization came to different peoples at different times and was implemented by them in different ways. Finger counting was bound to lead to bases of 5, 10, and 20. The Babylonians used 60. The Syriacs used two for their base. Some of our high-speed computers use the base two.

3. *NOTHING* IS DISCOVERED

Yet these impulses in themselves in the right direction, were not enough. Something highly significant was missing. If $\diagup$ was used to represent one and $\diagdown$ was used to represent two, how would one read $\diagup\diagdown$? Is it three,

twelve, one hundred and two, or one thousand and two? In retrospect it is easy to see that what is missing is a *place holder*, a symbol that will indicate the absence of a unit. If the one and the two symbols given before were meant to be used to represent one hundred and two, then some space-separating symbol is needed between them to signify the absence of the ten's unit. Perhaps  would do the trick.

Thus a group of *no members*, a collection of "no individuals," had to be recognized as a form of quantity. A symbol "*0*", zero, had to be invented for this connotation. Should the need for the invention of *0* seem incredible to some of us, it is only necessary to cite the fact that there is no *0* symbol in the Roman numeral system. Numbers were brought about by counting. But to count *0* may seem to be incredible. The Romans' oversight is understandable.

4. ESTABLISHING A BASE

Just what is meant by the *base*? Our system uses only the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. These are ten distinct symbols. The number ten itself, as well as all larger numbers, is represented by means of these symbols only. Thus,

$$\begin{aligned} 314 &= 3 \text{ hundreds and } 1 \text{ ten and } 4 \text{ units} \\ &= 3 \quad (100) \quad + \ 1\ (10) + \ 4 \end{aligned}$$

By anticipating the notion of exponents a bit, we may express this also as

$$314 = 3(10)^2 + 1(10) + 4$$

in which the symbol $(10)^2$ is an abbreviation for $10 \times 00 = 100$. In the number 506, the 0 prevents confusion with 56 or 5,006. The 506 consists of 5 hundreds, no tens, and 6. Similarly,

$$6{,}528 = 6(1000) + 5(100) + 2(10) + 8$$

or

$$6{,}528 = 6(10)^3 + 5(10)^2 + 2(10) + 8$$

EXERCISES (I-4)

1. Represent the following numbers by means of powers of 10:

a. 73 b. 840 c. 6,507 d. 6,057

2. What numbers do each of the following represent?

a. $5(10)^4 + 6(10)^2 + 7(10)$

b. $3(10)^2$

c. $4(10) + 3 + 7(10)^3 + 2(10)^2$

5. OTHER BASES

15840

To get the full import of a base in a number system, it is preferable to move away from the too familiar ten and to use other bases for awhile.

A system employing five as a base has only five distinct symbols: 0, 1, 2, 3, and 4. The number five itself, as well as all larger numbers, is represented by arrangements of these basic symbols. Thus,

five is represented by 10	(1 five and 0)
six is represented by 11	(1 five and 1)
fourteen is represented by 24	(2 fives and 4)

Base five		*Base ten*
32	$= 3(5) + 2$	= seventeen
213	$= 2(5)^2 + 1(5) + 3$	= fifty-eight

If we consider $(5)^2$, which is $5 \times 5 = 25$, as representing quarters, and (5) as representing nickels, we can look at the last illustration with more familiarity. In the five-system, $213 = 2$ quarters $+ 1$ nickel $+ 3$ cents, which totals, of course, fifty-eight cents. If our money system consisted only of these denominations, then this five-system would be of obvious advantage.

In a system with base seven, we have only seven symbols, 0, 1, 2, 3, 4, 5, and 6. The number 52 in this system represents 5 sevens and 2, which is equivalent to thirty-seven in our ten-system. Similarly,

$$325 = 3(7)^2 + 2(7) + 5$$

which comes to one hundred and sixty-six in our system.

The base twelve (duodecimal) has been urged upon us because the number twelve has more exact divisors than ten. Of course this requires twelve symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, plus two others to be adopted, one for ten and one for eleven. We shall indicate these respectively as t and e.

Base twelve			*Base ten*
11	$= 1(12) + 1$	=	thirteen
39	$= 3(12) + 9$	=	forty-five
$4e$	$= 4(12) + e$	=	fifty-nine
324	$= 3(12)^2 + 2(12) + 4$	=	four hundred and sixty
$t7$	$= t(12) + 7$	=	one hundred and twenty-seven

Where dozen and gross are used extensively, this system has something to offer. In the illustration 324 conveniently represents 3 gross, 2 dozen, and four.

The two-system (dyadic or binary) of the Syriacs of long ago has come into its own recently by its use in high-speed computers. This is because

there are but two symbols, 0 and 1, which can be associated with the presence or absence of an electric current. In this system,

$$\begin{aligned} 11 &= 1(2) + 1 && = \text{three in the ten's system} \\ 101 &= 1(2)^2 + 0(2) + 1 && = \text{five in the ten's system} \\ 110101 &= 2^5 + 2^4 + 2^2 + 1 && = \text{fifty-three in the ten's system} \end{aligned}$$

The great many places needed for a number of moderate size constitutes a weighty deterrent for everyday use of this system. However, computation with these numbers involves a minimum of memory, for all that we have to remember are the ordinary sums and products involving 0 and 1 and the new rule that $1 + 1 = 10$ in the binary system.

$$\begin{array}{r} (5 \times 3 = 15) \\ 101 \\ \underline{\times 11} \\ 101 \\ \underline{101} \\ 1111 \end{array} \qquad\qquad \begin{array}{r} (7 \times 3 = 21) \\ 111 \\ \underline{\times 11} \\ 111 \\ \underline{111} \\ 10101 \end{array}$$

EXERCISES (1–5)

1. Express the value of each of the following in our decimal system. The quantity in parentheses indicates the base to which the number preceding it is expressed.

a. 34 (5) c. 11,001 (2) e. $39t$ (12)
b. 201 (3) d. 56 (7)

2. A simplified method of changing from base 10 to another base is illustrated in the following change to base 6:

$$\begin{array}{r|l} 6 & 58 \\ \hline 6 & 9 \\ \hline 6 & 1 \\ \hline & 0 \end{array} \qquad \begin{array}{c} \text{Remainder} \\ 4 \\ 3 \\ 1 \end{array}$$

58 (base ten) = 134 (base six)

Check: $1(6)^2 + 3(6) + 4 = 58$.

3. Change each of the following decimal numbers to the base indicated in parentheses:

a. 72 (5) c. 382 (12)
b. 146 (7) d. 5618 (2)

4. You probably recall the addition and multiplication tables which, in part, looked like this:

Addition

+	1	2	3	·	·
1	2	3	4	·	·
2	3	4	5	·	·
3	4	5	6	·	·
·	·	·			

Multiplication

×	1	2	3	·	·
1	1	2	3	·	·
2	2	4	6	·	·
3	3	6	9	·	·
	·	·	·		

Construct similar tables for the following bases:

a. 5 b. 2 c. 6 d. 12

5. Use the tables of exercise 4 to obtain the results of the following, assuming that the numbers are expressed in the bases indicated in parentheses:

a. $43 + 32$ (5)	e. $132 + 24$ (6)
b. 43×32 (5)	f. 132×24 (6)
c. 1011×110 (2)	g. $59 + 47$ (12)
d. $1011 + 110$ (2)	h. 59×47 (12)

The results may be checked by converting them and the original numbers to the decimal system.

6. The Syriacs used just two symbols, $| = 1$ and $\rho = 2$. They used the additive principle and wrote their numbers from left to right. Express the numbers 3, 4 and 5 in this system.

7. In what essential respect is the Syriac binary system different from the binary system illustrated in the preceding problem?

8. The Babylonians used the symbol $\vee$ to represent 1 or 1 multiplied by some integral power of 60. They also used $<$, which represented 10 or 10 multiplied by some integral power of 60.

   a. Express the value of each of these *cuneiform* symbols in concise form.

   b. What value or values could the following have: $< \vee$? The number is written from left to right and the additive principle is employed.

6. NATURAL NUMBERS

From a historical viewpoint, the arithmetical processes used in computation were developed in response to practical needs. The test for accuracy and sufficiency was the practical test of the market place. Coherence and logic were considerations that came much, much later. In geometry they came earlier than in arithmetic when Euclid gathered together many known geometrical facts into a logical system.

The numbers 1, 2, 3, $\cdots$ used in the process of counting are called **natural numbers.** Later we shall have occasion to refer to them also as **positive integers.** The level of abstraction that is involved in the symbolization should not be overlooked. The number 2 is assigned to any collection of two objects or elements. The nature of the objects, their material aspects, have no bearing on the *twoness* of the group.

The natural numbers constitute a sequence of successive symbols, instead of pebbles, which can be matched with the members of any group to be counted. The last symbol matched is the *number* of the group. Thus, if the last symbol matched is "8," we say that there are 8 objects in the group.

7. GENERALIZED NUMBERS

Before beginning to speak of general principles of number computation, we must invent still other symbols of an even higher order of abstraction. Thus, if it is desired to speak of *any number whatsoever*, one may not use the symbol 7 to represent this thought, for 7 is not any number whatsoever. To satisfy this need, we may employ the letters of the alphabet once again. Any one of the letters, say, n, can be used to represent any number whatsoever.

What do we mean by *addition?* At this point we can be guided only by our intuitional notion derived from practical circumstances. The addition of two groups of *similar* terms that are represented by numbers means to us that they have been combined into one group. The process is indicated by the symbol + and is illustrated in Fig. I–1.

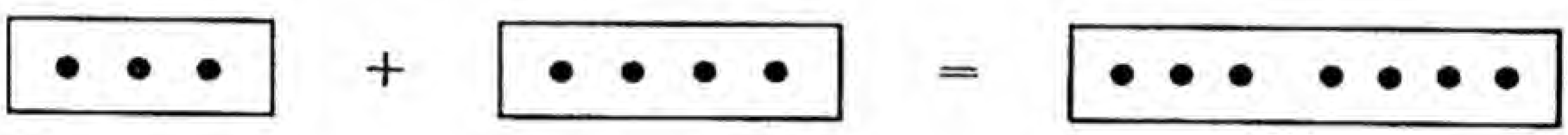


Fig. I–1

The concept of *multiplication* may be considered as repetitive addition and symbolized by "×" or "·". This is illustrated in Fig. I–2.

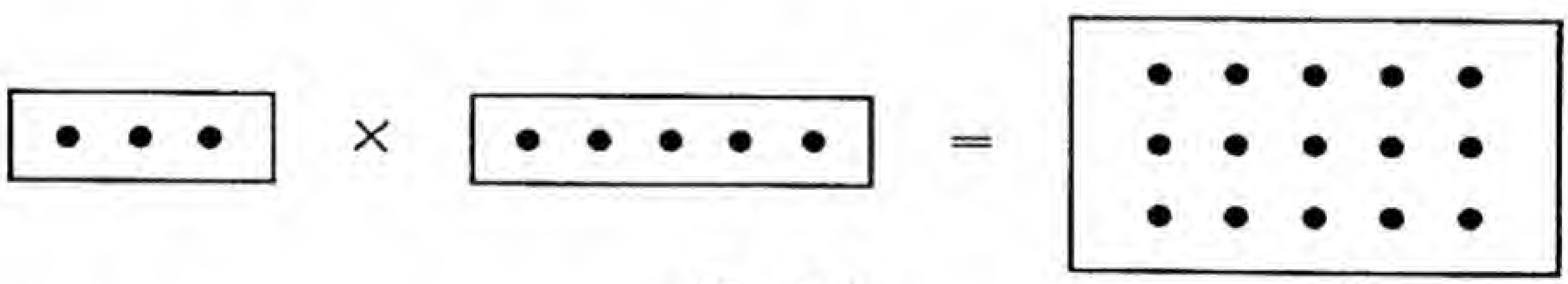


Fig. I–2

8. FUNDAMENTAL PRINCIPLES

The five fundamental assumptions of arithmetic are now stated simply and yet generally as:

(1) $a + b = b + a$	Commutative Law of Addition
(2) $ab = ba$	Commutative Law of Multiplication
(3) $a + (b + c) = (a + b) + c$	Associative Law of Addition
(4) $a(bc) = (ab)c$	Associative Law of Multiplication
(5) $a(b + c) = ab + ac$	Distributive Law of Multiplication with respect to addition

(For economy, we abbreviate "$a \times b$" as "ab".)

Let us examine the implications of these assumptions. The commutative assumptions, or *postulates*, set up rules permitting the change in direction of operation in addition and multiplication. Thus the postulates are in-

tended as generalizations of such situations as $3 + 4 = 4 + 3$ and $3 \times 4 = 4 \times 3$. Of course these specific statements are not likely to surprise anyone. We shall see, however, that the role of these as well as other postulates is vital in the logical foundation of arithmetic and mathematics generally.

The associative postulates refer to the grouping of the terms in addition and multiplication. In addition, for instance, the postulate indicates that it is immaterial whether the first term is added to the sum of the two following terms or whether the sum of the first two terms is added to the following term. Association in addition and multiplication are illustrated by the following:

$$(1) \qquad 3 + (2 + 7) = 3 + 9 = 12$$

or

$$(3 + 2) + 7 = 5 + 7 = 12$$

$$(2) \qquad 3 \cdot (2 \cdot 7) = 3 \cdot 14 = 42$$

or

$$(3 \cdot 2) \cdot 7 = 6 \cdot 7 = 42$$

Thus $3 \cdot 2 \cdot 7$ and $3 + 2 + 7$ have unique meanings, and there is no need to employ parentheses.

Of course we often take advantage of these guideposts without bothering about the implications. Thus in $4 \cdot 7 \cdot 5$ we are likely to think $(4 \cdot 5) \cdot 7 = 20 \cdot 7 = 140$. By changing the order, i.e., commuting the *factors*, we select a more convenient group to obtain the final product. Is not this likely to occur too when we work on $65 + 27 + 15$?

The distributive law will appear to be more familiar in the context, for example, of tripling a compound quantity such as 5 hours and 2 minutes or 5 feet and 2 inches. Thus

$$3(5 \text{ hr.} + 2 \text{ min.}) = 15 \text{ hr.} + 6 \text{ min.}$$

The multiplier 3 is *distributed* multiplication-wise over each of the elements of the compound quantity. In our decimal system we may look upon all numbers larger than 9 as compound numbers. Thus $23 = 2 \text{ tens} + 3$. So, $3 \cdot 23 = 3(2 \text{ tens} + 3) = 6 \text{ tens} + 9 = 69$. Of course this is done simply as

$$\begin{array}{r} 23 \\ \underline{\times 3} \\ 69 \end{array}$$

where distribution is the root of the process.

In $3 \cdot 27$ the distribution must be followed by the sum of similar terms.

$$\begin{aligned} 3 \cdot 27 = 3(2 \text{ tens} + 7) &= 6 \text{ tens} + 21 \\ &= 6 \text{ tens} + 2 \text{ tens} + 1 \\ &= 8 \text{ tens} + 1 \\ &= 81 \end{aligned}$$

All this is conveniently organized in the usual vertical form. This is presented in somewhat drawn-out fashion in order to reveal clearly the nature of the final form:

$$\begin{array}{r} 27 \\ \times 3 \\ \hline 21 \\ 60 \\ \hline 81 \end{array} \qquad\qquad \begin{array}{r} 27 \\ \times 3 \\ \hline 21 \\ 6 \\ \hline 81 \end{array} \qquad\qquad \begin{array}{r} 27 \\ \times 3 \\ \hline 81 \end{array}$$

The first stage shows the two products that result from the distribution, and their sum. In the second, the 0 is omitted after the 6 and may easily be understood. In the last stage, the 2 tens of the 21 may be "carried" mentally to reach the utmost in simplification.

In working with sums and products, we note that other assumptions are being tacitly employed. For example, if the sum or product of two numbers were not themselves numbers, we would have little to talk about. We assume, therefore, *that the sum and the product of numbers are themselves numbers.* Significant implications follow almost immediately, for if the sum of two numbers is a number, it follows that there is no largest number in our system unless we impose certain artificial restraints because any number, no matter how large, can always be added to one or another of the other numbers or even to itself. Thus, we have endless sequence, and sooner or later we shall have to consider more carefully the implications of this tacit introduction of the concept of infinity.

We have seen that

$$M(c + d) = cM + dM$$

Suppose, however, that M represents the sum of a and b; $M = a + b$. By making still another assumption, i.e., that

Equal quantities may be substituted for each other,

we get, by *substitution* of $a + b$ for M

$$(a + b)(c + d) = c(a + b) + d(a + b)$$

This indicates further that both c and d are to be distributed over the sum of a and b. In general terms this is the usual method of multiplying numbers arithmetically. Consider

$$\begin{aligned} 32 \cdot 21 &= (3 \text{ tens} + 2)(2 \text{ tens} + 1) \\ &= 2 \text{ tens}(3 \text{ tens} + 2) + 1(3 \text{ tens} + 2) \end{aligned}$$

or by commuting,

$$32 \cdot 21 = 1(3 \text{ tens} + 2) + 2 \text{ tens}(3 \text{ tens} + 2)$$

In brief, the digits of either number are separately distributed over the digits of the other. In the more familiar vertical arrangement, this is, in easy stages:

$$\begin{array}{r} 32 \\ \times 21 \\ \hline 2 \\ 30 \\ 40 \\ 600 \\ \hline 672 \end{array} \qquad\qquad \begin{array}{r} 32 \\ \times 21 \\ \hline 32 \\ 640 \\ \hline 672 \end{array} \qquad\qquad \begin{array}{r} 32 \\ \times 21 \\ \hline 32 \\ 64 \\ \hline 672 \end{array}$$

The first stage shows all four partial products; the second stage completes each distribution in one line, and in the third stage, the 0, which can easily be understood, is omitted.

Involved in the foregoing stages also are some properties of 0, which constitute its technical definition:

> *A number is unchanged by the addition of 0, and the product of any number by 0 is 0.*

Symbolically, we write

$$a + 0 = 0$$

and

$$a \cdot 0 = 0$$

A comment is necessary with respect to the substitution postulate. If we say that "6 + 4" is the sum of two even numbers, the statement is true. But if we substitute the number "7 + 3" for "6 + 4," we have the false statement that "*7 + 3*" *is the sum of two even numbers*. The substitution of "8 + 2", instead, keeps the first statement true.

The number *ten* has many forms, some of which are explicitly stated in the preceding paragraph. In a statement concerning the form of a representation of the number ten, a substitution of another form of the number ten may or may not affect the truth of the statement. This is not the intended use of the *substitution postulate*. The postulate refers only to those cases where the form of the representation of a number is irrelevant to the truth of the statement. Thus "7 + 3" or any other form of "ten" may be substituted for "6 + 4" in *6 + 4 is larger than 9*, $6 + 4 = 2 \cdot 5$, *6 + 4 is an even number*, and $3(6 + 4) = 30$. While fine and troublesome distinctions are conceivable, we shall not be bothered by such situations.

EXERCISES (I-8)

1. In the examples here point out the places in the computation where the various postulates are utilized. More than one postulate will be utilized in any one case.

a. 38 + 472 b. 520 × 34 c. 27 × 16

2. Use the distributive postulate to perform the following products mentally:

   a. $38 \times 101 = 38(100 + 1)$ b. 23×201
   c. 42×99 d. 51×7 e. 83×13

3. a. Express the result of $m(p + q)$ in as many different forms as possible.
   b. Do the same for $m(pq)$.

4. What is the fundamental error in the following:

$$5(3 \cdot 7) = 15 \cdot 35 = 525$$

5. a. Complete the multiplication in $(a + b)(c + d)$ so that no parentheses are needed in the final result.
   b. Do the same for $(a + 2)(b + 3)$ and for $(x + 2)(x + 3)$.

6. In the light of the processes indicated in the parts of exercise 5, try your hand at mentally calculating the products of the following:

   a. $23 \times 14 = (20 + 3)(10 + 4)$ b. 32×16 c. 47×23

9. GENERALIZED ARITHMETIC

Having introduced a *generalized* arithmetic through using letters to represent numbers, we illustrate the fundamental laws with the generalized numbers.

$3a + 5a = a(3 + 5) = 8a$	Distributive and commutative laws. This illustrates that only similar terms may be combined into a single term.
$3a + 5b = 3a + 5b$	Not similar terms.
$3a \cdot 5b = (3 \cdot 5)(a \cdot b) = 15ab$	Commutative and associative laws.
$5x(2y + 3) = 10xy + 15x$	Distributive and commutative laws.

EXERCISES (I–9)

1. Perform as many of the indicated operations as is permissible. Indicate in each case the postulate(s) that is (are) employed.

   a. $6(5 + 2a)$
   b. $8m + 3p + 2m + 5p$
   c. $xy + xz$
   d. $6a(2b + 3c)$
   e. $4m \cdot 3k$
   f. $5ab \cdot 6cd$
   g. $(3a + 1)(2a + 3)$

I–9 REVIEW

1. Name the postulate(s) which justifies each of the following statements. Each letter represents a natural number or zero.

   a. $a + M$ is a number.
   b. aM is a number.
   c. $x + (y + z) = y + (x + z)$.
   d. $17 + 49 + 13 = 30 + 49$.
   e. $15(42) = 600 + 30$.
   f. $(5 \cdot 7) \cdot 9 = 5 \cdot (7 \cdot 9)$.
   g. $8 \cdot 9 + 8 \cdot 7 = 8(9 + 7)$.

2. Complete each of the following:

a. $2a(3b + 4) = (?)$
b. $ax + ay + b(x + y) = (x + y)(\ ?\ )$
c. $ax + ay = a(\ ?\ )$
d. $16 \cdot 7 + 16 \cdot 12 = 19 \cdot (?)$
e. $5u \cdot 4v \cdot 3w = (?)$
f. $(2a + 3)(3b + 5) = (?)$

3. The base used for the number 35 is the smallest possible natural number. Express the value of the number in the decimal system.

4. Compare the value of 47 that has base b with 47 in the number system with base $b + 1$, where both bases are permissible.

5. Find the product of 24 and 13 by:
   a. The doubling and halving method.
   b. The doubling method.

6. In addition to the data already given earlier with respect to the Roman number system, we know that L is used to represent 50. Express each of the following in Roman numerals:

a. 40 b. 60 c. 54 d. 56 e. 49

7. Represent 24 in Egyptian hieroglyphics.
8. Find the sum of $7(10)^2 + 5(10)$ and $4(10) + 6$.
9. Find the sum and product of the binary numbers 101101 and 11011.
10. Express the decimal number 48 in the base 3 system.
11. Find the value of the product $(x + y)(z + w)$.

10. INEVITABLE CONSEQUENCES—EXPONENTS

It is easy enough to imagine that, in the course of operations with the new "literal" numbers, situations will arise which cry out for simplification. This is not only a matter of physical economy; indeed, it is also a matter of great mental advantage because it provides concentrated conceptions. The full significance of this thought will gradually unfold in the panorama of ever newer symbolism that lies in the pages ahead. At the moment we are concerned with the situation where, in some product, all or many of the factors are identical. Let:

cc be written as c^2 and read as *c square*
ccc be written as c^3 and read as *c cube*
$cccccc$ be written as c^6 and read as *c to the sixth power*

The smaller number, to the upper right of the letter, is used to indicate the number of factors and is called the **exponent**. The repeated factor, the c in the above illustration, is called the **base**. In $3c^2$, which represents $3 \cdot c \cdot c$, the 3 is called the **coefficient**. When there is no indicated coefficient, as in c^9, a coefficient of *1* is understood; $c^9 = 1c^9$.

As could be expected, the introduction of new definitions taken in con-

junction with the previously established conclusions and postulates is bound to lead to new consequences. Since

$$a^3 \cdot a^5 = aaa \cdot aaaaa$$

it follows that

$$a^3 \cdot a^5 = a^8$$

In general,
$$a^m \cdot a^n = \underbrace{(a \cdots a)}_{m \text{ factors}}\underbrace{(a \cdots a)}_{n \text{ factors}} = a^{m+n}$$

Thus *in multiplication, if the bases are the same, the base is kept and the new exponent is the sum of the original exponents.* If coefficients appear, we have, by means of the associative law,

$$3t^2 \cdot 5t^3 = (3 \cdot 5)(t^2 \cdot t^3) = 15t^5$$

Once the application of the principle is recognized, a product is easily obtained at sight:

$$4y^3 \cdot 5y^4 = 20y^7$$

EXERCISES (I–10)

1. Find the result in each of the following:

a. $5x^2 \cdot 4x$
b. $6a^2b \cdot 3ab^2$
c. $4x^2y \cdot 5x^3$
d. $3x^2(x + 2)$
e. $5m^3(m^2 + 2m)$
f. $(3x + 2)(4x + 1)$
g. $(x^2 + 2)(x^2 + 3)$
h. $10^4 \cdot 10^5$
i. $5^2 \cdot 5^3$
j. $6 \times 10^{14}(3 \times 10^5)$
k. $(cd)^3(cd)^5$
l. $(1.1 \times 10^{17})(1.2 \times 10^{19})$
m. $7^3 \cdot 7^4$
n. $(a^2)^3$
o. $(n^x)(n^y)$

2. Explain or illustrate why $a^mb^m = (ab)^m$.
3. What convention is needed to avoid ambiguity with the symbol "ab^x"? Consider a^xb and $(ab)^x$ in your answer.
4. What is wrong with $(2^x)(3^y) = 6^{x+y}$?
5. Find the value of x in each of the following:

a. $3^{15} \cdot 4^{15} = 12^x$
b. $4^6 \cdot 2^4 = 2^x$
c. $8^a \cdot 4^b = 2^x$

6. Find the value of each of the following:

a. $(x + 1)^2$
b. $(a + 3)^2$
c. $(2a + 3)^2$
d. $(20 + 1)^2$
e. $(31)^2$
f. $(45)^2$
g. 62^2

11. SCIENTIFIC NUMBERS

An interesting use of exponents may be seen now:

$$100 = 10 \cdot 10 = 10^2$$
$$1000 = 10 \cdot 10 \cdot 10 = 10^3$$
$$1{,}000{,}000 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^6$$

It appears that the exponent of the last 10 indicates the number of zeros that follow the 1 in the original number. That is $10^8 = 100{,}000{,}000$. This indicates a very simple way of writing the very large numbers which occur so frequently in science these days. For example, the light year, which is the distance that light will travel in one year, is about 6 million million miles.

$$6{,}000{,}000{,}000{,}000 = 6 \times 1{,}000{,}000{,}000{,}000 = 6 \times 10^{12}$$

It is fairly apparent that the exponential form is easier to comprehend than the arithmetic representation. It is also easier to use in computation. For example, the earth is about 35,000 light years from the center of our galaxy. How many miles is this?

$$\begin{aligned} 35{,}000 \text{ light years} &= 35 \cdot 1000(6 \cdot 10^{12}) \\ &= 35 \cdot 10^3(6 \cdot 10^{12}) \\ &= 210 \cdot 10^{15} \\ &= 2.10 \cdot 10^2 \cdot 10^{15} \\ &= 2.1 \times 10^{17} \text{ miles} \end{aligned}$$

To reach the final form, the commutative and the associative postulates as well as the new law of exponents had to be employed. Large as these numbers may seem, they are dwarfed numerically by many other numbers such as 4×10^{33} which represents, in ergs, the total energy output of the sun in one second.

EXERCISES (I—11)

1. Find the energy output, in ergs, of the sun in one year, assuming no diminution in energy.
2. Show that $1200 \times 10^5 = 1.2 \times 10^8$.
3. The frequency of long radio waves is of the order of 10^4 cycles per second, whereas for gamma rays it is of the order of 10^{20}. How many times as large is the latter?
4. a. Express exponentially the distance in miles of 24,000 light years.
   b. Find the product: $3.1 \times 10^{14} \times 2.6 \times 10^{19}$.

12. REMOVING A BARRIER

Sooner or later, one begins to wonder whether 0 is not being overlooked as a possible exponent just as it had been unrecognized in the development of the number system.

The original meaning of the exponent, as an indication of the number of times the base is being used as a factor, is, of course, a restraining feature. For, how can a base be used as a factor "no times"? But again and again we shall see that such an impediment may be overcome quite easily by proceeding not from the perceptual and intuitive basis but from a philosophy of extension which takes as its guide a **principle of consistency.**

We shall search for a definition of b^0 which shall be consistent with previously established positions. Often, in such cases, there are many approaches. We indicate two of them now. Let us start with a full expression of the meaning of b^5. We shall do the same for $b^4, b^3, \cdots$

$$\begin{aligned} b^5 &= 1 \cdot b \cdot b \cdot b \cdot b \cdot b \\ b^4 &= 1 \cdot b \cdot b \cdot b \cdot b \\ b^3 &= 1 \cdot b \cdot b \cdot b \\ b^2 &= 1 \cdot b \cdot b \\ b &= 1 \cdot b \end{aligned}$$

We note that by reducing the exponent by one successively, we drop one of the factors b. Consistency requires that if we go on to b^0, we must continue in the same spirit and remove a factor b once more. We are thus led to a definition that

$$b^0 = 1$$

Since b is any number whatsoever, this seems to indicate that

any number to the zero power is 1

Thus, 9^0, 146^0, and $3{,}999{,}999^0$ are all 1. This seems like a bitter pill to take. The zero power is in the role of the great equalizer.

Let us try another approach that, perhaps, will afford greater comfort.

$$a^n \cdot a^0 = a^n$$

The conclusion follows from the Law of Exponents in multiplication. The answer a^n is the same as the first factor a^n on the left-hand side. We observe in this general example that a quantity multiplied by a^0 is unchanged in the process. This is a characteristic only of the number 1. *If we insist on abiding by our previous knowledge, and if we wish to admit a^0 into membership in our number system, then again we have no choice but to define*

$$a^0 = 1$$

We shall see shortly that unfortunately there must be *one exception* to this, and that is when the base (be it b or a) is itself 0.

EXERCISES (I–12)

1. Find the value of each of the following:

a. $6b^0$
b. $(6b)^0$
c. 9×10^0
d. xy^0
e. x^0y
f. $5(a + b)^0$
g. 2^0x
h. $(m^0)^2$
i. $(19^2)^0$

2. Find the value of x:

a. $a^0b^0 = (ab)^x$
b. $5^0 \cdot 2^0 = 10^x$

I–12 REVIEW

1. Express each of the following in simplest form:

a. $3a^2 \cdot 2b \cdot 5ab$
b. $3t \cdot 5 \cdot 2t^4$
c. $8a^2b^2 \cdot 5a^3b^4$
d. $5^a \cdot 2^a$
e. $ah^m \cdot bh^t$
f. $5^6 \cdot 1^6$

2. Express the value of each of the following without parentheses:

a. $12y^2(y^3 + 5y^2 + 2)$
b. $10a^0b^2 + 10a^2b^0$
c. $(x^2 + 5)(x^2 + 3)$
d. $(3m^2 + 1)(4m^2 + 3)$
e. $(h^3 + 1)^2$

3. If $a = 5$ and $b = 3$, find the value of

a. $3a^2 + b^0$
b. $a^2b^2 + a^2$
c. $10a^0b^2 + 10a^2b^0$
d. $(a^4 + b^4)^0$

4. Write each of the following in factored form (the *monomial* factor in front of the parentheses should be the largest possible factor that is common to all the terms of the original expression):

a. $x^2 + x^3$
b. $2y + 6y^2$
c. $ab^2 + a^2b$
d. $h^5 + h^3 + h^2$
e. $1.2 \times 10^{14} + 3.1 \times 10^{14}$

5. All the numbers in this exercise are written in the system with base four. Find the values of 10^n for $n = 1, 2, 3$, and 10, and without the use of exponents, express the results in the base-four system. Check your results by translating to the decimal system.

6. In the Theory of Relativity, Einstein developed the fact that mass and energy at rest are related by the famous formula $E = mc^2$ where m is a mass in grams, c is the velocity of light in centimeters per second, and E is in ergs. Find the number of ergs in a mass of 4 grams if c equals 3 million million centimeters per second.

13. THE FRACTION—BEGINNINGS

It was postulated earlier that ab is a natural number if a and b are natural numbers. If we represent the product by c, we may write

$$ab = c$$

The numbers a and b are called *factors*, or *divisors*, of c. From the viewpoint of either a or b, this relationship is also presented in two alternative ways:

$$a = \frac{c}{b} \qquad \text{and} \qquad b = \frac{c}{a}$$

The first is read as a *is equal to* c *divided by* b. The second is read in similar fashion. The number c/b is called a *fraction*. The process of finding a, the value of c/b, is called *division*. The fraction is also indicated in the following alternative ways:

$$\frac{c}{b}, \quad c \div b$$

When it happens that a natural number d has as its factors only the numbers 1 and itself, that is, only when

$$d = 1 \cdot d$$

we say that d is a **prime number.** If d were itself 1, the previous statement would be ambiguous. Consequently we speak of prime numbers only if the numbers are larger than 1.

Much of the foregoing discussion brings to mind a portion of the technical definition of 0, which was that

$$a \cdot 0 = 0$$

where a is any natural number.

We study this relationship from the viewpoint of the fraction. There are two cases. First,

$$0 = \frac{0}{a}$$

This may be stated as *0 divided by any natural number is* ***0***.

The second case is

$$a = \frac{0}{0}$$

Since a is *any* natural number, we must conclude that the value of the fraction is **indeterminate.**

On the other hand, the fraction $b/0$ is meaningless if b is not 0, since $a \cdot 0 = 0$ and not b.

EXERCISES (I–13)

1. Find all the factors of each of the following:

a. 30 b. 70 c. 105 d. 24

2. List all the prime numbers that occurred in the preceding exercise.
3. Classify each of the following as 0, indeterminate, or meaningless:

a. $\frac{5}{0}$ b. $\frac{0}{5}$ c. $\frac{0}{0}$

4. Complete the factorization of 630 until all the prime factors are found by starting with

a. $630 = 2 \cdot 315$ b. $630 = 3 \cdot 210$

5. Suppose that $M = pqr$ and also that $M = stu$ and that all the lower case letters are prime numbers. What conjecture or conjectures may be made concerning the factors of the natural number M?

14. ANOTHER EXTENSION—THE RATIONAL NUMBER

The fraction b/a is as yet a number only if, the *denominator*, is a factor of b, the *numerator*. Thus, since $3 \cdot 7 = 21$, the fraction $21/7$ is a number since it is only another representation of the number 3. In general, if

$$ax = b$$

then

$$x = \frac{b}{a}$$

At the moment there are two restrictions. First, a must be a factor of b, and second, a is not 0.

Practical considerations, such as one-quarter of a loaf of bread, $\frac{1}{4}$, or three-quarters of a loaf of bread, $\frac{3}{4}$, gave short shrift to the first of the limitations centuries ago. We shall remove the limitations by means of considerations which will reveal once again the spirit of mathematical extension.

So far, the letter x in the preceding expressions is defined as a number only if a is a factor of b. Thus, if $b = 10$ and $a = 2$, then $x = 5$, for $2 \cdot 5 = 10$. But, if $a = 3$ instead, $x = 10/3$ is not as yet a number, since 3 is not a factor of 10.

We may begin to remove the limitation right now. Specifically, from

$$3x = 2$$

we define

$$x = \tfrac{2}{3}$$

a number such that

$$3(\tfrac{2}{3}) = 2$$

in accordance with the expressed condition of the equation. Until now, in

our development, there has been no number which when tripled yields 2. We have now invested the symbol "⅔" with this property.

In general, the symbol b/a is a *number*, or a *ratio*, which fulfills the condition that

$$a \cdot \frac{b}{a} = b$$

where a and b are integers. The number b/a is also called a rational number.

EXERCISES (I–14)

1. Show that the definition of a rational number includes the special cases of $1 = 2/2$ and $8 = 8/1$.

2. The following equations may be used to define x and y as **reciprocal numbers:** $ax = b$ and $by = a$.

Express x and y as fractions and compare them.

3. If in $ax = b$, a is a factor of b and b is a factor of a, and both are natural numbers, what is x?

4. Each of the following defines a rational number x:

(1) What is the value of x in each case?

(2) What condition must x satisfy?

a. $5x = 3$ b. $4x = 1$ c. $7x = 5$

5. The symbol x represents a rational number in each of the following cases:

(1) Indicate that number.

(2) Indicate all the restrictions that must be placed on the other letters involved.

a. $mx = p$ b. $(a + b)x = c$ c. $Mx + Nx = Q$

6. a. List all the factors of 12.

b. A natural number N is a prime number, or just *prime*, if it has no factors other than 1 and N. Numbers that are not prime are said to be **composite.** List the prime factors whose product is 12. What relationship exists between the factors in (a) and the prime factors just listed?

7. It has been known for quite some time that prime numbers come in pairs such as 3 and 5, 17 and 19, and 41 and 43. Find three additional pairs.

It is thought that there are infinitely many pairs of this kind, although no proof of this supposition has been found.

8. Consider the following for a systematic search for primes (*Eratosthenes sieve*): Write the natural numbers from 1 to, say, 30. First eliminate all the even numbers from the list; then the multiples of 3, then of 5, and so forth.

To date the list of primes consists of about 10,000,000, most of which have to be found by advanced techniques.

9. Euclid proved for us that there is an *infinite number of primes*. To see that this is so, suppose that the list is finite and that the primes may be represented by a

finite number of letters $x, y, z, \cdots, u$. Show how the number $p = xyz\cdots u + 1$ proves that the finite supposition is false.

10. Curiosity, as well as other reasons, led to a search for a formula that could propagate primes. One of the most famous of these, short lived to be sure, was that of **Fermat:**

$$2^{2^n} + 1 \quad \text{for } n = 1,2,3,4,5,\cdots$$

a. Find the primes that correspond to $n = 1,2$, and 3.

b. Find the number, not prime, that corresponds to $n = 5$, and show that it is divisible by 641.

11. Fermat also noted that prime numbers which are of the form $4n + 1$ can be represented as the sum of two squares. For example: $13 = 4 \cdot 3 + 1$ and $13 = 2^2 + 3^2$. Find two other prime numbers for which this is true and represent them accordingly.

12. Another startling supposition, known as the *Goldbach conjecture*, is that even numbers (other than 2) seem to be the sum of two primes. For example: $8 = 5 + 3$ and $14 = 3 + 11$. Find three other illustrations.

This conjecture, too, has not been proven or disproven.

15. REGULATIONS FOR THE NEW NUMBER; EQUALITIES

Let us continue the process of removing the limitation on the rational number. To include such symbols in the number fold, it is necessary to require that they conform to the basic laws of numbers. With this principle as a guide, let us determine what this means specifically in terms of the operations of arithmetic.

How, for example, shall we add 2/5 and 3/7? It may seem easy to say 5/12, as some unfortunately do. But this turns out to be in contradiction to our basic principles which, for the sake of consistency, must be imposed on these new numbers.

This process of extension requires, at this point, two further postulates to deal with **equalities.** We shall assume that

> *the product of equal numbers by the same or equal numbers will result in equal numbers*, and *the addition of the same or equal numbers to equal numbers will result in equal numbers.*

Symbolically, if

$$N = N$$

then

$$kN = kN$$

and

$$k + N = k + N$$

where k is any number whatsoever.

These, as well as the other postulates, represent our idealization of relations between numbers as abstracted from our experience and insight.

EXERCISES (I–15)

1. Explain the validity of each of the following:
   a. If $x = y$, then $x^2 = xy$ and also $xy = y^2$.
   b. What additional conclusion is suggested by the results in (a)?
2. If $x = y$, then $2x = 2y$. Derive this conclusion in two different ways.

16. HOW TO ADD THE NEW NUMBERS

Suppose that we seek the result of

$$\frac{a}{b} + \frac{c}{d}$$

where both are rational fractions. How do we go about reaching a decision? Well, our resources consist of the definition of the rational number, the basic assumptions of arithmetic, and now those of equalities.

We can represent the two fractions by x and y, respectively:

$$x = \frac{a}{b} \qquad \text{and} \qquad y = \frac{c}{d}$$

or, by our definition,

$$bx = a \qquad \text{and} \qquad dy = c$$

We are in search of a value for $x + y$. This may suggest, at first, the addition of the last two equalities. If we did that, we would have $bx + dy$, from which we would have a tough time extracting just $x + y$. But if we arranged to have the coefficients of x and y the same before we make the addition, the extraction will be possible. Since the original coefficients are b and d, each can become bd by multiplying by d and b, respectively. So, from

$$bx = a \qquad \text{and} \qquad dy = c$$

we get, by the indicated multiplications,

$$dbx = ad \qquad \text{and} \qquad dby = cb$$

The addition of these equalities gives us

$$dbx + dby = ad + cb$$

which, in the light of the distributive postulate, may be written as

$$db(x + y) = ad + cb$$

This in turn, yields, by the definition of rationals,

$$x + y = \frac{ad + cb}{db}$$

or, by substitution,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

This not only provides us with a rule of procedure but it also indicates that the *sum of two rational numbers is itself a rational number.*

The following are illustrative of this conclusion:

a. $$\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5 + 4 \cdot 3}{3 \cdot 5} = \frac{10 + 12}{15} = \frac{22}{15}$$

(Hereafter, the first step may be done at sight.)

b. $$\frac{1}{7} + \frac{5}{6} = \frac{6 + 35}{42} = \frac{41}{42}$$

c. $$\frac{3a}{2} + \frac{5b}{3} = \frac{9a + 10b}{6}$$

d. $$\frac{k}{2} + \frac{3k}{5} = \frac{5k + 6k}{10} = \frac{11k}{10}$$

In the last illustration, $5k + 6k = (5 + 6)k = 11k$. This indicates again that the distributive postulate leads us to the concept of the addition of similar terms. As one would readily expect, the denominators themselves may be literal terms, as in:

e. $$\frac{3}{n} + \frac{4}{p} = \frac{3p + 4n}{np}$$

EXERCISES (I-16)

1. You probably recall that fractions were represented to youngsters by diagrams which subdivided wholes such as squares, rectangles, and circles.
   a. Use some of these devices to represent $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{3}{5}$.
   b. By means of some pictorial devices illustrate the results of each of the following: $\frac{1}{3} + \frac{1}{2}$; $\frac{1}{4} + \frac{1}{3}$; $\frac{2}{3} + \frac{3}{4}$.
2. Find the sums:

a. $\frac{2}{5} + \frac{4}{3}$	e. $\frac{6}{5x} + \frac{3}{2y}$
b. $\frac{5}{a} + \frac{7}{b}$	f. $\frac{a}{b} + \frac{c}{d}$
c. $\frac{3}{m} + \frac{6}{2p}$	g. $\frac{x}{y} + \frac{y}{x}$
d. $\frac{3x}{4} + \frac{2x}{5}$	h. $\frac{3a + 2}{5} + \frac{a + 3}{2}$

3. The fractions a/b and c/d were taken as rational fractions in the previous discussion.

a. What restriction does this place on b and d?

b. What restriction is consequently implied for their sum?

4. Write the equations that the following rational numbers must satisfy according to the definition of the rational number. All letters excepting x and y represent natural numbers.

a. $x = \dfrac{a}{c}$ c. $x = \dfrac{ab}{cd}$

b. $y = \dfrac{m}{p}$ d. $y = \dfrac{a+b}{c+d}$

5. By the same procedure used in the text, determine the value of $x + y$ in exercises 4(a) and 4(b).

17. EQUIVALENT RATIONAL NUMBERS

Several corollaries suggest themselves at this point, each of which will be of value to us. If in $ax = b$, a has the value of 1, then

$$1x = b \qquad \text{and so} \qquad x = \frac{b}{1}$$

Since b is a natural number, this indicates that we can consider any natural number to be a rational number with the denominator actually 1. Thus, $5 = 5/1$ and $13/1 = 13$, and so forth.

Further, from what we have assumed earlier, it follows that if we start with

$$ax = b$$

we may obtain by multiplication, ($k \neq 0$),

$$kax = kb$$

and then

$$x = \frac{kb}{ka}$$

Since $x = b/a$ according to the first equation, it follows that

> *the value of a fraction is unchanged if both numerator and denominator are multiplied by the same number*

This is one of the most important conclusions concerning fractions. By using different multipliers, one can convert any fraction to an infinite variety of equivalent fractions with differing denominators. For example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{7}{14} = \frac{19}{38} = \cdots$$

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{3m}{5m} = \cdots$$

Of equal interest and value is the reverse of this: the removal of a common multiplier, or factor, from both numerator and denominator. This is illustrated in the following:

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{3}$$

$$\frac{5}{10} = \frac{5 \cdot 1}{5 \cdot 2} = \frac{1}{2}$$

$$\frac{3a^2}{6a} = \frac{3 \cdot a \cdot a}{3 \cdot 2 \cdot a} = \frac{a}{2}$$

Consider now the following sum:

$$\frac{1}{6} + \frac{3}{4} = \frac{4 + 18}{24} = \frac{22}{24} = \frac{11}{12}$$

The sum 22/24 was reducible to 11/22 because of the presence of the common factor 2 in both numerator and denominator. If this condition could be anticipated, some effort might be saved. Examination of the illustration reveals the fact that both denominators in the original sum contained the factor 2. To take advantage of this condition, we need only take as our *least common denominator* (LCD) the number 12 instead of 24. The prime, irreducible factors of 12 are 2, 3, and 2, which include the factors of both denominators, the 6 and the 4. Thus, both sixths and fourths may be changed to twelfths as follows:

$$\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

Similarly,

$$\frac{1}{15} + \frac{4}{21} = \frac{7 + 20}{105} = \frac{27}{105} = \frac{9}{35} \qquad (\text{LCD} = 5 \cdot 3 \cdot 7 = 105)$$

$$\frac{5}{6a} + \frac{2}{3a} = \frac{5 + 4}{6a} = \frac{9}{6a} = \frac{3}{2a} \qquad (\text{LCD} = 2 \cdot 3 \cdot a)$$

$$\frac{5}{xy^2} + \frac{3}{x^2y} = \frac{5x + 3y}{x^2y^2}$$

$$\frac{2a + 3b}{20} + \frac{3a + b}{12} = \frac{3(2a + 3b) + 5(3a + b)}{60} = \frac{21a + 14b}{60}$$

EXERCISES (I-17)

1 Simplify each of the following:

a. $\frac{12}{15}$

b. $\frac{3x}{6y}$

c. $\frac{x^2}{x^3}$

d. $\frac{6a^2b}{21ab^2}$

e. $\frac{14(a + 2)}{21(a + 2)}$

2. Find the results in each of the following:

a. $\frac{3}{5} + \frac{1}{10}$

b. $\frac{5}{12} + \frac{7}{18}$

c. $\frac{1}{12} + \frac{1}{4}$

d. $\frac{1}{a} + \frac{1}{a^2}$

e. $\frac{1}{a^2b} + \frac{1}{ab^2}$

f. $\frac{1}{x^2} + \frac{1}{x^3}$

g. $\frac{2x + 1}{10} + \frac{x + 3}{15}$

h. $\frac{a + b}{25} + \frac{2(a + b)}{35}$

i. $\frac{5}{m + p} + \frac{3}{(m \mp p)^2}$

j. $\frac{5}{8a^2} + \frac{7}{6a^3} + \frac{1}{4a^2}$

k. $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$

3. An expression such as $14a + 14b$ is usually written in the factored form $14(a + b)$, where the *binomial* factor $a + b$ is prime. The coefficient 14 need not be factored further.

Factor each of the following:

a. $6m + 6$
b. $xy + xz$
c. $x^2 + x^3$
d. $a^2b + ab^2$
e. $6mp + 12p^2$
f. $12p + 32$
g. $16(k + 1) + 12(k + 1)^2$
h. $6m^2 + 12m + 18$
i. $\frac{1}{2}r^2h + \frac{1}{2}rh^2$

4. A fraction is said to be in its simplest form if the numerator and denominator are *relatively prime* to each other. The numbers P and Q are relative primes if neither contains a factor of the other (other than 1, of course). For example, 15 and 14 are relatively prime although neither one is a prime number.

Simplify where possible:

a. $\frac{a + 4}{4}$

b. $\frac{a + 4}{a}$

c. $\frac{a^2 + 4a}{a}$

d. $\frac{6x^2 + 3x}{12x^2}$

e. $\frac{7a + 7b}{a + b}$

f. $\frac{6m + 9}{10m + 9}$

g. $\frac{x^2 + x^3}{5x + 5}$

h. $\frac{12(s + t)^3}{18(s + t)}$

i. $\frac{5(a + 2)(a + 3)}{10(a + 3)(a + 4)}$

5. If the LCD of two fractions is 12 and one of the denominators is 4, what may we conclude the other denominator to be?

6. It was indicated earlier that the fraction is a ratio of two integers or just a ratio. In speaking of the ratio m/n, also written $m:n$ and read as *m to n*, the order is as indicated, and the fraction is usually put into its lowest terms. Find the ratios of each of the following:

a. 12 to 15
b. 24 to 36
c. 45 to 50
d. 6 lb. to 8 lb.
e. 10 in. to 12 in.
f. 2 ft. to 6 yd
g. 40 min. to 2 hr.
h. $6a$ to $10a^2$

7. A **proportion** is an equality between two ratios, as in $2/3 = 4/6$, or $2:3 = 4:6$. From our knowledge of fractions this must mean that the two fractions can differ only in the presence of a common factor or of common factors in both numerator and denominator. Thus,

$$\frac{2}{3} = \frac{2 \cdot 2}{2 \cdot 3}.$$

We may say, then, that a proportion has the general form

$$\frac{a}{b} = \frac{k(a)}{k(b)}$$

Determine whether the following are proportions:

a. $\frac{6}{8} = \frac{9}{12}$

b. $\frac{16}{20} = \frac{28}{35}$

c. $\frac{12}{27} = \frac{8}{12}$

d. $\frac{3a}{2p} = \frac{9ap}{6p}$

8. The analysis of a proportion in exercise 7 leads to a very valuable observation. In $a/b = k(a)/k(b)$, we note that the product of either numerator with the opposite denominator is the same kab. By taking these two products, we obtain $kab = kab$. This may be remembered by the thought that **in a proportion, the cross-products are equal.**

a. Use this new fact to check on the proportions in exercise 7.

b. Use this fact to determine the missing term in the following proportions:

$$\frac{4}{9} = \frac{12}{?} \qquad \frac{12}{18} = \frac{?}{21} \qquad \frac{48}{?} = \frac{36}{42}$$

18. DETERMINING PRODUCTS OF RATIONAL NUMBERS

What shall be the meaning or the result of multiplication of rational numbers? What, for example, is the result of $\frac{2}{3} \cdot \frac{4}{5}$? To determine this, let us take the fractions a/b and c/d again and call them x and y, respec-

tively. Our goal is to find a value for xy that is consistent with the principles accepted or developed thus far.

$$x = \frac{a}{b} \quad \text{and} \quad y = \frac{c}{d}$$

or, $$bx = a \quad \text{and} \quad dy = c$$

Then $$(bd)(xy) = ac \qquad \text{by multiplication of equalities}$$

Consequently $$xy = \frac{ac}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{by substitution}$$

The product is a rational number consisting of the respective products of the numerators and denominators. By way of illustration, consider the following:

$$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$$

$$\frac{2x}{y^2} \cdot \frac{x}{y} = \frac{2x^2}{y^3}$$

$$\frac{1}{a+b} \cdot \frac{2}{a+b} = \frac{2}{(a+b)^2}$$

$$\frac{3}{5} \cdot \frac{1}{6} = \frac{3}{30} = \frac{1}{10}$$

The last illustration indicates a situation where an opportunity exists for simplification. The initial result, 3/30, is reducible because of the presence of the common factor 3 in both numerator and denominator. Since multiplication conjoins factors and does not create them, the common factor 3 must have been present in at least one of the numerators and one of the denominators. That being the case, it is usually best to remove the common factor at the start, as in the following:

$$\frac{\overset{1}{\cancel{3}}}{5} \cdot \frac{1}{\underset{2}{\cancel{6}}} = \frac{1}{10} \qquad \text{Dividing numerator and denominator by 3.}$$

$$\frac{\cancel{a}b}{3} \cdot \frac{5}{\cancel{a}} = \frac{5b}{3} \qquad \text{Dividing numerator and denominator by } a.$$

EXERCISES (I–18)

1. Find the value of each of the following:

a. $\frac{3}{7} \cdot \frac{2}{5}$ b. $\frac{18}{35} \cdot \frac{15}{21}$

c. $\frac{2a^2}{15} \cdot \frac{10}{a^3}$

d. $\frac{a^3}{x^2} \cdot \frac{x^3}{a^2}$

e. $\frac{5(a+b)}{2} \cdot \frac{4}{10(a+b)}$

f. $\frac{3x+6}{7} \cdot \frac{10}{5x+10}$

g. $\frac{x^2+x^3}{y^3+y^2} \cdot \frac{1+y}{x^2}$

2. Explore or illustrate the validity of the statement

$$\left(\frac{M}{P}\right)^n = \frac{M^n}{P^n}$$

3. Find the product of $4/5$ and $2/3$ by operating with equations exclusively.
4. a. In $ax = b$, what conclusion must be reached concerning x if $a = 0$ and $b \neq 0$ (b is not equal to 0)?
   b. What shall we say concerning $b/0$?

19. DIVISION OF RATIONAL NUMBERS

The rational number has brought division to our attention. It seems proper to inquire at this point as to the procedure in dividing rational numbers themselves. Once again we take a general course with respect to $(a/b) \div (c/d)$, in which each letter represents a natural number. Let

$$\frac{c}{d}x = \frac{a}{b}$$

so that

$$x = \frac{a}{b} \div \frac{c}{d}$$

This represents an extension of application of one of our first equations of the form $mx = p$, which leads to $x = p/m$.

Multiplying both sides of the first equation by bd, we get

$$bcx = ad$$

and so

$$x = \frac{ad}{bc} \qquad \text{(a rational number)}$$

By substitution,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$$

This result supplies the familiar rule of *inverting the second fraction and multiplying*. Again, illustrations follow:

a. $$\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \cdot \frac{2}{1} = \frac{4}{3}$$

b. $$\frac{2}{3} \div 2 = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

c. $$\frac{3}{x} \div \frac{2}{x^2} = \frac{3}{x} \cdot \frac{x^2}{2} = \frac{3x}{2}$$

When we invert a rational number so that a/b becomes b/a, we say that either one is the **reciprocal** of the other. This helps us see that division, like multiplication, is distributive with respect to addition. Note that

$$\frac{b+c}{a} = (b+c) \div a = (b+c) \cdot \frac{1}{a} = \frac{1}{a}(b+c) = \frac{b}{a} + \frac{c}{a}$$

So,

a. $\dfrac{6x+3}{3} = 2x + 1$ (and not $6x + 1$ or $2x + 3$)

b. $\dfrac{63}{3} = 21$

c. $\dfrac{5a + a^2}{a} = 5 + a$

EXERCISES (I–19)

1. Find the results of each of the following:

a. $\dfrac{3}{x^2} \div \dfrac{6}{x^3}$

b. $\dfrac{a}{b} \div \dfrac{a^2}{b^2}$

c. $\dfrac{a^2}{b^2} \div \dfrac{a}{b}$

d. $\dfrac{2m+6}{15} \div \dfrac{5m+15}{12}$

e. $M \div \dfrac{1}{2}$

f. $K^2P^2 \div \dfrac{K^3}{P}$

g. $\dfrac{8y+10}{10}$

h. $\dfrac{T^2 + T^3}{T^3}$

i. $\dfrac{1}{2}(12m + 4)$

j. $\dfrac{1}{k}(k^2 + k)$

k. $\dfrac{1}{3} \cdot 15 \cdot 6$

l. $\dfrac{1}{3}(15 + 6)$

m. $\dfrac{1}{m} \cdot 6m \cdot m$

2. a. Show that $(a/c) \div (b/d)$ is equivalent to $(a/b) \div (c/d)$. State result in words.

b. Utilizing the conclusion in (a), find the values of the following at sight:

(1) $\dfrac{10}{21} \div \dfrac{5}{7}$ (2) $\dfrac{x^2}{y^2} \div \dfrac{x}{y}$ (3) $\dfrac{(a+b)^3}{x^2} \div \dfrac{a+b}{x}$

3. A length such as 5 inches means the unit 1 inch is taken 5 times That is, 5 in. = 5(1 in.). So, (5 in.)(3 in.) = $5 \cdot 3$(1 in.)(1 in.) = 15(1 in.)^2 = 15 in.^{2}, or 15 sq. in.

Find the results of each of the following:

a. (8 in.)(3 in.)(2 in.)

b. $\dfrac{12 \text{ sq. in.}}{4 \text{ in.}}$

c. 15 sq. in. $\div$ 3 sq. in.

d. $12\dfrac{\text{lb.}}{\text{sq. in.}} \cdot 10$ sq. in.

e. $\dfrac{x \text{ cu. in.}}{y \text{ sq. in.}}$

f. $30\dfrac{\text{ft.}}{\text{sec.}} \cdot 15$ min.

g. $K\dfrac{\text{lb.}}{\text{cu. in.}} \cdot Q$ cu. in.

h. $20\dfrac{\text{gm.}}{\text{cu. cm.}} \div \dfrac{1}{5 \text{ cu. cm.}}$

4. Find the values of the following by operating with equations exclusively. That is, instead of $\frac{3}{4}$ use $4x = 3$.

a. $\frac{2}{3} \div \frac{4}{5}$ b. $\frac{4}{5} \div \frac{2}{3}$

5. We have had occasion to use the postulate that *equal quantities multiplied by equal quantities yield equal quantities.*

a. Show how *equal quantities divided by equal quantities yield equal quantities* is, in effect, a restatement of the same postulate.

b. What exception must be made in (a)?

6. We have *solved* equations of the type $ax = b$ on the basis of the definition of the rational number. The observation in exercise 5(a) shows another approach. Use this approach in solving

a. $12y = 30$ and b. $my = m^2$

for y.

I–19 REVIEW

1. Factor each of the following obtaining prime factors where possible:

a. 42
b. 182
c. 165
d. $ab + ac$
e. $\pi x + \pi y + \pi z$
f. $5d^2 + 5d^3$

2. Simplify each of the following:

a. $\dfrac{65}{91}$
b. $\dfrac{bc^3}{bc^2}$
c. $\dfrac{\pi r^2 h}{\pi r h^2}$
d. $\dfrac{ax + a^2}{3x + 3a}$

3. Find the results of each of the following:

a. $\dfrac{3}{7} + \dfrac{4}{5}$
b. $\dfrac{a}{b} + \dfrac{a}{c}$
c. $\dfrac{a}{b} \div \dfrac{a}{c}$
d. $\dfrac{ab^2}{c} \div \dfrac{c^3}{a^2b}$
e. $\dfrac{5}{8} \div \dfrac{3}{4}$
f. $\dfrac{1}{3}(3a + 6)$
g. $\dfrac{6a^2}{5} \div \dfrac{2a^3}{15}$
h. $\dfrac{5.6 \times 10^{11}}{3.4 \times 10^4}$
i. $\dfrac{ab^2 + a^2b}{ab}$
j. $\dfrac{5u}{12v^2} \cdot \dfrac{4v}{15u^2}$
k. $\dfrac{5a}{3} \div \dfrac{7a}{6}$
l. $\dfrac{15}{v} \div \dfrac{3}{w}$
m. $\dfrac{3x}{b} \div \dfrac{5x}{ab^2}$

4. Find the value of x in each of the following proportions:

a. $x:3 = 4:6$

b. $x:5 = 2:3$

c. $\frac{10}{x} = \frac{2}{3}$

d. $\frac{a}{x} = \frac{b}{a}$

e. $\frac{2}{3} = \frac{1}{x}$

5. The employment of exponents helps to simplify the expression of the product of the prime factors when one or more of the factors occurs more than once. For example, $8 = 2^3$. Do the same for the following:

a. 144 b. 648 c. 1800

6. It has been conjectured by Bertrand and proved by Tchébicheff that there is at least one prime number for integral values of $n > 7/2$ in the interval between n and $2n$, or more exactly, between n and $2n - 2$. For example, for $n = 6$, $2n = 12$ and $2n - 2 = 10$. Between 6 and 10, there is at least one prime number which is 7. Verify this theorem for values of n from 7 to 11, inclusive.

7. The reader must have conjectured by now that the factors of any natural number are unique. More specifically,

any natural number greater than 1 can be factored into a product of prime numbers in only one way.

This is the **fundamental theorem of arithmetic,** which we will not prove here.

a. Justify the use of the phrase "greater than 1" in the statement of the theorem.

b. What postulate makes the order of the prime factors irrelevant?

8. a. If it is known that 17 is a factor of 8177 and also that $8177 = 37 \cdot 221$, what deduction may be made concerning the numbers 17, 37, and 221?

b. If $M = ab$ and the prime number p is a factor of M, what conclusions may be drawn concerning the numbers a, b, and p?

20. MORE INVERSES

We pause momentarily for perspective. We started with the basic operations, addition and multiplication, undefined except in context and circumscribed by a series of assumptions. We inverted the operation of multiplication in a simple way, $2 \cdot ? = 6$, and we were led inexorably to the definition of the rational number and to its operations. Division entered the scene in the guise of an inverse operation. Has not addition also an inverse? Surely. *Subtraction*, of course. We turn in this direction now, and in the spirit of an earlier occasion, attempt to remove barriers that may lie in the way of inverting the operation of addition. The equation

$$2 + x = 5$$

sets the stage for the inversion. For, to inquire *what number added to 2 gives 5?* creates the need to operate with the 5 and 2 to give the answer 3. We can symbolize this by denoting that

$$x = 5 - 2$$

In general, where

$$a + x = b$$

we may indicate $\qquad x = b - a$

Our arithmetic of addition provides us with all the needed answers to such equations but, so far, only when

$$b > a \qquad \text{and} \qquad b = a$$

The symbol $>$ means *greater than.*

Our number system as yet does not provide meaning for such expressions as $5 - 7$ and $3 - 12$. The restriction against $b < a$ (*b is less than a*) interferes with the complete generality of the preceding equations, and efforts ought to be made to remove the limitation.

Of course our ancestors, with their experience of debts, losses, and matters of similar connotation, accomplished the same objective by written qualifiers, special symbols, or other devices. We shall proceed, however, with our logical course.

EXERCISES (I–20)

1. Using the "inequality" symbols, $>$ and $<$, insert these between the following pairs of numbers, writing each correctly in two ways:

a. 5 and 7 b. 12 and 13 c. $\frac{1}{2}$ and $\frac{1}{3}$

2. Do the same as in exercise 1 for:

a. x^2 and x^3 if $x > 2$
b. x and x^2 if $x > 0$ but $x < 1$

This double condition in (b) is abbreviated as

$$0 < x < 1 \qquad \text{or} \qquad 1 > x > 0$$

3. For what natural number values of b are each of the following true?

a. $3 < b < 5$
b. $4\frac{1}{2} < b < 5\frac{1}{2}$
c. $10 < b + 3 < 15$
d. $6\frac{3}{4} > b > 6\frac{1}{4}$

4. It was pointed out in the preceding discussion that $x = b - a$ is, as yet, defined only for $b > a$ and $b = a$. Illustrate these possibilities.

5. It was also mentioned in the discussion that our ancestors overcame the limitations against cases such as 5 − 7 in various ways, some of which may even subsist to this day. Can you cite or invent such possible devices?

21. A NEW NUMBER

We have seen that for values of b less than a, x is undefined in $a + x = b$. Since 0 is the smallest number we have so far, we may say more generally that x is as yet undefined in $a + x = 0$. This limitation may be removed by creating (that is, defining) a new number, a **negative number** that will satisfy the last equation. Thus, for any rational number a, we define the negative number $-a$ such that

$$a + (-a) = 0$$

In particular, this means that

$$3 + (-3) = 0,\ 8 + (-8) = 8, \text{ and } \tfrac{2}{3} + (-\tfrac{2}{3}) = 0$$

Of course those who wish to be guided by practical analogies can find many relevant illustrations such as a "rise of 5° in temperature followed by a fall of 5° will constitute a net change of 0°."

This new definition expands the number system once again. Thus we have laid the groundwork for an open number system:

$$\cdots -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5, \cdots$$

By contrast, we refer to the natural numbers as **positive numbers** and write for $1, 2, 3, 4, \cdots$, the numbers $+1, +2, +3, +4, \cdots$.

As always, we want to look into the matter of operating with new numbers. We shall be guided in our decisions by all previous assumptions and deductions.

A number now may have three possible references. With respect to 5, for example, we may consider sometimes $+5$, at other times -5, and finally, just the numerical value 5 without regard to the sign. To be specific about the last case, we need a new definition. We say that the **absolute value of a number** is the value of the number without regard to its sign. A couple of vertical bars, $|\quad|$, symbolizes this new name. Thus,

$$|+5| = 5 \qquad \text{and also} \qquad |-5| = 5$$

EXERCISES (I–21)

1. Find the value of M where possible:

a. $6 + M = 9$
b. $15 + M = 12$
c. $3 + |M| = 5$
d. $6 + |M| = 5$
e. $P + M = 0$

22. NEW ADDITION

In considering operations with positive and negative numbers, we are confronted with expressions such as $(-3) + (+2)$ and $(+3) + (-2)$. Two of the signs in each of these cases are associated with the numbers, and the other sign connotes the operation of addition. The two uses are distinguished by the parentheses. This situation may be simplified considerably by the introduction of a desirable convention. We propose that the sign of addition, $+$, shall be omitted when the numbers to be added have an indicated sign, $+$ or $-$, between them. Thus $-3 + 2$ represents the sum of -3 and $+2$, and $+3 - 2$ represents the sum of $+3$ and -2. Similarly, $3 - 2 - 4$ represents the continued sum of $+3$, -2, and -4.

In considering addition of signed numbers, one soon recognizes the existence of four possible cases. The order of the terms in any case is inconsequential, since we are going along with the commutative postulate.

$$3 + 5 = ?$$

If this is an expression with natural numbers, the sum is 8 and the $+$ sign refers to the nature of the operation rather than to the sign of the number 5. However, from the new viewpoint, we are concerned with the sum of two positive numbers. Since, the positive numbers have been identified with the natural numbers, consistency demands the same answer to the new case. So,

$$3 + 5 = 8 \qquad \text{(still)}$$

Thus *the sum of two positive numbers is defined to be a positive number.* Had we decided that the answer was -8, we would introduce immediately the kind of inconsistency that we are trying to avoid.

We must now determine the mode of addition of all possible combinations of positive and negative numbers. The **Principle of Consistency**, by which we continue to be guided, requires our conformity with all previous postulates, definitions, and theorems concerning numbers and equalities.

The theorem of addition is obtained as follows:

$a - a = 0$	definition of negative number
$b = b$	postulate of identity
$(a + b) - a = b$	addition of equalities

We can illustrate this general conclusion by a series of examples:

a. If $a = 5$, $b = 2$, then by substitution:	$7 - 5 = 2$
and by the commutative postulate:	$-5 + 7 = 2$
b. If $a = 5$ and $b = -2$, then by substitution and (a):	$3 - 5 = -2$
so, also:	$-5 + 3 = -2$
c. If $a = 5$ and $b = -7$, then by substitution and (b):	$-2 - 5 = -7$

The conclusions are more familiarly stated as:

a. *The sum of two numbers of the same sign is a number with the same sign whose absolute value is the sum of the absolute values of the two numbers; and*

b. *The sum of two numbers with different signs is a number whose absolute value is the difference of their absolute values and whose sign is the sign of the number with the larger absolute value.*

The following are applications of the above conclusions:

$$9 + 3 = 12 \qquad 3x - 5x = -2x$$
$$9 - 3 = 6 \qquad -5a^2 - 2a^2 = -7a^2$$
$$-9 + 3 = -6 \qquad 7(a + b) - 3(a + b) = 4(a + b)$$
$$-9 - 3 = -12 \qquad -8xy + 6xy - 5xy = -7xy$$

EXERCISES (I–22)

1. Find the results of each of the following:

a. $18 - 21 - 12$
b. $-32 + 16 + 14$
c. $5x^2 - 7x + 3x^2 - 2x$
d. $-3ab + 5ab$
e. $6m - 2m^2 - 4m + 5m^2$
f. $-3(a + b) - 4(a + b)$
g. $2m^2n + 3mn^2 - 6m^2n$
h. $\frac{1}{3} - \frac{1}{2}$
i. $\frac{3}{4} - \frac{2}{3} - \frac{1}{2}$

2. Indicate the condition under which each of the following is valid:

a. $p + q = |p + q|$
b. $p + q = -|p + q|$
c. $p + q = |p| + |q|$
d. $|p + q| \leq |p| + |q|$
(where the sign $\leq$ "means less than or equal to")

3. In $x + a = b$, with a and b both positive, we now have $x = b - a$, irrespective of which of the two, b or a, is the larger. This suggests the possibility of solving the equation for x by adding $-a$ to both members of the equation, mentally in most cases. For example, in $x + 7 = 18$, by adding -7 to both sides we get $x = 11$. In $x - 7 = 18$, by adding $+7$ to both sides, we get $x = 25$.

Solve the following for the numerical value of the literal quantity:

a. $y + 4 = 9$
b. $h + 5 = 3$
c. $x - 4 = 3$
d. $m - 9 = 2$
e. $2m + 7 = 13$
f. $8 + 3m = 15$
g. $2t - 12 = -3$
h. $3k - 20 = 0$

23. SUBTRACTING SIGNED NUMBERS

Turning to the subtraction of signed numbers, we wonder what is meant by $5 - (-7)$? We have defined subtraction as the inverse of addition. Thus

$$b + x = a \qquad \text{leads us to} \qquad x = a - b$$

where $a - b$ refers to the subtraction of the signed numbers a and b. However, since

$$b + [a + (-b)] = b + (-b) + a = a$$

we see by comparison with the first equation above that

$$x = a + (-b)$$

and then, by substitution, that

$$a - b = a + (-b)$$

This means that *the subtraction of the signed number b is equivalent to the addition of* $-b$, which we now illustrate concretely as:

$$\begin{aligned} 8 - (+3) &= 8 - 3 = 5 \\ 8 - (-3) &= 8 + 3 = 11 \\ 8 - (+9) &= 8 - 9 = -1 \\ -3u - (-5u) &= -3u + 5u = 2u \end{aligned}$$

EXERCISES (I–23)

1. Find the differences by taking the second term from the first:

a. $15, -23$
b. $-15, -8$
c. $21, 5$
d. $5x^2, -7x^2$
e. $-3ab, 5ab$
f. $6y(z + 2), 9y(z + 2)$
g. $-(\frac{2}{3}), -(\frac{3}{4})$

2. Explain why $M - (P - Q)$ is the equivalent of $M + (Q - P)$.
3. Using the conclusion in exercise 2, simplify each of the following:

a. $5a - (3 - 2a)$ b. $3m^2 - (m^2 - 6m)$ c. $7x - (2x + 3)$

4. Having established an equivalence between addition and subtraction of any signed number, we see that postulate regarding addition of equal quantities includes the subtraction of equal quantities. However, it is more often stated as a separate postulate:

equal quantities may be subtracted from equal quantities leaving equal quantities

Solve each of the following:

a. $3x - 2 = 8$
b. $5M + 4 = 19$
c. $7a - 8 = 21$
d. $6p + 14 = 82$
e. $\frac{7}{8}y + \frac{1}{4} = \frac{1}{3}$

5. a. Provide reasons for

$$\begin{aligned} 0 - (-n) &= 0 + n \\ -(-n) &= n \end{aligned}$$

b. Prove that $-(+n) = -n$

24. PRODUCTS OF SIGNED NUMBERS

We determine the rules of multiplication in the same manner as in the previous cases. We note first that the product of two positive numbers is positive, in keeping with their identification with the natural numbers. For further guidance we turn to such commitments as the distributive postulate and the definitions of negative numbers and zero.

$$a \cdot 0 = 0$$
$$a(b - b) = 0$$
$$ab + (a)(-b) = 0$$

and so
$$(a)(-b) = -(ab)$$

The following are specific illustrations of this conclusion:

a. If $a = 2$ and $b = 3$, then by substitution: $(2)(-3) = -6$
 and by the commutative postulate: $(-3)(2) = -6$
b. If $a = -2$ and $b = 3$, then by substitution and (a): $(-2)(-3) = -(-6) = 6$

The product of two numbers of the same sign is positive. The product of two numbers of different signs is negative. The absolute value of the product is the product of the absolute values of the numbers.

25. DIVIDING THE NEW NUMBERS

Division has been defined as the inverse operation of multiplication. It is fairly simple to see from the conclusion in the previous article, as well as from the illustrations there, that the rule of division of signed numbers is identical with that of multiplication after the substitution of the word "quotient" for the word "product." A few illustrations of both operations are in order:

$$3(-7) = -21 \qquad -2a(-3a^2) = 6a^3$$

$$\frac{-21}{-7} = 3 \qquad 3m(2m - 3) = 6m^2 - 9m$$

$$-4(-5) = 20 \qquad \frac{6a^2 - 3a}{3a} = 2a - 1$$

$$\frac{20}{-5} = -4 \qquad (t + 2)(t - 3) = t^2 + 2t - 3t - 6$$

$$3(-5x) = -15x \qquad = t^2 - t - 6$$

Nothing has as yet been stated explicitly about the matter of division where exponents are involved. The inverse approach, as may have been

recognized in one of the preceding illustrations, is adequate for this situation.

Since $$x^2x^3 = x^5$$
then $$\frac{x^5}{x^2} = x^3$$
and, in general, $$\frac{x^m}{x^n} = x^{m-n}$$
where $m > n$ and $x \neq 0$.

In dividing two exponential expressions with the same base, the result is that base with a power equal to the difference of the exponents taken in the same order.

By way of further illustration:

a. $\frac{a^7}{a^2} = a^5$ b. $\frac{6a^5}{3a^4} = 2a$ c. $\frac{8^6}{8^4} = 8^2$

EXERCISES (I–25)

1. Find the value of each of the following:

a. $5(-2)(-3)$
b. $-7(-4)(-5)$
c. $-3(2a + 1)$
d. $-5t(2 - 3t)$
e. $-a(a - 3)$
f. $-(b - 5)$
g. $-(t + 3)$
h. $-6m^2n(3mn^2)$
i. $\frac{24}{-8}$
j. $\frac{-24}{8}$
k. $\frac{-6}{-8}$
l. $\frac{-a^3}{a^2}$
m. $\frac{15a^4}{-3a^2}$
n. $(a - 3)(a + 2)$
o. $(3a - 1)(2a + 5)$
p. $(x - y)(x + y)$
q. $2a(3a^2 - 2a + 1)$
r. $\left(\frac{-a}{c}\right)\left(\frac{c^2}{a}\right)$
s. $\left(\frac{x - y}{5}\right)\left(\frac{10}{y - x}\right)$
t. $(5a - b)(3a + 2b)$

2. A product such as (28)(99) can be done mentally if visualized as $28(100 - 1)$. Try this on:

a. 7(99) b. 15(98) c. 35(98) d. 8(999)

3. If the statement $(m - n)(m + n) = m^2 - n^2$ is read from right to left, we have a means of saving mental energy very often in computing the difference of two squared numbers. Thus, $17^2 - 14^2 = (17 - 14)(17 + 14) = 3(31) = 93$. Try this on:

a. $29^2 - 25^2$ b. $16^2 - 15^2$ c. $38^2 - 36^2$ d. $141^2 - 139^2$

Note the interesting special case when $m - n = 1$.

4. Perform the indicated operations:

a. $\dfrac{s^5}{s^3}$

b. $\dfrac{x^3y^4}{x^2y^2}$

c. $\dfrac{m^6}{m^6}$

d. $\dfrac{21x^9}{18x^2}$

e. $\dfrac{6a^5 + 4a^4}{2a^2}$

f. $5^8 \div 5^4$

g. $2^{10} \div 2^7$

h. $\dfrac{5.4 \times 10^{21}}{1.8 \times 10^{16}}$

i. $\dfrac{3.9 \times 10^{26}}{1.1 \times 10^{15}}$

5. If $-x = a$, then $x = -a$. (a) Show the validity of this conclusion by use of the multiplication postulate for equalities. (b) How would the conclusion be validated from the division viewpoint? (c) Can the conclusion be justified otherwise?

6. Solve the following equations:

a $-6x + 8 = 18$

b. $3a - 5 = -11$

c. $24 = 8 - 2x$

d. $\dfrac{2b}{3} - 5 = 8$

e. $12 - \dfrac{3k}{4} = 20$

f. $5h + 2 = 2h + 8$

g. $9m - 3 = 16 - m$

I–25 REVIEW

1. Find the value of the unknown that satisfies each of the following equations:

a. $3a - 5 = 8a + 20$

b. $5y + 3 = 19 - 3y$

c. $\dfrac{4y}{3} + \dfrac{3}{5} = \dfrac{3}{10}$

d. $\dfrac{1}{x} + \dfrac{1}{3x} = 1$

e. $\dfrac{5}{2y} = \dfrac{5}{3y} + \dfrac{1}{3}$

f. $\dfrac{12}{5x} = \dfrac{2}{15}$

g. $\dfrac{3a + 2}{5} = \dfrac{a}{2}$

h. $h(2 - 3h) = 3(4 + h - h^2)$

2. For what values of T are the following true?

a. $|T - 2| = 3$

b. $\left|\dfrac{2T}{3} - 5\right| = 3$

3. For what values of n are the following true?

a. $\dfrac{1}{n} < \dfrac{1}{n^2}$

b. $5 \leq 2n - 3 \leq 11$

4. Without performing the indicated multiplications, explain why $(ak + al)(u + v) = (k + l)(au + av)$.

5. Perform the indicated operations and express the results in the simplest form possible:

a. $3a(-2a^2)(-5a^3)$

b. $6a^2 - 3a - 5a^2 + 7a$

c. $\dfrac{-24a^3b}{-18ab^2}$

d. $(h - 3)(h + 3)$

e. $8t - (3 - 4t)$

f. $(-2)^3 - (-1)^2$

g. $\left(\dfrac{3}{4} \cdot \dfrac{1}{2}\right) \div \dfrac{3}{8}$

h. $3.5 \times 10^{12} - 1.9 \times 10^{12}$

i. $(5w - 3)(2w - 1)$

j. $\left(\dfrac{3}{4}\right)^2 - \left(\dfrac{1}{4}\right)^0$

k. $(3h - 2)^2$

l. $-a(1 - a)$

m. $5(h - 3) - 2(3h + 1)$

n. $(3x - 2) - (2 + 3x)$

o. $\dfrac{3x + 6y}{h} \div \dfrac{5x + 10y}{h^2}$

p. $\dfrac{3h + 1}{5} + \dfrac{2h - 1}{3}$

q. $\dfrac{3}{x} + \dfrac{1}{5} - \dfrac{2}{x}$

r. $\dfrac{5a - 3}{2} - \dfrac{2a + 1}{3}$

s. $\dfrac{1}{a - b} + \dfrac{1}{a + b}$

6. If $a/b = (a + c)/(b + d)$ prove that $a:b = c:d$.

7. Find the negatives of each of the following quantities:

$$\frac{2}{-3}, \frac{-2}{3}, 3x - 2, \text{ and } a - b$$

8. Subtract the second quantity from the first:

a. $3a - b, 2a - 5b$

b. $x^2 - x - 1, 2x^2 - x + 3$

26. BARRIERS GO DOWN AGAIN FOR EXPONENTS

The division of exponential terms, $x^m \div x^n$, is restricted to those cases where m is larger than n. This is a condition not unlike those we met originally in connection with

$$ax = b \qquad \text{and} \qquad x + a = b$$

where x was defined only for certain values of a and b. These limitations were removed by the inclusion of the rational number and the signed number, respectively, in the number field.

Why, then, should x^m/x^n be defined only for $m > n$? Unrestricted operation and greater generality are sufficient motives for new definitions to remove the limitations. Let us try this by starting with $m = n$.

$$\frac{x^m}{x^m} = ?$$

There are two possible views of this expression. One stems from arithmetic. If x is not 0, the arithmetic view is that any number divided by itself is 1.

The other view stems from the impulse to use the rule of exponents in spite of the untoward result. This means that we subtract the exponents and get x^0, a brand new expression.

$$\frac{x^m}{x^m} = \begin{cases} 1 & \text{by the arithmetic approach} \\ x^0 & \text{by the exponential approach} \end{cases}$$

Consistency demands that we identify the two results as being equal. We define, therefore, that

$$x^0 = 1$$

as incidentally we surmised earlier, also on the basis of consistency. *Any number, other than 0, raised to the 0 power is just 1.*

$$(93)^0 = 1; \quad (\tfrac{1}{2})^0 = 1; \quad (3a)^0 = 1; \quad 3a^0 = 3; \quad (a + b)^0 = 1$$

27. AND AGAIN

Now suppose that $m < n$. What inkling can we get with respect to

$$\frac{x^2}{x^5} = ?$$

Again we take two views of this fraction. On the basis of the fact that we may remove common factors from the numerator and denominator, this fraction reduces to $1/x^3$. This is referred to briefly as dividing both numerator and denominator by x^2. By following boldly the rule of exponents in connection with division, we get, on the other hand, x^{-3}.

$$\frac{x^2}{x^5} = \begin{cases} \dfrac{1}{x^3} & \text{by simplification} \\ x^{-3} & \text{by exponent rule} \end{cases}$$

Consistency demands that we identify the two answers.

$$x^{-3} = \frac{1}{x^3}$$

In general,

$$x^{-m} = \frac{1}{x^m} \qquad \text{for } x \neq 0$$

Concretely, this indicates that

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$

Now x and $1/x$ are reciprocals of each other, and since

$$x^{-m} = \frac{1}{x^m} = \left(\frac{1}{x}\right)^m$$

we may say that

a base to an integral power is equal to the reciprocal of that base to the power with the opposite sign.

This is particularly convenient when the original base is a fraction, as in

$$\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^4 = \frac{81}{16}$$

The base ten was used advantageously with positive exponents in scientific notation applications. We return to these but use negative exponents. We note that

$$\begin{aligned}
10^{-1} &= \frac{1}{10} & &= 0.1\\
10^{-2} &= \frac{1}{10^2} = \frac{1}{100} & &= 0.01\\
10^{-3} &= \frac{1}{10^3} = & &= 0.001\\
10^{-4} &= \frac{1}{10^4} & &= 0.0001\\
10^{-5} &= \frac{1}{10^5} & &= 0.00001\\
&\ldots\ldots & &\\
10^{-n} &= & &= \underset{n \text{ places}}{0.000\cdots1}
\end{aligned}$$

Reversing these observations, we see that

$$0.000003 = 3 \times 0.000001 = 3 \times 10^{-6}$$

In this way, very small numbers, which abound in modern science, are represented conveniently by negative exponents of 10. A quantum of light energy is 5×10^{-12} erg. Planck's very important constant in modern physics is 6.6×10^{-27}. Imagine writing and operating instead with 0.0000000000000000000000000066, rather than 6.6×10^{-27}!

EXERCISES (I-27)

1. Find the value of each of the following:

a. $(6m)^0$

b. $6m^0$

c. 6^0m

d. $h^3\left(h^2 + \frac{1}{h^2}\right)$

e. $-2k^0(k - 1)$

f. $-2k^0(k - 1)^0$

g. $2^4 - 2^0 + 2^2$

h. $3^2 \cdot 3^0 \cdot 3$

i. $4^2\left(\frac{1}{4^0}\right)$

j. $\frac{1}{2^0} \cdot \frac{1}{3^0}$

2. Simplify the following:

a. $\dfrac{d^4}{d^5}$

b. $\dfrac{-24e^2}{36e^3}$

c. $\dfrac{-b^2}{-b^5}$

d. $\dfrac{(-b)^2}{(-b)^5}$

e. $\dfrac{2^6}{2^4}$

f. $\dfrac{6.2 \times 10^4}{2.2 \times 10^9}$

g. $\left(\dfrac{3}{4}\right)^2$

h. $\left(\dfrac{x}{y}\right)^a\left(\dfrac{x}{y}\right)^{-a}$

i. $\dfrac{10u^{-5}}{2u^{-2}}$

j. $\dfrac{3^{-4}}{3^3}$

3. Rewrite each of the following so that no negative exponents appear:

a. $8x^{-2}$

b. $(4x)^{-3}$

c. $\left(\dfrac{u}{v}\right)^{-2}$

d. $6x^{-2}y^3$

e. $-8(mn)^{-2}$

f. $6(s + t)^{-3}$

g. 2.7×10^{-14}

h. $ab^{-1} - a^{-1}b$

i. $6x^{-3} \cdot 4x^5$

j. $12v^{-2} \cdot 6v^{-2}$

k. $\dfrac{14a^{-5}}{7a^{-9}}$

4. Find the value of each of the following:

a. $\left(\dfrac{3}{4}\right)^{-2}$

b. $12\left(\dfrac{1}{2}\right)^{-3}$

c. $4^{-2} + 4^0 - 4^{-3}$

d. $\left(\dfrac{1}{2}\right)^{-3} + \left(\dfrac{1}{3}\right)^{-2}$

5. In scientific notation it is customary to have only one figure in front of the decimal point in the first factor, as in 3.8×10^{-4}. Express each of the following in this form:

a. 0.000062

b. 0.0000037

c. $0.000081 \times 0.0000036$

6. Represent each of the following in decimal notation:

a. The wave length of gamma rays is about 3×10^{-9}mm.

b. The wave length of X-rays is about 3×10^{-7}mm.

28. NEWER NUMBERS—IRRATIONALS

We have gone a long way with this matter of exponents, from a countable number of factors, as in x^4, to a newer view for x^{-4}. Why stop here? Why cannot the exponents be rational fractions? The drive toward comprehensiveness requires the consideration, the inclusion into the fold, of cases such as $a^{2/3}$.

Before, to find a basis for the definition of the negative exponent, we contrived an exercise which, when pursued by established procedures, resulted in a negative exponent. We can try the same tactic for the fractional exponent. Let us start with $x^{1/2}$ and raise this meaningless symbol to the second power.

$$(x^{1/2})^2 = x^{1/2}x^{1/2} = x$$

If this in itself is not startling, consider the case when $x = 2$.

$$(2^{1/2})^2 = 2$$

We have "something" here which, when squared, yields 2. We had better pause momentarily and look at some squares.

Integers:	$\cdots -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ 4,\ \ 5,\ \cdots$
Their squares:	$16,\ \ 9,\ \ 4,\ \ 1,\ 0,\ 1,\ 4,\ 9,\ 16,\ 25,\ \cdots$

There does not appear to be an integer whose square is 2. Perhaps there is a rational fraction whose square is 2, but there are too many cases of this kind to investigate numerically, so instead we try a very general approach. That, of course, is by means of algebraic notation.

Suppose that there is a rational number a/b whose square is exactly 2. For convenience let us assume that the fraction is in its lowest terms. This means that a and b have no common factor. If they have, that common factor is to be removed in advance of what follows. So,

$$\left(\frac{a}{b}\right)^2 = 2$$

or

$$\left(\frac{a^2}{b^2}\right) = 2$$

Clearing of fractions, we have

$$a^2 = 2b^2$$

Now, if two quantities are equal to each other, they must possess the same factors. Since the right-hand side of the last equation has the factor 2, there must be a factor 2 on the left-hand side as well. But $a^2 = aa$. So, there is no choice. The number a must have the factor 2. This means that $a = 2\times$ (some number). Therefore we shall let $a = 2k$, where k is just some number. This gives us, by substitution,

$$(2k)^2 = 2b^2$$

or

$$4k^2 = 2b^2$$

which, simplified, yields

$$2k^2 = b^2$$

The condition is now reversed. There is a factor 2 on the left-hand side. So, the right-hand member of the equation must also have a factor 2.

Since $b^2 = bb$, it follows that 2 must be a factor of b. But this unequivocally contradicts the initial condition that a and b have no factors in common. There is but one conclusion possible: There is no rational number whose square is 2.

Once again we find our number system lacking. The choice is clear. Create a new number to close the gap. Let us call this new number "irrational" by way of contrast to the part of the number system developed thus far. Of course there is need to invent a new symbol too. Before we do so, we recall that the squares of both the positive and negative numbers are positive. We have need, therefore, of introducing both positive and negative irrational numbers. Thus, we let

$\sqrt{2}$ represent the positive irrational whose square is 2

and

$-\sqrt{2}$ represent the negative irrational whose square is 2

These are read respectively as the positive or the negative "square root of 2." Symbolically this means that

$$(\sqrt{2})^2 = 2$$

and

$$(-\sqrt{2})^2 = 2$$

or

$$\sqrt{2}\sqrt{2} = 2$$

Earlier, we had

$$2^{1/2}2^{1/2} = 2$$

Consequently

$$\sqrt{2}\sqrt{2} = 2^{1/2}2^{1/2}$$

Consistency requires that we *define*

$$\sqrt{2} = 2^{1/2}$$

and, in general

$$\sqrt{a} = a^{1/2}$$

where $a > 0$.

The restriction $a > 0$ is necessary. A glance at our short table of squares of numbers indicates that squares of all integers, and of rational numbers too, are positive. That is, $(-5)^2 = +25$ and $(+5)^2 = +25$. This does not mean that we shall not explore this other possibility. But, one case at a time.

The symbol $\sqrt{}$ is read as *square root*. The a is called the *radicand*.

Inasmuch as no rational number will be equal to $\sqrt{3}$, any evaluation can be only approximate. Thus,

$$(1.7)^2 = 2.89$$
$$(1.73)^2 = 2.9929$$
$$(1.732)^2 = 2.999824$$

With the inclusion of more and more decimal places, we can get as close to 3 as we wish, although in fact we shall never arrive at 3 exactly. For, if

we did so, it would derive from the square of a decimal number with a finite number of decimal places, and a decimal number with a finite number of decimal places is a rational number.

We know that $(-1.7)^2$, $(-1.73)^2$, and so forth, will give precisely the same results as above. The $\sqrt{3}$ is approximately $+1.732$ and the $-\sqrt{3}$ is approximately -1.732. We write

$$\sqrt{3} \approx 1.732 \qquad (\approx \text{ is read } \textit{is approx.})$$

$$-\sqrt{3} \approx -1.732$$

$$\pm\sqrt{3} \approx \pm 1.732$$

Why, one may ask, need we restrict ourselves to squares? Why not investigate other powers? Indeed, we should.

Numbers	$\cdots -3,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3, \cdots$
Numbers cubed	$\cdots -27,$	$-8,$	$-1,$	$0,$	$1,$	$8,$	$27, \cdots$
Numbers to 4th power	$\cdots 81,$	$16,$	$1,$	$0,$	$1,$	$16,$	$81, \cdots$
Numbers to 5th power	$\cdots -243,$	$-32,$	$1,$	$0,$	$1,$	$32,$	$243, \cdots$

Our knowledge of exponents leads to the following conclusions, analogous to the $2^{1/2}$ of a few lines back.

$$(2^{1/3})^3 = 2; \quad (2^{1/4})^4 = 2; \quad (2^{1/5})^5 = 2$$

Clearly these indicate another host of irrationals of the sort we just developed in conjunction with the square root. We shall extend our notations to include these new possibilities. Let:

a. $\sqrt[3]{2}$ represent the irrational number whose cube is 2, so that

$$\sqrt[3]{2} = 2^{1/3}$$

b. $\sqrt[4]{2}$ represent the positive irrational number whose fourth power is 2

so that
$$\sqrt[4]{2} = 2^{1/4}$$
and so forth.

In general,

$$\sqrt[n]{a} = a^{1/n}$$

where $\sqrt[n]{a}$ is read as the "nth root of a" and the n is the *index* of the root. Looking back, we see now that the sign $\sqrt{\ }$ really stands for $\sqrt[2]{\ }$, where the index 2 is always understood.

Now, in $\sqrt{a}$ we had to make the restriction $a > 0$, since, guided by our present knowledge of the number system, no squares of numbers are negative.

Well, by the same token, we know that all even powers of numbers are positive. So, again, we must make the qualification,

$$\sqrt[n]{a} = a^{1/n} \qquad a > 0 \text{ for even values of } n.$$

However, odd powers of positive numbers are positive, and odd powers of negative numbers are negative, so that no qualification is necessary for odd indices. Various illustrations follow (rational results are for convenience only):

$$625^{1/4} = \sqrt[4]{625} = 25$$

$$-(625)^{1/4} = -\sqrt[4]{625} = -25$$

$$(-625)^{1/4} = \sqrt[4]{-625} = \text{undefined, as yet meaningless}$$

$$64^{1/3} = \sqrt[3]{64} = 4$$

$$(-64)^{1/3} = \sqrt[3]{-64} = -4$$

$$-(64)^{1/3} = -\sqrt[3]{64} = -4$$

It is desirable to recapitulate the three necessary stipulations in the preceding illustrations:

1. As yet, even roots of negative radicands are undefined. The $\sqrt{-3}$ and $\sqrt[4]{-16}$ are meaningless at the moment.

2. An even root of a positive radicand is taken as positive number to avoid ambiguity as well as to conform to the general stipulation regarding signs: $\sqrt[4]{16} = 2$ and $-\sqrt[4]{16} = -2$.

3. Odd roots of positive as well as negative radicands are defined and are unambiguous, being positive when the radicand is positive and negative when the radicand is negative: $\sqrt[5]{32} = 2$ and $\sqrt[5]{-32} = -2$.

4. In general, $\sqrt[n]{a^n} = |a|$ for even values of n

and $\sqrt[n]{a^n} = a$ for odd values of n

So far our fractional exponents all had the numerator 1. We can readily expand to the other cases. The fraction $2/3$, for example, can be viewed as a product in two ways, either as $2(1/3)$ or $(1/3)2$. Thus,

$$a^{2/3} = (a^2)^{1/3} \quad \text{or} \quad (a^{1/3})^2$$

Therefore $a^{2/3} = \sqrt[3]{a^2}$ or $(\sqrt[3]{a})^2$

Illustratively, $8^{2/3} = \sqrt[3]{8^2}$ or $(\sqrt[3]{8})^2 = 4$ either way

Finally, suppose that the exponent is a negative fraction. Well, we have already defined that $a^{-n} = 1/a^n$ for integral exponents. We extend this

case, again in the interests of consistency, to include all the negative rational exponents. Consequently

$$a^{-p/n} = \frac{1}{a^{p/n}} = \frac{1}{\sqrt[n]{a^p}}$$

with the same stipulation that was made earlier, that is, $a > 0$ for even values of n. As a specific illustration,

$$16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{8}$$

Surely, more needs to be said about our new irrationals. What are the means of operating with them in terms of the fundamental operations?

$$\sqrt{2} + \sqrt{3} = ?$$

Irrationals with different radicands can not be added or subtracted because of our restriction of similar terms in these operations.

$$6\sqrt{5} + 3\sqrt{5} = 9\sqrt{5}$$

Here the terms are similar. Or, by the distributive postulate, we have

$$(6 + 3)\sqrt{5} = 9\sqrt{5}$$

These illustrations indicate that addition may not be possible when the radicands are different or when the indices are unlike. Thus

$$\sqrt{2} + \sqrt{3} = \sqrt{2} + \sqrt{3}$$

and

$$\sqrt{2} + \sqrt[3]{2} = \sqrt{2} + \sqrt[3]{2}$$

We now consider multiplication illustratively.

$$\sqrt[3]{3} \cdot \sqrt[3]{5} = \sqrt[3]{15}$$

since

$$3^{1/3} \cdot 5^{1/3} = (3 \cdot 5)^{1/3} = 15^{1/3}$$

We make note of the fact, explicitly, that we have retained the laws of exponents for this realm as we have been doing with established conclusions for all extensions.

Irrational numbers can then be multiplied when the indices are the same. If the indices are not the same, multiplication is still possible, although a little more complicated. Now consider an aspect of this operation, factoring:

$$\sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$$

since

$$18^{1/2} = 9^{1/2} \cdot 2^{1/2}$$

Thus an irrational number may be factorable, and this may be advantageous when one of the factors is a rational number. The seeming dissimilarity of the terms, in the next illustration, is dissipated when one of the terms is factored and *simplified.*

$$\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4}\sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

It may be expected that division of irrationals is possible since division is the inverse of multiplication. Indeed the method can be determined on this basis. However, let us take again the exponential view of the case.

$$\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$$

since

$$\frac{15^{1/2}}{3^{1/2}} = \left(\frac{15}{3}\right)^{1/2} = 5^{1/2} = \sqrt{5}$$

As in multiplication, if the indices are not the same, division is still possible although somewhat complicated and generally inconsequential. If the radicands do not divide exactly, there is little gained by this approach. The next method will show a preferable attack.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{or} \quad \tfrac{1}{2}\sqrt{2}$$

(Note: $\sqrt{2}/\sqrt{2} = 1$.)

In obtaining the objective, we note the use of a basic principle of fractions carried over from the rational field of numbers. This procedure is referred to as *rationalizing the denominator*, which describes the change that was accomplished in the original fraction. Where approximate evaluation is necessary, it is much simpler to take one-half the value of the square root of 2 than it is to divide 1 by the square root of 2, as in the last illustration. We consider another variation of the irrational fraction.

$$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

since

$$\left(\frac{3}{4}\right)^{1/2} = \frac{3^{1/2}}{4^{1/2}}$$

Thus the root of a fraction may be considered as the quotient of the root of the numerator and denominator separately. The illustration shows a specific advantage in the recognition of this possibility. Should the de-

nominator of the radicand fraction not be a perfect square, the condition is easily remedied.

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3} \cdot \frac{3}{3}} = \frac{\sqrt{6}}{3}$$

EXERCISES (I-28)

1. Find, where possible, the value of each of the following:

   a. $8^{1/3}$
   b. $-(4)^{-1/2}$
   c. $16^{3/4}$
   d. $(-16)^{-1/4}$
   e. $100^{5/2}$
   f. $49^{1\frac{1}{2}}$
   g. $16^{-0.25}$
   h. $(-27)^{2/3}$
   i. $(125)^{1/3} - 4^{-2}$
   j. $(8^{-2})(\frac{1}{64})^{-2}$
   k. $(8)^{1/5}(4)^{1/5}$
   l. $(12)^{1/3}(18)^{1/3}$
   m. $a^{-2p}a^{3p}$
   n. $8^{2/3} \cdot 16^{1/2}$

2. If $x = 4$, find the values of:

   a. $9x^{1/2}$
   b. $(9x)^{1/2}$
   c. $9^{1/2}x$
   d. $9^{1/2} - x^{1/2}$
   e. $\left(\frac{9}{x}\right)^{1/2}$
   f. $\left(\frac{9}{x}\right)^{-1/2}$
   g. $\left(\frac{1}{9x}\right)^{-1/2}$

3. For what values of p are each of the following true?

   a. $\sqrt{p^2} = p$ b. $\sqrt{p^2} = |p|$ c. $\sqrt{p^2} = -p$

4. Show that

$$M^{k-p} = \frac{1}{M^{p-k}}$$

5. Show that $\sqrt[3]{2}$ is an irrational number.

6. Represent each of the following with fractional exponents:

   a. $\sqrt[3]{t^2}$
   b. $(\sqrt[4]{m})^3$
   c. $\sqrt[3]{h}\sqrt[3]{h}$
   d. $\sqrt[2]{m^{1/4}}$
   e. $\sqrt{9x}$
   f. $\dfrac{\sqrt{a}}{\sqrt[3]{a}}$
   g. $\sqrt{x^2y}$
   h. $\sqrt[5]{a-b}$
   i. $\sqrt{x^2 - 4}$

7. Simplify where possible:

a. $\sqrt{18}$
b. $\sqrt{x^3}$
c. $\sqrt[3]{16}$
d. $\sqrt[5]{-64}$
e. $\sqrt[4]{-32}$
f. $\sqrt{a^5b^4}$
g. $\sqrt{200} + \sqrt{18}$
h. $5\sqrt[3]{24} - 2\sqrt[3]{81}$
i. $\sqrt[4]{a^4} - \sqrt[5]{a^6}$
j. $5\sqrt{75} - 10\sqrt{12}$

8. Rationalize each of the following:

a $\dfrac{1}{\sqrt{3}}$
b. $\sqrt{\dfrac{1}{3}}$
c. $\dfrac{5}{\sqrt{5}}$
d. $\dfrac{a}{\sqrt{b}}$
e. $\sqrt{\dfrac{3}{5}}$
f. $\dfrac{6}{\sqrt[3]{3}}$
g. $\dfrac{8 + \sqrt{2}}{\sqrt{2}}$
h. $\dfrac{\sqrt{5}}{\sqrt{7}}$
i. $\dfrac{10}{\sqrt{8}}$
j. $\dfrac{1}{\sqrt[3]{9}}$

9. The statement has been made in the text that a decimal number with a finite number of decimal places is a rational number. Show that this is so.

10. Justify the method of rationalization generally by showing that

$$\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

where, of course, $b \neq 0$.

11. Prove that

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

where $b \neq 0$.

12. Show that $\sqrt[n]{a}\ \sqrt[n]{b} = \sqrt[n]{ab}$.

I-28 REVIEW

1. Find the value of each of the following:

a. $(-125)^{1/3}$
b. $16^{-3/4}$
c. $\sqrt{4.0 \times 10^{-12}}$
d. $(1.3 \times 10^{-7})^{-2}$
e. $\sqrt[3]{16} - \sqrt[3]{54}$
f. $2\sqrt{50} + 3\sqrt{98}$

2. Express the values of the following as a single fraction and without the use of negative exponents:

a. $x^{-2} + x^2$

b. $5x^{-1} + \dfrac{1}{x^{-1}}$

c. $(2a)^{-3} - (a)^{-2}$

d. $a^{-2}b + ab^{-2}$

e. $\sqrt{\dfrac{1}{x^2}} - 2x^0$

f. $(a^{-1} + b^{-1})(a^{-1} - b^{-1})$

3. Express the values of the following with rational denominators and with integral radicands:

a. $\dfrac{5}{\sqrt{5}}$ b. $\dfrac{1}{\sqrt{x^3}}$ c. $\dfrac{2}{\sqrt[3]{2}}$ d. $\sqrt{\dfrac{5}{8}}$ e. $h^{-4/3}$

4. Prove that $\sqrt{2} + 3$ is an irrational number. (The first step is the same as that used to show that $\sqrt{2}$ is irrational.)

5. With respect to $\sqrt[m]{p}$, list the various restrictions that have been placed on m, p, and $\sqrt[m]{p}$ itself.

6. Earlier we expressed natural numbers as powers of 10. The same may be done for numbers between 0 and 1. For example, $0.24 = 2(10)^{-1} + 4(10)^{-2}$.

a. Express in the same way: 0.385, 0.0403.

b. Express similarly the binary numbers 0.11, 0.10011.

c. Find the decimal values of the binary numbers in part (b).

7. The number $2n + 1$ is odd for all natural number values of n.

a. Check this for various values of n.

b. Use $2n + 1$ to prove that the square of any odd number is odd.

29. EVER FORWARD

We have been forced to restrict our radicands to positive values for even roots. This is contrary to the spirit of completeness that has pervaded our approach to the number system. Why not $\sqrt{-1}$?

We have no number which, when squared, yields a negative result. We can either arbitrarily discard this case or *create* a new number in response to this challenge. The latter course is the one we have been following. We make, therefore, the following definition:

Let i be a new **number** *such that its square is -1.*

$$i^2 = -1 \qquad \text{or} \qquad i = \sqrt{-1}$$

Unfortunately, the letter i is taken from the word *imaginary*, which was and still is the name given to the number. This stigma belies the real usefulness of such numbers in modern technology and reflects only the limited view of the originators of the definition.

The mathematical view, as before, is to select such additional definitions of imaginary numbers as to impose complete consistency of their operations

with all previous postulates, definitions, and derived conclusions of the number system. By extending the notion of factorability, we have

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$$

and

$$\sqrt{-3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$$

One should note that in addition to i and i^2, we have, in the light of the maintenance of the laws of exponents,

$$i^3 = i^2\,i = -i$$
$$i^4 = i^2\,i^2 = 1$$
$$i^5 = i$$

and so forth.

All integral powers of i will then be either ± 1 or $\pm i$. It need never be necessary, therefore, to go beyond the first power for imaginary numbers.

Because of our concept of similar terms, we have

$$2i + 3i = 5i \qquad \text{and} \qquad 5 + 2i = 5 + 2i$$

By analogy with products of irrationals, we have also

$$\sqrt{-9}\sqrt{-4} = 3i \cdot 2i = 6i^2 = -6$$

However, this example indicates that we must make some qualification. For, had someone taken the product before utilizing the imaginary notation, we would have had, instead, $\sqrt{36}$. This would give us 6 and not -6. In fact other ambiguities may enter if we do not impose some qualifications. Suppose that someone performed the following factoring:

$$\sqrt{-4} = \sqrt{-1}\sqrt{-1}\sqrt{-1}\sqrt{4}$$

This would lead to the unhappy conclusion that

$$2i = -2i$$

These illustrations indicate that we should express immediately in terms of i any imaginary number we meet. (We could make a more formal restriction that $\sqrt{a}\sqrt{b} = -\sqrt{ab}$ when both a and b are negative.) Further, we see that in factoring $\sqrt{-a}$, where a is positive, the factor $\sqrt{-1}$ should be used only once, or $\sqrt{-a} = i\sqrt{a}$.

The preceding illustration of nonsimilar terms, $5 + 2i$, tacitly introduces a more general number of the type of

$$x + iy$$

This is called a *complex number* and consists of a *real part* x and an *imaginary part* y. By contrast, the rational and irrational numbers taken together are called **real numbers.**

When x and not y is 0 as in $3i$, we say that we have a *pure imaginary number*. When y but not x is 0 as in 5, we have a *real number* x. When both x and y are 0, we have the real number 0. In effect then, *the rational number system is completely included in the complex number system.*

As with all other new numbers, we take a look at operations with complex numbers.

a. $(a + bi) + (c + di) = (a + b) + (c + d)i$ Commutative and distributive laws
$2 + 3i + 5 - 7i = 7 - 4i$

b. $(a + bi)(c + di) = (\text{ac} - \text{bd}) + (\text{bc} + \text{ad})\text{i}$

$$\begin{aligned}(2 + 3i)(5 - 7\text{i}) &= 10 + 15i - 14i - 21i^2 \\ &= 10 + 15i - 14i + 21 \\ &= 31 + i\end{aligned}$$

c. $\dfrac{5 + 2i}{1 + i} = \dfrac{5 + 2i}{1 + i} \cdot \dfrac{1 - i}{1 - i} = \dfrac{7 - 3i}{2} \qquad \left(\dfrac{1 - i}{1 - i} = 1\right)$

The quantities $1 - i$ and $1 + i$ are called *conjugates* of each other. In general, $a + bi$ and $a - bi$ are complex conjugate numbers.

We note that the sum, difference, product, and quotient of complex numbers are, in general, complex numbers.

EXERCISES (I-29)

1. Represent each of the following in the imaginary unit i in simplest form:

a. $\sqrt{-9}$

b. $\sqrt{-25} - \sqrt{-16}$

c. $\sqrt{-8}$

d. $5\sqrt{-18} + \sqrt{-8}$

e. $\sqrt{-a^2}$

f. $\sqrt{-m^3}$

g. $5\sqrt{12} + \sqrt{-27}$

h. $\sqrt{-1/4} + \sqrt{-4/9}$

i. $2\sqrt{-a^2} - 3\sqrt{-a^2}$

j. $5i^3 + 6i^5$

2. Remove the imaginary term from the denominator:

a. $\dfrac{10}{\sqrt{-4}}$

b. $\dfrac{7}{i}$

c. $\dfrac{8}{i^3}$

d. $\dfrac{12}{1 - i}$

e $\dfrac{1}{2 + 3i}$

f. $\dfrac{6 + i}{i + 2}$

g. $\dfrac{\sqrt{-16}}{\sqrt{-4}}$

h. $\dfrac{1}{1 + 1/i}$

3. a. Show that the product of a complex number by its conjugate is always a positive real number.
   b. If $m + \sqrt{n}$ is a irrational number, with m and n positive rationals, then $m - \sqrt{n}$ is its conjugate irrational number. Show that the product of two conjugate irrationals is a rational number.
4. Rationalize the denominator in each of the following:

a. $\dfrac{14}{3 + \sqrt{2}}$ d. $\dfrac{6}{\sqrt{3} + \sqrt{2}}$

b. $\dfrac{6}{5 - 2\sqrt{3}}$ e. $\dfrac{\sqrt{5}}{\sqrt{5} - \sqrt{2}}$

c. $\dfrac{1}{\sqrt{5 - 1}}$ f. $\dfrac{1}{a + b\sqrt{c}}$

5. Perform the indicated operations and simplify the result:

a. $(3 - 2i) + 2(5 + i)$ d. $(2 + 3i)(4 + i)$
b. $(6 + i) - 3(2 - i)$ e. $(a + bi)(a - bi)$
c. $(8 + i) - (6 + i)$ f. $(a + bi)^2$

6. Show that the conjugate of the product of two complex numbers is equal to the product of their conjugates.

7. Discuss the fallacy: $i = i, \sqrt{-1} = \sqrt{-1}, \sqrt{\dfrac{-1}{+1}} = \sqrt{\dfrac{+1}{-1}}, \dfrac{\sqrt{-1}}{\sqrt{1}} = \dfrac{\sqrt{+1}}{\sqrt{-1}}, \dfrac{i}{1} = \dfrac{1}{i}; i^2 = 1.$

30. THE ENTRANCE OF GEOMETRY

One of the most familiar occurrences of numbers is in conjunction with diverse instruments and scales. Rulers, tape measures, thermometers, barometers, and many, many other similar objects attest to the correlation of numbers with points on a line. It is necessary, therefore, that we take note of the entry of these geometric elements.

Now, the elemental components of any field cannot be defined because of their inherent simplicity. A definition utilizes, as a synonym, a simpler word. It is immediately redundant to imagine the existence of a word simpler than the simplest in any related list of words. To search a dictionary for the synonyms of the word "residence," for example, would uncover such words as building, home, dwelling, abode, and something like "the place where one lives." In this context the last phrase and the word "place" are used functionally or descriptively. The end of the list of synonyms is in sight. After all, can an infinite regression possibly exist in a vocabulary? Basic, elemental, *primitive terms* such as "place" must be left *undefined.*

It is a fact that we have already employed some undefined terms such as *number*, *equal*, *addition*, and *multiplication*, as well as a series of terms necessary for logical communication such as *is*, *not*, *and*, *but*, *if*, *all*, and others. The meanings of these words have been acquired through the postulates that employ them and govern their use.

In the same way, in turning our attention to the straight line, we state forthwith that *space*, *figure*, *straight line*, *point*, and *distance* are and will remain undefined.

On the other hand, the **postulates are the assumed propositions** on which our deductions are based. They are as basic to the whole deductive system of propositions as are the undefined terms to the development of the concepts. *Geometry requires a foundation of postulates in the same way that arithmetic requires them.* Thus we postulate first that *the points* on a straight line can be numbered so that the *absolute value* of the difference between any two numbers *measures a distance* between the corresponding points.

A (at 1), B (at 4)

0 1 2 3 4 5 6

$$AB = |1 - 4| = |4 - 1| = 3 \text{ units}$$

31. ACHIEVING UNITY

With this postulate as our foundation, we are ready to construct a geometric analogue of our number system. Starting with any point on a line (hereafter meaning a straight line) and with any distance as a unit, we can determine a series of points with which our positive and negative integers may be directly associated.

$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

By means of equal divisions of the unit and of the line, points may be found with which to associate the rational numbers as $2/3$ and $-3/4$. The *irrational number points*, however, or just *irrational points*, are not so easily located, since rational divisions will not suffice. At the moment, only approximations, very close points, will be possible. Theoretically accurate determinations will have to await further development of geometric principles.

After the location of all (a mental *all*) rational and irrational points, a complete **one-to-one correspondence** is established between all the real numbers and all the points on a line. To each point there is associated one and only one real number, and to each real number there is associated one and only one point. We shall see later that this is also the basis of the development of *analytic geometry*. At the moment we are concerned only with the geometrical representation of the number system that has been developed thus far.

32. A LARGER DOMAIN

The points of the single line in the preceding article have been completely exhausted in the representation of the real numbers. To provide for the imaginary and complex numbers, it is necessary to introduce another line, or more technically, a second "axis" (see Fig. I-3).

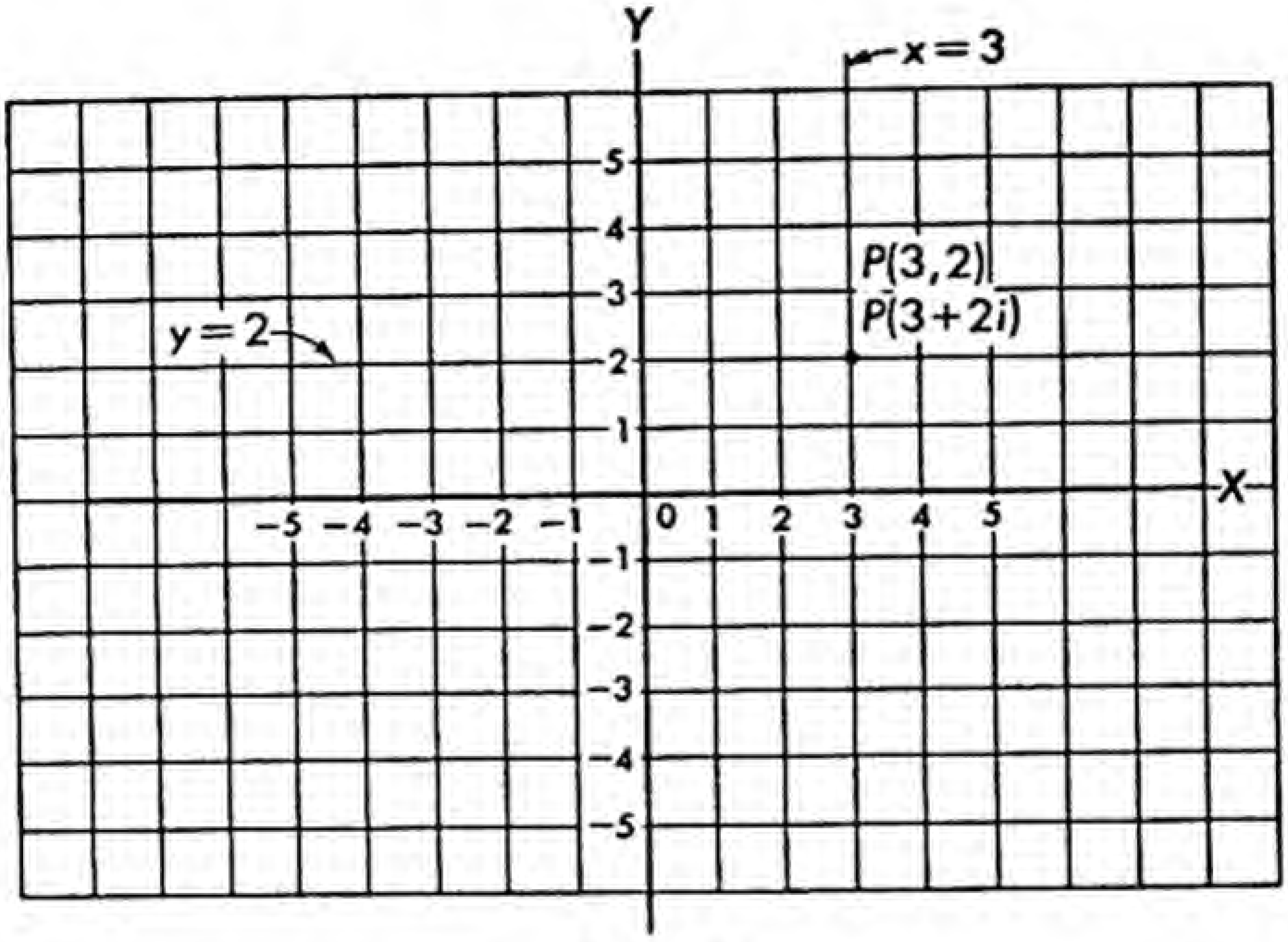


Fig. I-3

Let us call the first axis, the horizontal one, the *X-axis*, or *axis of real numbers*. We introduce a *Y-axis* at 90° to the first. (The notion of "90 degrees" will shortly be treated more adequately.) The point of intersection of the two axes will be called the *origin*, which is taken to be the *O* point for each. We can lay off our real units on both axes in the same way as we did for the single line.

We superimpose a *grid* of lines on these axes by drawing lines at 90° to each of the axes. Next we extend the first representation of real numbers. Thus $x = 3$ will no longer be the point 3 on the X-axis but will be the entire vertical line at 90° to the X-axis through the point which formerly was 3. Similarly $y = 2$ is the entire line at 90° to the Y-axis through the point which is two units on the Y-axis above the origin. These are indicated in the drawing (Fig. I-3). It is easy to recognize that our scheme is identical with that of the lines on road and geographic maps.

The point P, where the lines $x = 3$ and $y = 2$ meet, is to be considered as the representation of the number pair ($x = 3$, $y = 2$) simultaneously, or just (3,2).

Now consider any complex number

$$x + iy$$

If the x and y are selected as above, we can have a unique representation for each and every complex number. Thus the particular point P (Fig. I-3) may be used to correspond to the complex number $3 + 2i$. Other illustrations are shown in Fig. I-4.

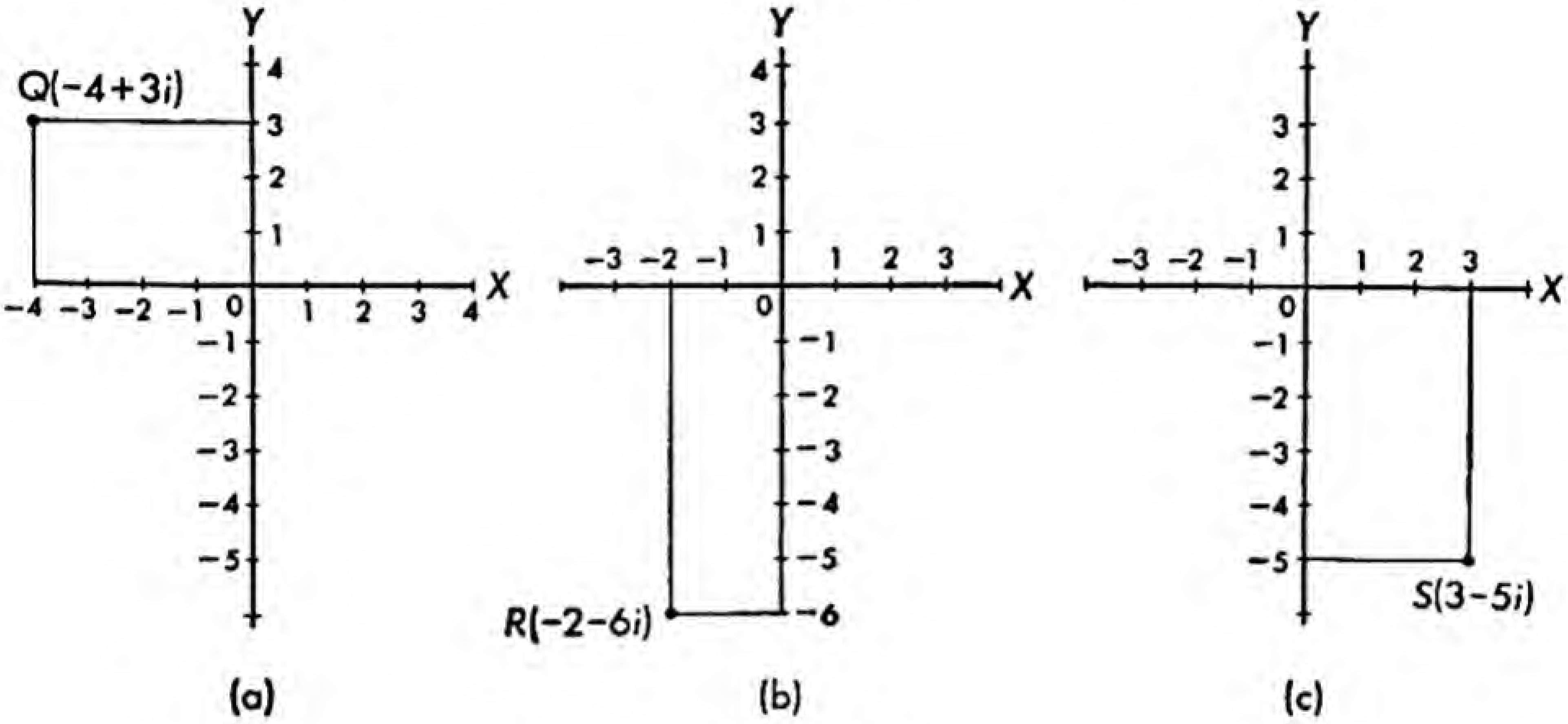


Fig. I–4

Since the X-axis itself represents $y = 0$, it follows that the points on this axis represent, as before, all the real numbers; for example, $5 + 0i = 5$. Also, the Y-axis represents the fact that $x = 0$. Thus all points on the Y-axis represent the pure imaginaries, $2i = 0 + 2i$.

In brief, the suggested graphical system has a point representation of the whole number system as presently conceived. The advantages of this relationship are yet to be realized.

EXERCISES (I-32)

1. Sketch the following graphically:

a. $5 + 4i$
b. $3 + 3i$
c. $-1 + 4i$
d. $-3 - 2i$
e. $5 - 4i$
f. $3a + 2ai$
(Use an arbitrary a as a unit.)

2. Using the representation of number pairs (x, y) for complex numbers $x + iy$, represent each of the following graphically:

a. $(2, 0)$
b. $(0, 3)$
c. $(5, 1)$
d. $(-3, 2)$
e. $(4, -3)$
f. $(-5, -6)$

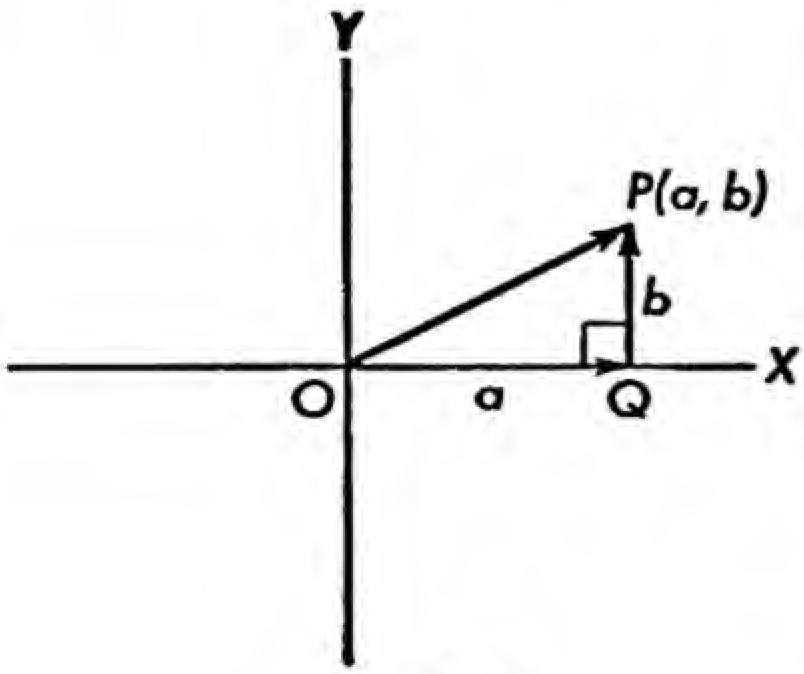


Fig. I–5

3. The complex number $3 + 2i = (3, 2)$ may also be represented by the **vector** "$\overrightarrow{OP}$". A vector is a quantity that has magnitude and direction. The magnitude of the vector $\overrightarrow{OP}$ (see Fig. I–5) is the length $|OP|$, and the direction will refer to the angle that OP makes with the positive portion of the X-axis.

$\overrightarrow{OQ}$ is called the *horizontal component* of $\overrightarrow{OP}$, and $\overrightarrow{QP}$ is the vertical component. Since (a, b) is $a + bi$, we define

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$

That is, a vector is equal to the sum of its horizontal and vertical components.

Sketch each of the following vectors, indicating the vertical and horizontal components:

a (5, 2)
b. (−3, 4)
c. $-5 - 6i$

4. a. With the real number pairs representing complex numbers, show that $(a, b) + (c, d) = (a + c, b + d)$.

b. State the result in terms of horizontal and vertical components.

5. Show how the conclusions in exercise 4 may be used to add two vectors graphically.

I–32 REVIEW

1. Find the sum of the following vectors graphically:

a. $(1 + 4i), (2 + 2i)$
b. $(5 - 3i), (-2 - 2i)$
c. $(-4 + 2i), (-3 - i)$

2. If (a, b) and (c, d) represent complex numbers, prove that $(a, b)(c, d) = (ac - bd, bc + ad)$.

3. We define (0, 0) as the zero element of the complex numbers.

a. Where is this on the graph?

b. Prove: if $(a, b) = (c, d)$, then $a = c$ and $b = d$.

(The postulates of equality are assumed to hold for complex numbers too.)

c. State in words the conclusion in (b).

4. Show that

a. $\sqrt{36} \neq \sqrt{-9}\sqrt{-4}$
b. $\sqrt{a^2} \neq \sqrt{-a}\sqrt{-a}$

(The symbol $\neq$ is read "is not equal to.")

33. DEDEKIND'S DEFINITION OF IRRATIONALS

In previous text we assigned rational numbers to the points of a line through the use of an arbitrary unit and through equal subdivisions of the unit.

The $\sqrt{2}$, for example, may be conceived as the number which divides the rational number system and its corresponding points on a line into two classes (Fig. I-6). In one class, M, we include the negative numbers, 0, and all the positive rationals whose squares are less than 2. In class N, we place all the rational numbers whose squares are greater than 2.

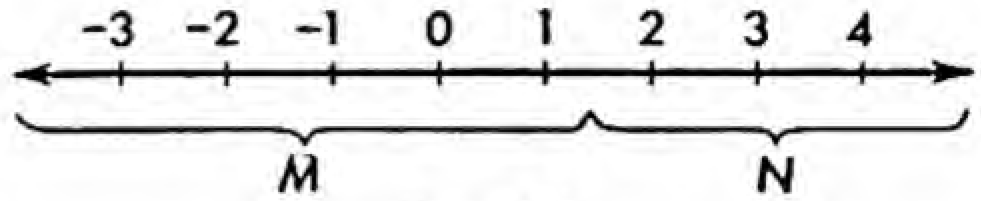


Fig. I-6

It would be well to recall that between any two rational numbers, no matter how close, there is an infinite number of other rationals. For example, between 98/1000 and 99/1000 it is possible to insert 981/10,000, 9801/100,000, $\cdots$ interminably. Or, decimally, between 0.098 and 0.099 one can select endlessly such numbers as 0.0981 and 0.09800001.

This attribute of rational numbers is described by saying that the rational numbers are **everywhere dense.** Within the immediate neighborhood of a rational number or rational point, no matter how small, there are an infinite number of other rationals or points. There just is no *next* larger or smaller rational number or rational point.

In spite of the "everywhere denseness" of the rational numbers and points there are *gaps* in the picture. For, if we return to our classes M and N mentioned before, we note that class M has no largest rational number and class N has no smallest rational number. The $\sqrt{2}$ represents a **cut** or separation between the two classes. In this manner, by means of a cut, Dedekind defined the irrational number.

EXERCISES (I-33)

1. The average of two numbers p and q is $\frac{1}{2}(p + q)$. Show how this definition may be used to indicate that there are an infinity of rationals between any two rationals.

2. In another sense, the Dedekind cut may be defined as a classification of the rational numbers into two classes P and Q such that every element of one class is larger than every element of the other.
   a. Show how 0 may be defined as a Dedekind cut.
   b. Define P and Q so that P will have 2 as its largest element. Has Q a least element?
   c. Redefine P and Q so that Q has 2 as a least element. What may be said of the class P?
   d. If P and Q define an irrational element, what may be said concerning largest and least elements of the two classes?

3. Another interesting approach to the definition of an irrational number is through a sequence of **nested intervals.**

An interval is taken with rational end points. Within this another interval is taken in the same way and so on indefinitely. The length of the nth interval tends toward 0. (A fuller account of sequences and limiting tendencies lies ahead of us.)

It is postulated that a sequence of intervals so selected defines a unique point.

By way of illustration, consider the following intervals with their indicated end points:

I_1: 1, 2	I_4: 1.414, 1.415
I_2: 1.4, 1.5	I_5: 1.4142, 1.4143
I_3: 1.41, 1.42	....

By continuing in this fashion, one defines the $\sqrt{2}$.

a. Indicate a few of the nested intervals that may be used to define $\sqrt{3}$.

b. Let I_n and I'_n represent the nth intervals of two irrational numbers whose end points are respectively (a_n, b_n) and (a'_n, b'_n). Let $a_n + a'_n$ and $b_n + b'_n$ be the respective end points of the nth interval of $I_n + I'_n$. Show how this procedure may be used to define the sum of two irrational numbers such as $\sqrt{2} + \sqrt{3}$.

c. How would the product $\sqrt{2}\sqrt{3}$ be defined by means of nested intervals?

34. THE COMPARISON OF INFINITIES

Strangely enough it has been discovered that there are *more* irrational numbers than rational numbers. But, what meaning can *more* or *less* have when dealing with an infinitude of elements? How can such collections be compared?

Let us begin by considering the positive integers arranged as follows: (We omit the negative integers and 0 for convenience.)

A:	1	2	3	4	5	6	7	8	$\cdots\ n\cdots$
B:	2	4	6	8	10	12	14	16	$\cdots\ 2n$
C:	3	6	9	12	15	18	21	24	$\cdots\ 3n\cdots$
D:	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	$\cdots 10^n \cdots$

The ingenious procedure of comparison, as developed by Cantor, is that of **1-to-1 correspondence.** The correspondence is taken vertically. To every number in A there corresponds uniquely a number in B which is double that of the one above it in A. Conversely every number in A corresponds to one in B uniquely and is one-half of the number below it. Thus, to every number in the endless sequence in A, there corresponds one and only one in B, and to every one in B, there corresponds one and only one in A.

Similarly, A and C are in 1-to-1 correspondence, as are also A and D and every other pair of the foregoing sequences. If this matching process is imagined in its endless extension, no one of the sequences will outlast any of the others in spite of the fact that each sequence consists of selected elements of A.

The essence of size resides in this matching technique. We say that sequences (or, more generally, sets of any elements whatsoever) which can be placed in 1-to-1 correspondence have the same **cardinal number.**

Where the sets A to D are infinite in extent, they have a **transfinite cardinal number.** And, where the set can be put into 1-to-1 correspondence with the set of natural numbers, the set is said to be **denumerable** and is assigned, symbolically, the transfinite cardinal number $\aleph_0$, *aleph-null.*

It was a very great surprise to most people when Cantor went on to show that the set of rational numbers is no larger than the set of natural numbers. This is clearly shown in the following ingenious arrangement:

$$\begin{array}{ccccccc}
1 \longrightarrow & 2 & 3 \longrightarrow & 4 & 5 & 6 & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{5} & \frac{2}{5} & \cdots & & & &
\end{array}$$

By following the direction of the arrows, we have the sequence

$$1,\ 2,\ 1/2,\ 1/3,\ 2/2,\ 3,\ 4,\ 3/2,\ 2/3,\ 1/4,\ 1/5, \cdots$$

If we delete those values that have already made an appearance, we get

$$1,\ 2,\ 1/2,\ 1/3,\ 3,\ 4,\ 3/2,\ 2/3,\ 1/4,\ 1/5, \cdots$$

This procedure yields an ordering of rational numbers in which each one appears once and only once and in which each one has a unique position in the sequence. Consequently this last sequence has a first, a second, a third term, and so forth, which means that the sequence is in 1-to-1 correspondence with the natural numbers. The rational numbers are therefore denumerable and may be assigned the cardinal number $\aleph_0$. There are no more and no less rational numbers than there are integers.

We turn now to the real number system which includes, of course, all the rationals given above as well as the irrationals. For convenience once more, we restrict ourselves to the positive values. We are immediately faced with a new problem and that is to find a common method of representation for all the real numbers. The common basis is found in the decimal representation of numbers. Thus, $\sqrt{3} = 1.732\cdots$ It is desirable, therefore,

to follow suit and to represent rationals with endless decimal places. For example,

$$\tfrac{2}{3} = 0.66666666\cdots$$
$$\tfrac{3}{7} = 0.428571428571\cdots$$
$$\tfrac{1}{4} = 0.250000000000\cdots \qquad \text{or better} \qquad 0.2499999999\cdots$$

There is no apparent or ingenious approach to the ordering of these real numbers. Cantor's attack was through *indirect reasoning.* We suppose that the real numbers between 0 and 1 are denumerable and that some number is a first, another a second, and so forth. This could be indicated as follows:

1st number:	$.\,a_1a_2a_3a_4a_5\cdots$
2nd number:	$.\,b_1b_2b_3b_4b_5\cdots$
3rd number:	$.\,c_1c_2c_3c_4c_5\cdots$
and so forth	$\cdots$

If this is indeed possible, then this interminable list somewhere contains each and every real number without exception.

But Cantor constructed a real number that is not in the sequence. We start by taking A_1, any integer different from a_1, and B_2, any integer different from b_2, and so forth. We may avoid 0's and 9's to avoid confusion with cases like $2.50000\cdots$ and $2.499999\cdots$. In this manner we construct the number

$$.\,A_1B_2C_3D_4E_5\cdots$$

which differs from each and every number in the list above in at least one decimal place. It differs from the first number in the tenth's place, from the second number in the hundredth's place, and so forth.

Thus the supposition that the real numbers are denumerable is fallacious. The real numbers are not denumerable and must represent a higher order of infinity than the denumerable set. This new case is symbolized by the transfinite cardinal number C.

If we accept tentatively the intuitive concept of a line as a *continuous* figure with no gaps in it, then the line can be referred to as a *continuum,* and the real number system, which is in 1-to-1 correspondence with the points on the line, is also a continuum. The cardinal number C is taken from this word.

It is an interesting and challenging fact that while transfinite numbers larger than C have been shown to exist, it has not been shown whether transfinite numbers between $\aleph_0$ and C exist.

EXERCISES (I–34)

1. Show that the set of integers is denumerable (no restrictions to positive integers only).

2. Show that the set of rationals is denumerable.

3. Show that the positive rational numbers may be ordered by means of this stipulation: Select all rational numbers m/n which have the same $m + n$ and arrange these in decreasing size. Let each m/n group precede an $m + n + 1$ group. Illustrate.

4. If by the sum of two sets we mean the set consisting of the elements that are in either or both sets, show that the sum of two denumerable sets is a denumerable set. Illustrate.
(The sum of two sets is also called their **union,** which is symbolized by "$\cup$" and often by "+". The sum or union of the sets A and B is written as $\boldsymbol{A \cup B}$ or $\boldsymbol{A + B}$.)

5. If A is the set of all rational numbers and B is the set of all irrationals, what is the set $A \cup B$?

6. If A is the set of all rational numbers and B is the set of natural numbers, describe the set $A \cup B$.

7. What is the cardinal number of the set $A \cup B$ if A is the set of positive even integers and B is the set of positive odd integers?

8. By the concept of **product** or **intersection** of two sets, A and B, is meant the set of elements which the classes have in common. This product set is symbolized by $\boldsymbol{A \cap B}$ or $\boldsymbol{AB}$.

   a. If A is the set of even numbers and B is the set of multiples of three, describe the set $A \cap B$.

9. If A is the set of natural numbers and B is the set of even numbers, describe the set $A \cap B$.

10. Let N represent the cardinal number of a set of a finite number n of elements. By means of illustration or description support the plausibility of the following true conclusions:

a. $\aleph_0 + N = \aleph_0$ c. $\aleph_0 + \aleph_0 = \aleph_0$
b. $\aleph_0 + C = C$ d. $C + C = C$

11. Modify the proof of the nondenumerability of the real numbers between 0 and 1 by considering the numbers as written in the binary system.

12. Show that the set of the reciprocals of the integers is denumerable. Of course one reciprocal is undefined. Which?

13. If $A \cup B$ is the set of real numbers and B is the set of irrationals, what may be said of the set A?

14. If A is the set of positive odd integers and B is the set of positive even integers, describe the set $A \cap B$.

15. If A is the set of rational numbers and B is the set of irrationals, describe the set $A \cap B$.

16. If M is the set of Republicans and T the set of rich people, interpret

(a) $M \cap T$ (b) $M \cap T$ (c) TM (d) $T + M$

Devise diagrams to illustrate your conclusions.

II — GEOMETRY IN MATHEMATICS

1. POINT AND LINE

The mathematical concepts of point and straight line evolved gradually in human consciousness, aided by environmental experience and abetted by the compulsion to create intellectual order through generalized concepts.

The stars in the skies and the tiniest dots anywhere gave impetus to the abstraction of the mathematical point as the substanceless dot, while taut threads and the rays of light led to the mathematical straight line.

These were fruitful and suggestive thoughts, although it is impossible to reduce the physical point to the idealized mathematical point. We return, therefore, to the formal approach wherein the mathematical concepts of point and line, each undefined, are given relatedness through another postulate.

Any two distinct points are on one and only one straight line.

Thus if A and B are two distinct points (Fig. II–1), there is one and only one straight line AB containing them. Should another straight line exist, CD, distinct from AB, it follows immediately that CD and AB can have at most one point in common.

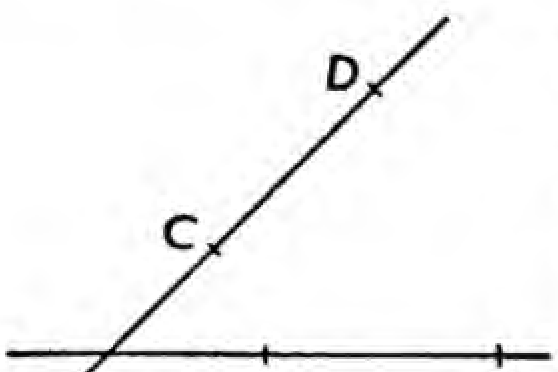


Fig. II–1

We have had occasion to refer to portions of lines. The word *interval* has been used to refer to a portion of a line that is included between two distinct points. The two *end* points are considered as belonging to the set of points. In this case the interval is called a **line segment.** If, on the other hand, we refer to a portion of a line which has only one end point included in the set of points concerned, we say that we have a **half-line** or **ray.** Thus, by way of illustration, the X-axis is a line; the part of the

X-axis between $x = 2$ and $x = 5$ inclusive is a line segment; and the positive portion of the X-axis with O included is a ray.

EXERCISES (II-1)

1. Consider the set of points on the X-axis. Suppose that m and n represent two fixed, real numbers that are distinct. Describe or name the intervals that are given by each of the following where x is described by:

a. $m \leq x \leq n$
b. $x \geq m$
c. $x > n$
d. $m \leq x < n$
e. $m < x < n$
f. $x \leq n$
g. $m < x \leq n$
h. x

2. THE ANGLE

To assure ourselves the existence of other lines in the plane, we add to our list of postulates *the existence of at least one point, P, not on the line AB* (Fig. II-2). This immediately introduces an infinity (of cardinality C) of other lines, since P determines another line with each point on AB (Fig. II-3).

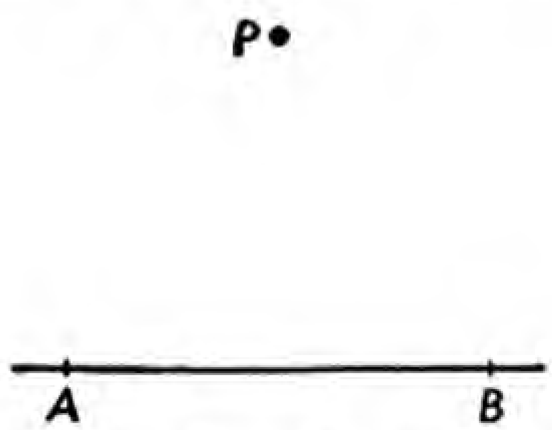


Fig. II-2

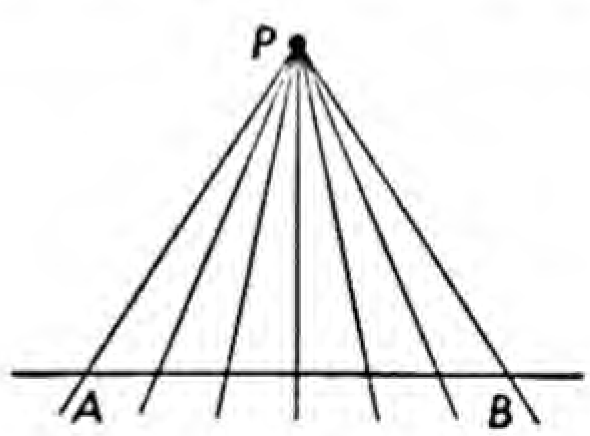


Fig. II-3

Each of the new lines possesses other points which in turn help to determine other lines. The set of points and lines are referred to as a *plane configuration* or just a *plane*. From this welter of lines and points, we can single out first the simple configurations formed by two rays with a common end point, as in Fig. II-4.

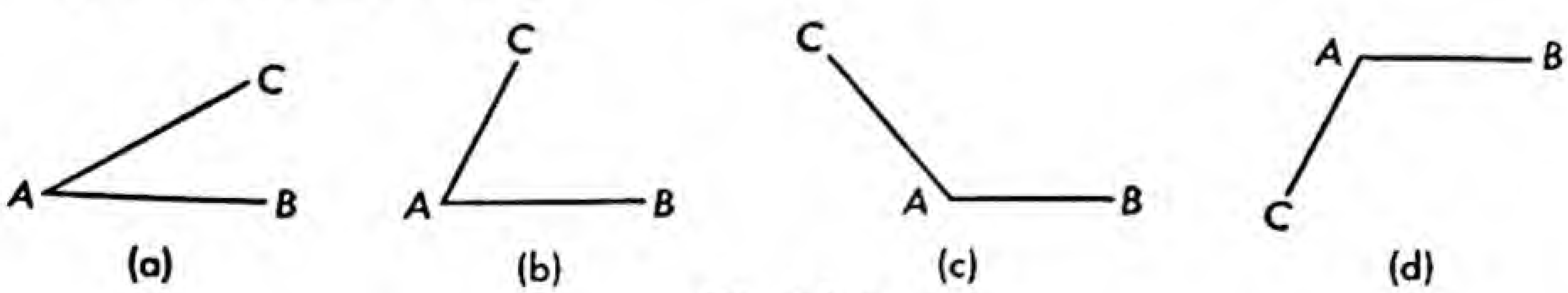


Fig. II-4

Of course, this configuration is called an **angle**. The point A is called the *vertex* and the half-lines CA and AB, the *sides* of the angle. In the preceding illustrations, all the angles, if appearing singly, can be referred to as $\angle A$, that is, "angle A". To avoid confusion, if need be, the angles may also be

read as $\angle CAB$ or $\angle BAC$, where the middle letter is always the vertex.

Since the ray CA may be chosen arbitrarily relative to the ray BA, we see the possibility of different angles of different *size*.

The sense of the existence of a quantity with respect to the angle raises the problem of measurement and a unit of measure. It should not be overlooked that units of measure are arbitrary. The inch, the pound, and the hour are not only instances of units of measure but are also remnants of many other units adopted in the past.

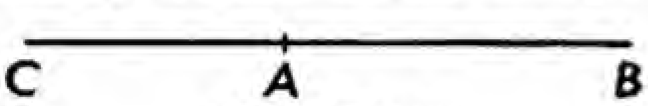


Fig. II–5

For the angle, we choose as a basis for determination the special angle where the sides of the angle lie on the line BC (Fig. II–5).

Since the angle CAB, $\angle CAB$, has its sides on one straight line, an obvious name for this case is the **straight angle.** In point of fact, we have two straight angles in this diagram, depending on whether we view the figure from above or below.

Consistency requires that whatever unit we assign to one straight angle, we also assign to the other. Otherwise we run the risk of a system of measurement that is relative to position and which is neither inconceivable nor useless. Rather, we seek at first a stable, rigid system. To guarantee this outlook, we need some other postulate such as:

All straight angles are equal to each other.

As with all measurement, units of subdivision are desirable. Too large a unit is cumbersome for some purposes, and too small a unit has a like objection at other times. The inch, for example, is far too small for astronomical purposes and far too big for microscopical purposes. The straight angle is too large a unit; subdivision is desirable.

The straight angle is conceived to be divided into 180 equal parts, each of which is called *one degree* (1°). The two straight angles in Fig. II–5 will contain 360° together. Each degree is considered as containing 60 equal *minutes* (60′), and each minute is divided into 60 equal *seconds* (60″).

This scale of 60's stems from the early Babylonians who found it arithmetically convenient because of the many integral divisors of 60.

The discussion of measurement of angles brings to mind the earlier discussion regarding the measurement of line segments which was based on the postulated concept of the numbering of points. The only difference is that here we are concerned with the numbering of half-lines.

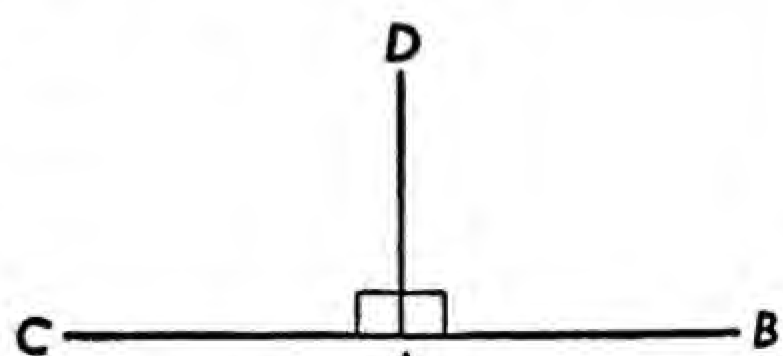


Fig. II–6

The line AD, which *bisects* the straight angle in Fig. II–6, cuts it into two equal 90° parts. AD is said to be **perpendicular** to CB, $AD \perp CB$. The 90° angle is called a **right angle.**

Angles between 0° and 90° are generally characterized as **acute angles.** Those between 90° and 180° are called **obtuse angles**, and those between 180° and 360°, **reflex angles.**

EXERCISES (II–2)

1. The French have used the *grade* system of measuring angles wherein the straight angle contains 200 grades. In military circles the straight angle is often taken as *3200 mils*. Describe acute, right, and obtuse angles in all three systems.
2. Change the following measurement to degrees and minutes:

   a. $62\frac{3}{8}°$ b. 15.75°

3. Find the value of each of the following:

   a. $38°47' + 26°52'$ b. $146°21' - 72°38'$

3. RECTANGULAR COORDINATES

We return to the two perpendicular axes of Chapter I, by means of which we have set up a **rectangular coordinate** system. This time we shall use only real values on both axes (Fig. II–7). An x-value is called an

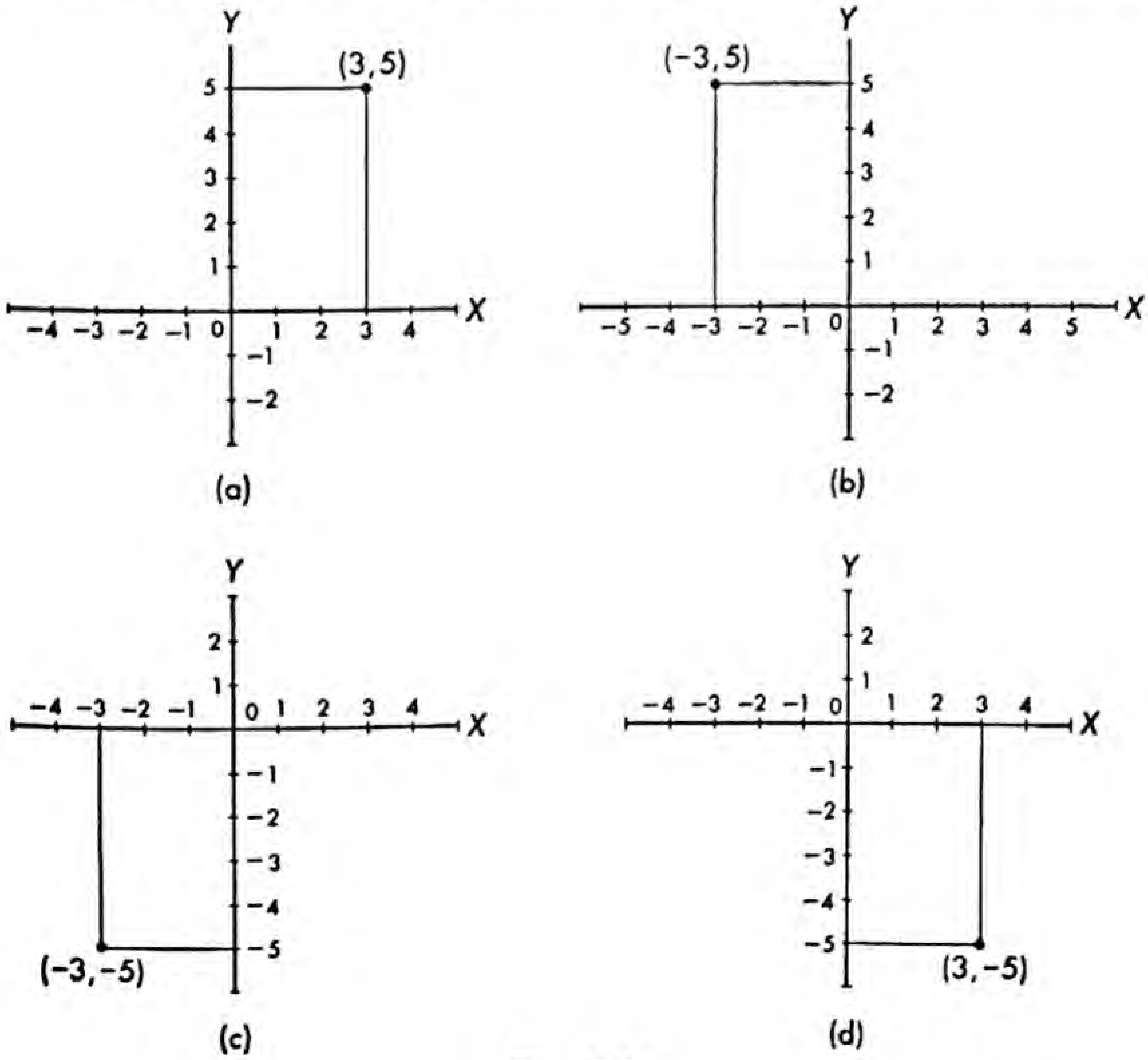


Fig. II–7

abscissa; a y-value, an **ordinate**; and the two together, **coordinates.** The ordered pair (3,5) in which $x = 3$ and $y = 5$ is represented by one and only one point.

Strangely enough, there are no more points in the plane than there are on either axis or even on a small segment of either axis. The cardinality of the plane is also C.

To illustrate this, let us restrict ourselves, again for convenience, to $0 < x < 1$ and $0 < y < 1$, that is, to the interior of a unit square in the first quadrant at the origin. The interior points of the square can be put into 1 to 1 correspondence with the points in the open-ended segment $0 < x < 1$. We could begin by considering the point ($\frac{1}{2}$, $\frac{1}{3}$). Because of the occurrence of irrationals, we once again represent all numbers decimally to an infinite number of places. So, ($\frac{1}{2}$, $\frac{1}{3}$) becomes $(0.4999\cdots, 0.333\cdots)$.

We determine a point on the X-axis in the given interval which corresponds uniquely to this point. This is done by taking the first digit of the abscissa, followed by the first digit of the ordinate; then the second of the abscissa, and so forth alternately. We obtain in this way the number $x = 0.439393\cdots$. This can be done for every point within the square, yielding each time a unique point in the selected interval on the X-axis.

In general, for $(.a_1a_2a_3\cdots, .b_1b_2b_3\cdots)$ we form $x = .a_1b_1a_2b_2a_3b_3\cdots$, and so we have a 1-to-1 correspondence between the points within the unit square and the points within the unit segment. Of course this could be carried out for the entire plane.

Since the complex plane is adequate for the representation of all complex numbers $x + iy$, all the complex numbers have the same cardinality as the points in the plane, and so too have the points within the unit segment.

It will be interesting to note one means of setting up a one-to-one correspondence between segments of unequal lengths. Consider, for example, the segments AB and XY, in Fig. II–8. Their respective points may be put in one-to-one correspondence simply by means of lines drawn through P. To any point a_1 on XY there corresponds uniquely the point b_1 on AB, and to any point b_2 on AB there corresponds the point a_2. Thus every point on one segment has its unique counterpart on the other.

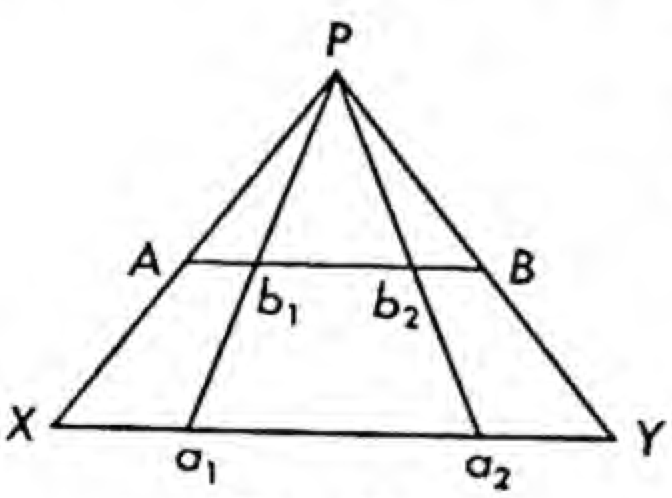


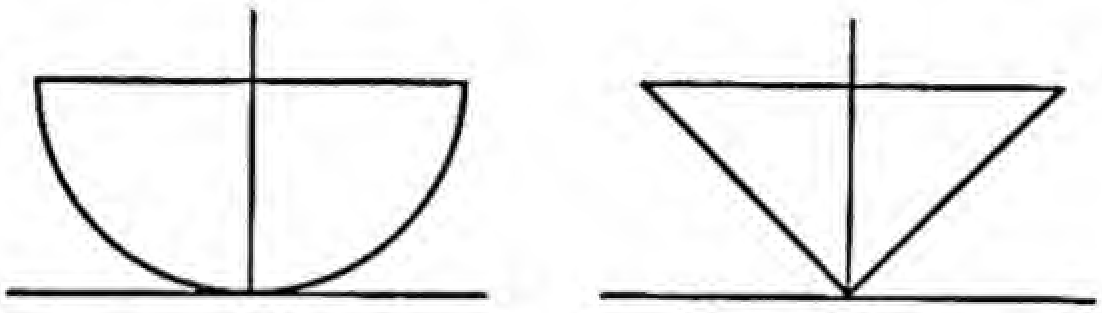
Fig. II–8

The line AB may easily be conceived as a small part of XY. This gives rise to the thought that the number of points in any segment of a line is the same as the number of points on the line. Both are of cardinality C. This resembles the circumstance in arithmetic where the number of integers, odd as well as even, are no more or no less than the even numbers alone. These cases underline a significant characteristic of an infinite set, that is, it may be put in one-to-

one correspondence with a part (subset) of itself. **The whole, in the infinite domain, is not greater than some of its parts.** This is a decisive distinction between an infinite and a finite set.

EXERCISES (II-3)

1. The diagrams in Fig. II-9 may be used to show how the points of a finite segment of the X-axis may be put into 1 to 1 correspondence with the entire X-axis. Describe how this can be done.



(a) (b)

Fig. II-9

2. Two line segments can be of different lengths but yet have the same number of points. We know that this is not a contradiction. Why is it not?

3. Examine the matter of the 1 to 1 correspondence between a finite class and a proper subset of this class. (A set M is a **subset** of set P if every element of M is an element of P. Set M is a **proper subset** if P contains at least one element not in M.)

4. Just as all numbers between 0 and 1 can be represented in the decimal system, so may they be represented in any other system with a finite base.

In the decimal system, a number such as 0.43256 represents

$$4(0.1) + 3(0.01) + 2(0.001) + 5(0.0001) + 6(0.00001)$$

or

$$4(10^{-1}) + 3(10^{-2}) + 2(10^{-3}) + 5(10^{-4}) + 6(10^{-5})$$

In general the number $0.a_1a_2a_3a_4 \cdots a_n \cdots$ is

$$a_1(10^{-1}) + a_2(10^{-2}) + a_3(10^{-3}) + a_4(10^{-4}) + \cdots + a_n(10^{-n}) + \cdots$$

This suggests the means of representation in any system. In the binary system, we have

$$a_1(2^{-1}) + a_2(2^{-2}) + a_3(2^{-3}) + a_4(2^{-4}) + \cdots + a_n(2^{-n}) + \cdots$$

The binary number 0.1101 is

$$1(2^{-1}) + 1(2^{-2}) + 0(2^{-3}) + 1(2^{-4})$$

which is the equivalent in the decimal system of

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{16} = \tfrac{13}{16}$$

a. Find the decimal equivalent of the binary numbers 0.1011, 0.0011, 0.010111.

b. How may we show the nondenumerability of the real numbers between 0 and 1 via the binary system?

4. DISTANCE IN THE PLANE

We have seen that the points on a straight line may be numbered and that the absolute value of the numerical difference of two of these numbers represents the **distance** between the corresponding points.

We extend this concept to the plane and state that the distance between any two points in the plane depends also on their respective coordinates, (x_1, y_1) and (x_2, y_2). Just how this is developed remains to be seen. At this juncture we are no longer in the position of making an arbitrary decision. Rather, the distance relationship must be consistent with all other relevant decisions, and in particular with the concept of distance on the one-dimensional line. However, although lacking in a definitive formula, we can anticipate the uniqueness of the distance between two points, and so we can speak legitimately of the distance between two points.

5. BASIC TRIANGLE RELATIONSHIPS

Geometric figures are all around us. We have had occasion to draw a few, although at the time we were not quite ready to examine them in their entirety. Fig. II–10 shows some other familiar examples. The first is the

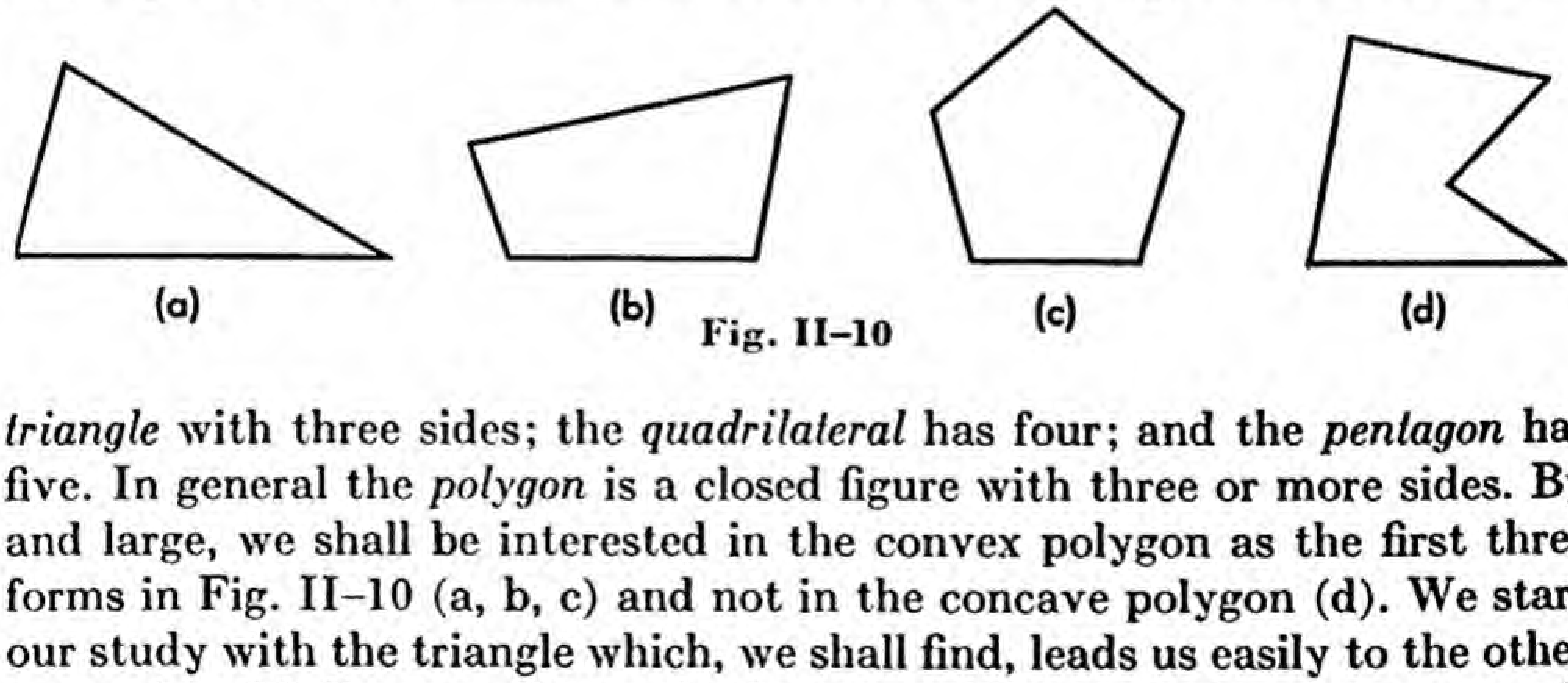


Fig. II–10

triangle with three sides; the *quadrilateral* has four; and the *pentagon* has five. In general the *polygon* is a closed figure with three or more sides. By and large, we shall be interested in the convex polygon as the first three forms in Fig. II–10 (a, b, c) and not in the concave polygon (d). We start our study with the triangle which, we shall find, leads us easily to the other cases. As with lines, angles, and other quantities of mathematics, we seek a basis for comparisons of triangles too. We can look for clues through an intuitive examination of some drawings.

The coordinates of the vertices of two triangles (Fig. II–11, a, b) are the same, while in the third case the coordinates are each the same multiple k, $k \neq 1$, of the previous coordinates.

Figures such as (a) and (b) are called *congruent* ($\cong$) if, by definition, "all the corresponding parts, sides, and angles are respectively equal."

The third figure (c) is said to be *similar* ($\sim$) to the first two, and this, by definition, shall mean that "the corresponding angles are equal to each other and that the corresponding sides are proportional." Clearly, congruence is a special case of similarity.

Modern technology is replete with instances of these cases. Congruent figures are essential to mass production wherein duplicate parts are a basic ingredient. Scale drawings, blueprint work, and model making are indicative of the significance of similar figures.

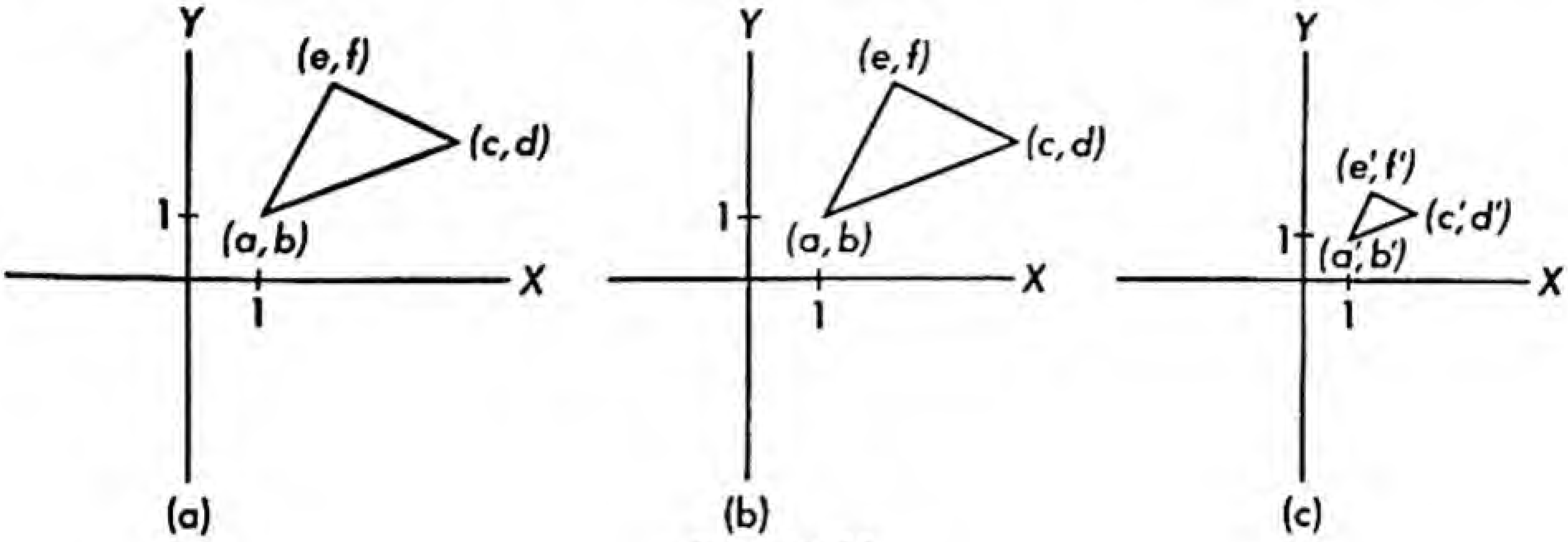


Fig. II–11

It will be recalled that *ratio* was used earlier as a quotient of integers. We may broaden this concept now to include the quotient of any real numbers, excepting, of course, that the denominator may not be 0. We also learned earlier that by *proportion* we mean the equality of any two ratios as in $a/b = c/d$.

Fortunately a determination of the similarity of two triangles does not require a full knowledge of all the parts of the triangles. Our knowledge of line segments and angles points to the possibility that any one triangle is determined by some combinations of less than six parts. Consequently we start with the following postulate as a basis for the explorations of properties and relations between triangles.

Two triangles are similar if their corresponding sides are proportional.

In general, by *corresponding angles* of similar triangles we shall mean those that lie opposite the corresponding sides.

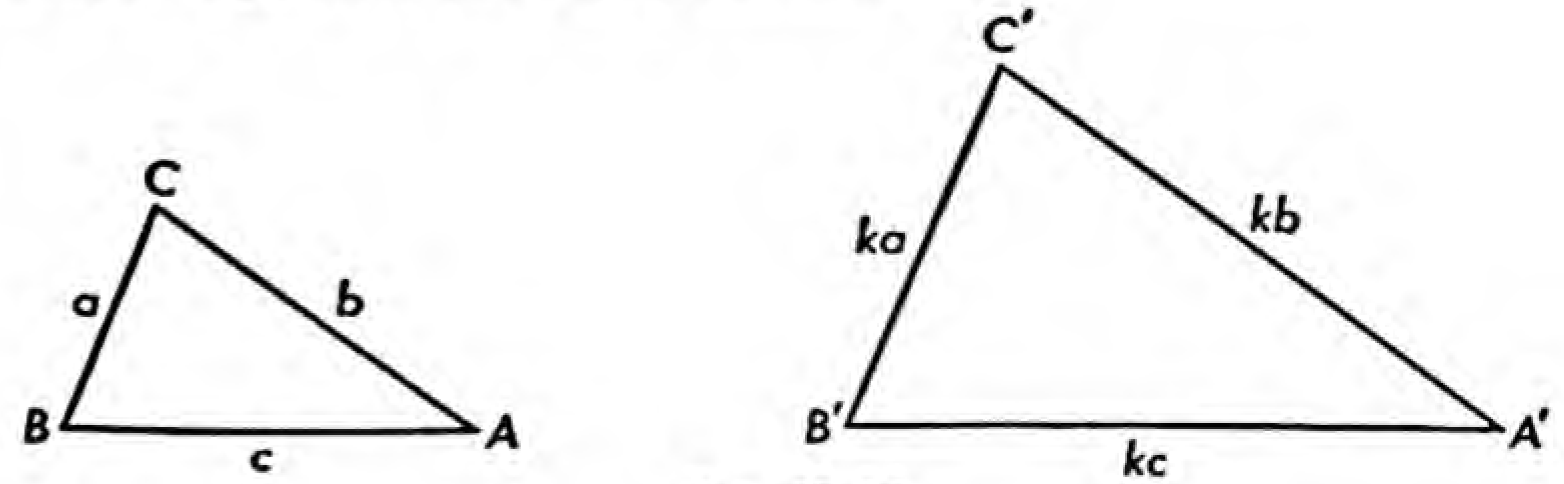


Fig. II–12

These two triangles (Fig. II–12) are similar, and the **ratio of similitude or proportionality** is k. Each pair of sides whose ratio yields k is to be

considered as corresponding sides, and the angles lying opposite these pairs are corresponding angles. By our postulate these triangles are similar. And by our definition of similar figures, the corresponding angles, such as B and B', are equal. We refer to the postulate as the "*sss*" case of similarity. Of course, as with all ratios, the order of comparison can be changed. There is no harm in speaking of the ratio of similitude as being $1/k$. When $k = 1$, the corresponding sides are equal and the triangles are congruent.

EXERCISES (II-5)

1. List a few specific illustrations or applications of congruent and similar figures.

2. We have seen that $ka - kb = k(a - b)$. Examine the validity, or extent of validity, of $ka - kb = k|a - b|$.

3. If $(a, 0)$ and $(b, 0)$ are any two points on the X-axis, compare the length of the segment they determine with that determined by

a. $(ka, 0)$ and $(kb, 0)$, b. $(a + k, 0)$ and $(b + k, 0)$.

4. a. Assume that the distance between the points (ma, mb) and (mc, md) is m times that for (a, b) and (c, d), respectively. Use this information to graph a pair of similar triangles on the same axes or on different axes but employing the same units. Specific numerical values may be selected.
   b. The conclusion in exercise 3b. suggests an approach for determining the coordinates of the vertices of a triangle congruent to any given triangle with known coordinate vertices. Illustrate.

5. The following sets of numbers represent the lengths of sides of triangles. Indicate which pairs are congruent or only similar, and, if similar, determine the factor of proportionality:

a. 3, 5, 7 b. 1.5, 2.5, 3.5 c. 7, 5, 3
d. 9, 25, 49 e. 25, 9, 49

6. By means of diagrams or other specific illustrations consider the question whether similarity or congruence of quadrilaterals is predictable on the basis of the lengths of the sides alone.

7. Another way of looking at the congruence of triangles for any given set of numbers representing the lengths of the sides is that if a triangle is possible at all with these measures, that triangle is unique. Thus, we have either one triangle or none at all.

Illustrate the possibility of *none at all.*

8. If it is known only that $\triangle PQR \sim \triangle ABC$, can the corresponding angles be selected? If not, what is the least amount of additional information that is needed? ($\triangle$ = triangle.)

9. Which of the following two propositions is valid?
   a. All pairs of congruent triangles are similar.
   b. All pairs of similar triangles are congruent.

6. DEDUCTIVE CONSEQUENCES

If we define an equilateral triangle as one with three equal sides, it follows from the earlier postulate that all equilateral triangles are similar (Fig. II–13). For, whatever be the ratio of two sides (say AB and $A'B'$),

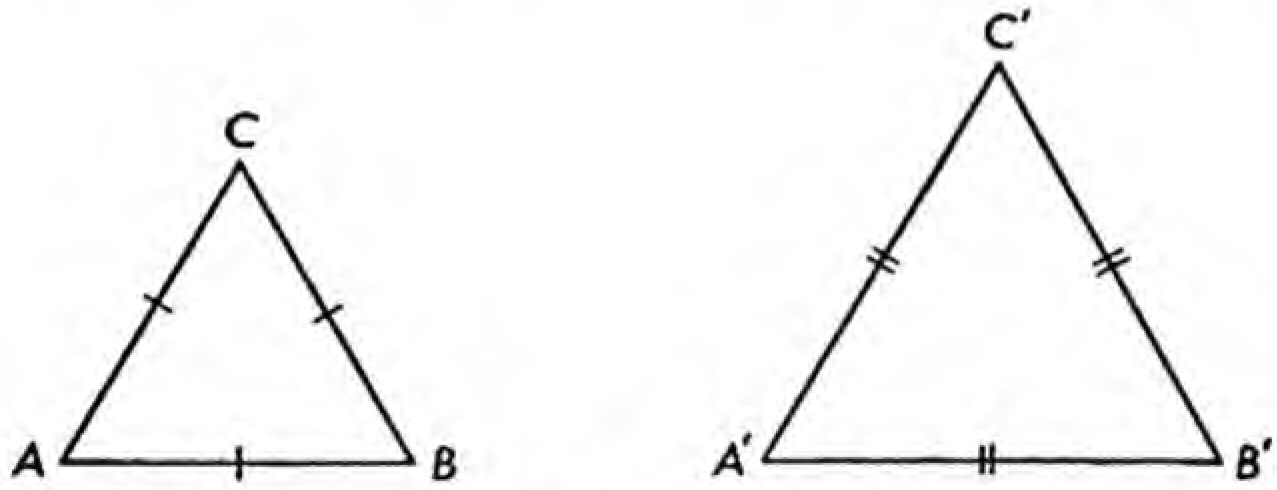


Fig. II–13

the same ratio must hold for the other corresponding sides which are the same as the first two. Diagrammatically, equal lines are indicated by the same number of dashes, as shown in Fig. II-13. Equal angles will be shown by the same number of equal arcs.

Of course the corresponding angles are equal. Since, in this case, any side of one triangle may be taken to correspond to any side of the other, it follows that $\angle A'$ may be taken to correspond to any angle of the other triangle. That is $\angle A' = \angle A = \angle B = \angle C$ or $\angle A = \angle A' = \angle B' = \angle C'$. In brief, the angles of the triangles are all equal to each other. Consequently an equilateral triangle may be said to be also *equiangular;* that is, all its angles are equal to each other.

We may take another view of the case where three noncollinear (not in one line) vertices are given (Fig. II–14).

It appears intuitively that two sides and the included angle uniquely determine a triangle. This seems to be so because the third side, indicated by the dotted line, is unique; consequently the other two angles of the triangle are also unique. Labeling this a *sas* configuration, we could expect that similarity, including congruence, would be determined by this combination of data.

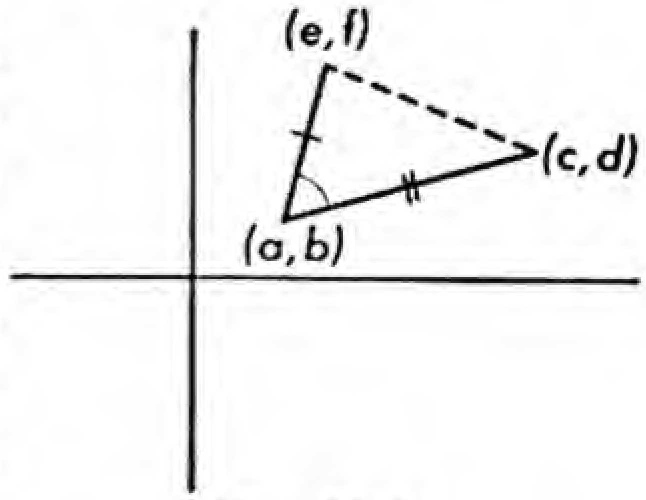


Fig. II–14

Let us then examine the question of the similarity of two triangles which have two sides of one proportional to two

sides of the other and of which their included angles are equal (Fig. II–15). Specifically, suppose that

$$\frac{A'C'}{AC} = \frac{A'B'}{AB} = k \qquad \text{and} \qquad \angle A = \angle A'$$

See Fig. II–15.

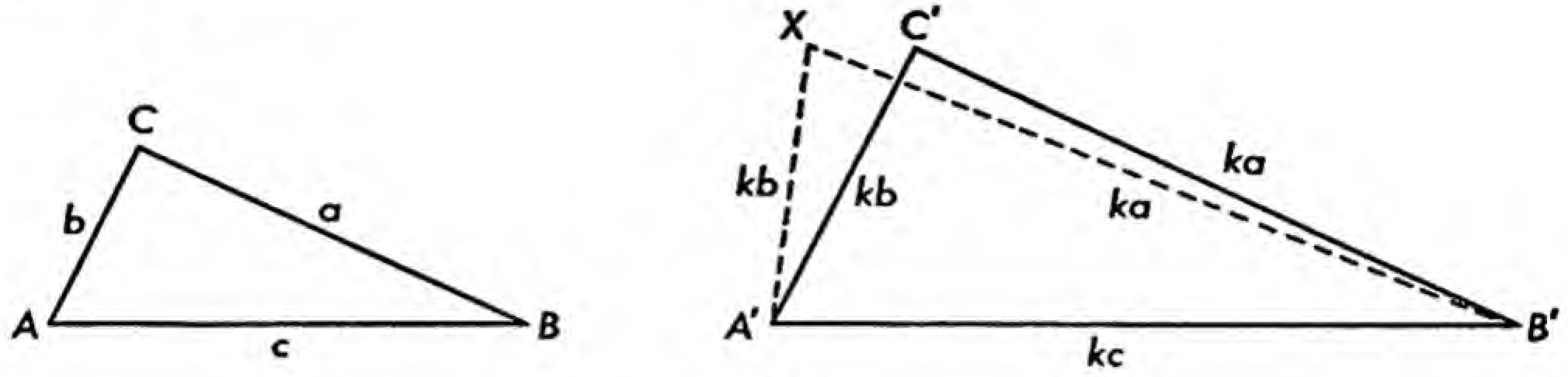


Fig. II–15

Should we assume that $\triangle A'B'C'$ is not similar to $\triangle ABC$, then there exists one that is, by virtue of proportional sides, *sss*. For convenience, and with no loss in generality, we take this triangle to be $A'B'X$, as indicated in the diagram. Since $A'B' = kc$, we have $A'X = kb$ and $B'X = ka$. Also, $\angle B'A'X = \angle A$. Because of the given data, this means that $\angle B'A'X = \angle A'$ too. The last equality makes it necessary for the point X to lie on the line $A'C'$, and since $A'X = A'C' = kb$, the point X and the point C' must be the identical point. Consequently the triangles $A'B'C'$ and $A'B'X$ are one and the same triangle. Thus, $\triangle ABC \sim \triangle A'B'C'$.

For brevity, we refer to this *two sides and the included angle* case of similarity as the sas case. Where the factor of proportionality is 1, the triangles are congruent.

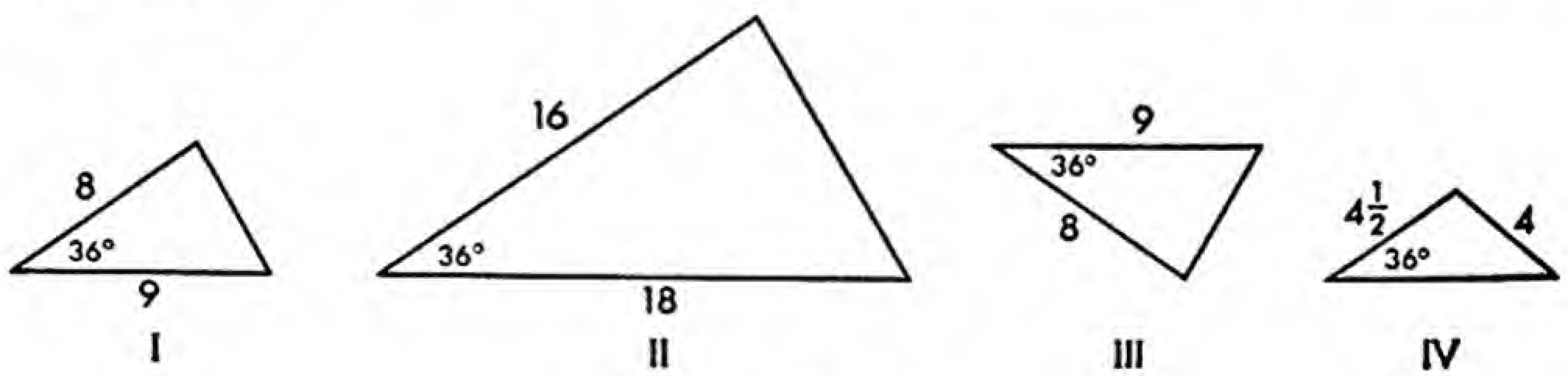
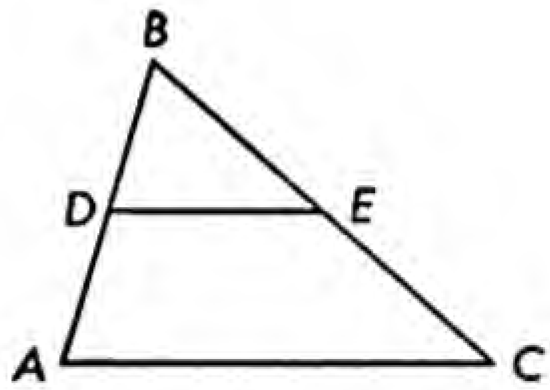


Fig. II–16

In Fig. II–16, triangles I, II, and III are similar, while I and III are also congruent. Triangle IV seems to have the right data, but their disposition in not *sas* as in the other cases.

As an interesting application of this theorem, consider $\triangle ABC$ (Fig. II–17a) with the line DE joining the midpoints D and E of the two sides,

respectively. The result is that the two triangles are similar, since they have $\angle B$ in common, and the sides including this angle are in the ratio of 1:2; $BD = \frac{1}{2}BA$ and $BE = \frac{1}{2}BC$. Now that the triangles are similar, one can say that DE must also be one-half its corresponding side; $DE = \frac{1}{2}AC$.



(a)

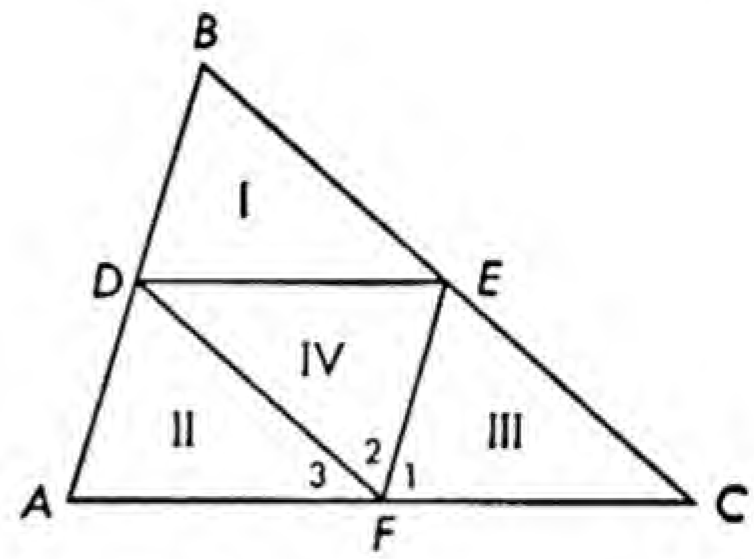


Fig. II–17 **(b)**

By the same token, if we join all three midpoints of the sides of a tri angle, triangles II and III in Fig. II–17 (b) are also similar to $\triangle ABC$. Furthermore, just as $DE = \frac{1}{2}AC$, so $DF = \frac{1}{2}BC$ and $EF = \frac{1}{2}BA$. Thus triangle IV $\sim \triangle ABC$ by the *sss* method.

In consequence all five triangles are similar to each other in Fig. II–17b. We are in a position now to reach a specific numerical generalization. If we check on the corresponding angles of these triangles, we note that $\angle 1 = \angle A$, $\angle 2 = \angle B$, and $\angle 3 = \angle C$. Inasmuch as $\angle 1 + \angle 2 + \angle 3 =$ a straight angle, it follows that

$$\angle A + \angle B + \angle C = 180°$$

Thus **the sum of the angles of any triangle is 180°.** Here is a first *constant* resulting from, of course, the combined effect of postulates, definitions, and derived conclusions (theorems).

This key theorem leads quickly to a number of other important theorems. We have seen earlier that an equilateral triangle is equiangular. Now we can say that each angle in such a triangle is 60°. Also a triangle can contain at most one right angle, in which case the other two angles total 90°. Two angles that total 90° are said to be **complementary,** and one is the **complement** of the other. So, the acute angles of a right triangle are complementary. Likewise, a triangle can contain at most one obtuse triangle.

From the fact that no triangle can contain two right triangles, we know that from a point not on a line, at most one perpendicular line is possible to the line. Since "at most one" does not necessarily mean "one," we consider whether there is one perpendicular.

We know that through P (Fig. II-18), lines may be drawn to line m. Let PX be any one of them. This forms angles 1 and 2 with line m. Since angles 1 and 2 comprise a straight angle, one of them must be acute and the other obtuse; otherwise both are right angles. If both are right angles, that proves our point. If both are not right angles, suppose that $\angle 1$ is the acute angle. Its complement, $\angle 3$, may be measured off at P as indicated. Consequently $PY \perp m$, since there are 180° in a triangle. So, **there is one and only one perpendicular from a point to a line.**

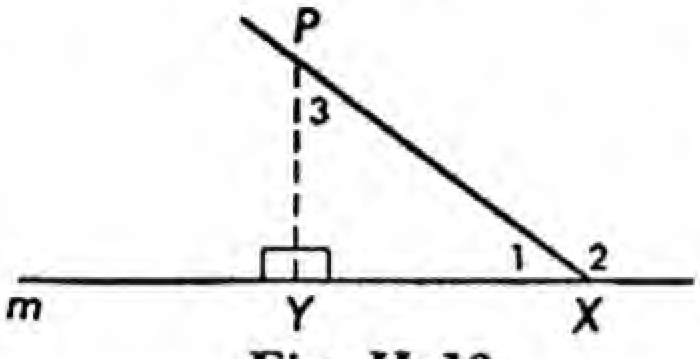


Fig. II-18

Two angles are **supplementary** when their measures total a straight angle or 180°. Because of the measure of an angle, it follows that an angle can have only one unique supplement. Thus a 150° angle is the supplement of a 30° angle, and $(180-x)°$ is the supplement of one of $x°$.

Should there exist one or more supplements of the same angle, then these must be equal. This occurs always when two straight lines intersect.

Both angles 1 and 2 (Fig. II-19) are supplementary to $\angle z$, and so $\angle 1 = \angle 2$. Such a pair of angles are called **vertical angles.** We have two important conclusions. One is that **supplements of the same or equal angles are equal,** and the other is that **vertical angles are equal.**

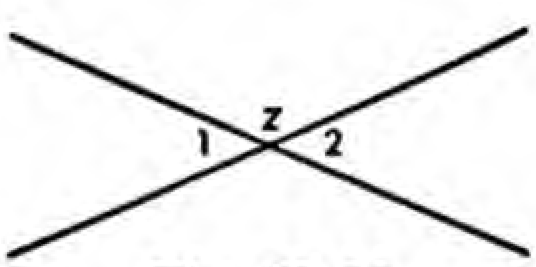


Fig. II-19

If we change our view from 180° to 90°, we have an analogous statement concerning complementary angles (Fig. II-20). **Complements of the same or equal angles are equal.**

To reach another important conclusion, consider any polygon of n sides and any point O within it (Fig. II-21). Let all vertices be joined with O, thereby forming n triangles. The sum of all the angles of all the triangles is $180\,n°$. If we subtract the sum of the angles about O, which is 360°, we get the sum of the angles of the polygon, which is $180\,n - 360 = 180\,(n-2)°$.


Fig. II-20

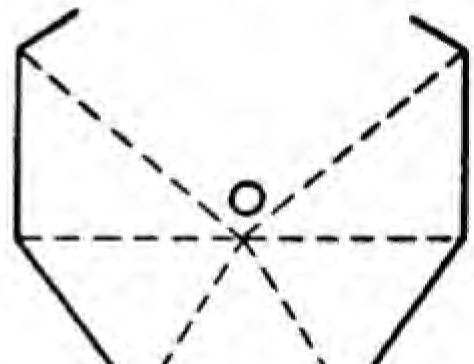


Fig. II-21

Thus the sum of the angles of any polygon is equal to 180° multiplied by two less than the number of sides. In particular we get the familiar fact that the sum of the angles of a quadrilateral is 360°. We can anticipate the

additional fact that a quadrilateral may contain as many as four right angles if it is equiangular. Of course this comes as no surprise to one who is familiar with the rectangle and square. What may be surprising, however, is that we have arrived at this juncture via a deductive conclusion.

EXERCISES (II-6)

1. It is possible to show that an equilateral triangle is equiangular because it is self-congruent.
   a. How may this be done?
   b. Since any triangle is self-congruent, does this mean that every triangle is equiangular?
2. Find the value of x wherever possible (see Fig. II-22).

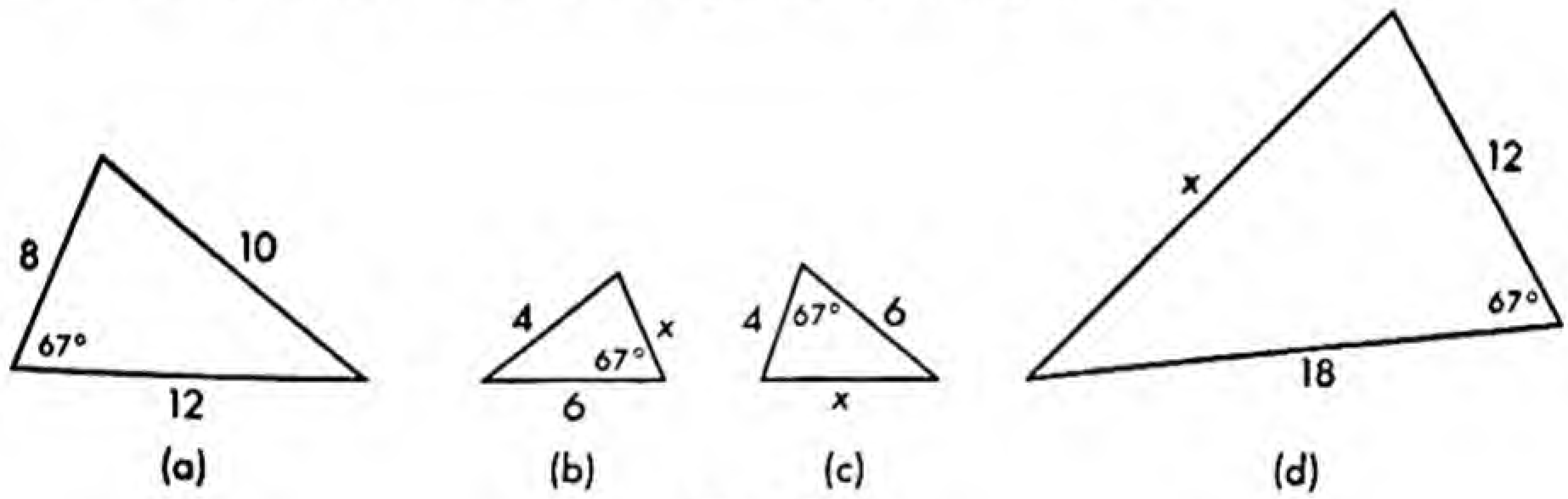


Fig. II-22

3. A median of a triangle is the line segment joining a vertex to the midpoint of the opposite side. Show that any two corresponding medians of two similar triangles are in the same ratio as the corresponding sides.
4. If P, Q, R, and S are each the midpoints of their respective segments, prove that $PQ = RS$ (see Fig. II-23).

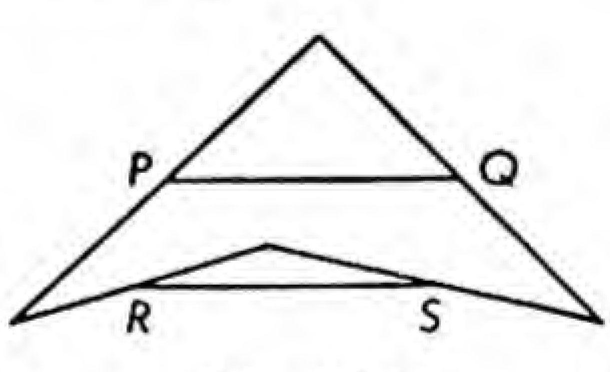


Fig. II-23

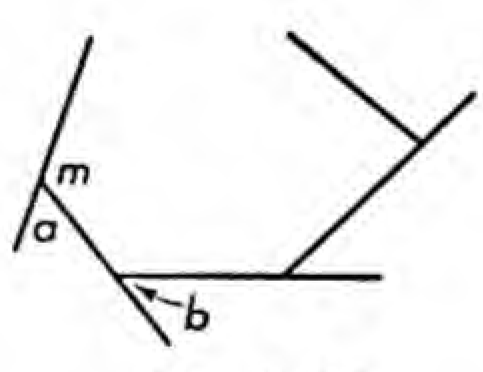


Fig. II-24

5. If the midpoints of the sides of a quadrilateral are joined successively, a new quadrilateral is formed in which the opposite sides are equal to each other. Prove that this is so.
6. Find the number of degrees in each angle of the following:
   a. An equiangular pentagon.
   b. An equiangular hexagon (six sides).
   c. An equiangular decagon (ten sides).

7. If each side of an n-sided polygon (Fig. II–24) is extended consecutively beyond each vertex, an "exterior angle," such as a or b, is formed at each vertex. If m is called an interior angle of the polygon.

a. What is the sum of the interior and the exterior angles?
b. What is the sum of the interior angles alone?
c. What, then, is the sum of the exterior angles?

8. Find the supplement and complement of each of the following where possible. Avoid negative results.

a. $37°$ b. $52°27'$ c. $136\frac{1}{4}°$
d. $m°$ e. $2m°$ f. $(90 - k)°$
g. $(180 - 3k)°$

9. An *altitude* of a triangle is the perpendicular segment drawn from a vertex of a triangle to the opposite side and terminated by the opposite side. Sketch the three altitudes in each of the following kinds of triangles:

a. Acute (all angles are acute)
b. Right c. Obtuse

10. Prove that an exterior angle of a triangle is equal to the sum of the two interior angles that are not adjacent to it.

11. If AB and ED are $\perp$ to BE (Fig. II–25), prove that $\angle A = \angle D$.

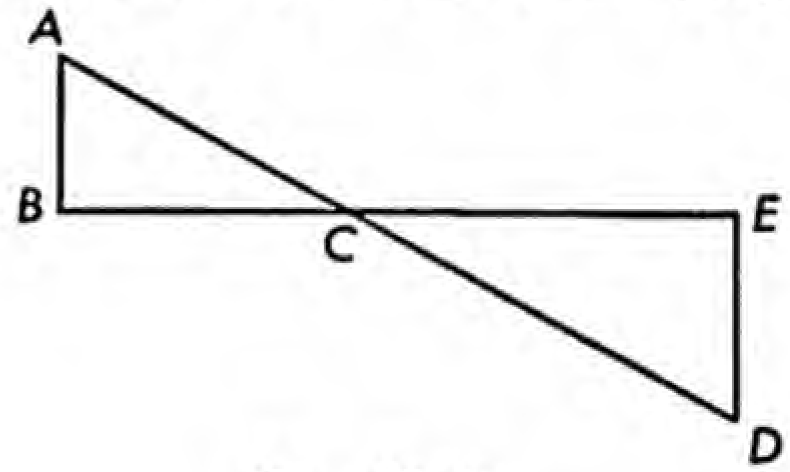


Fig. II–25

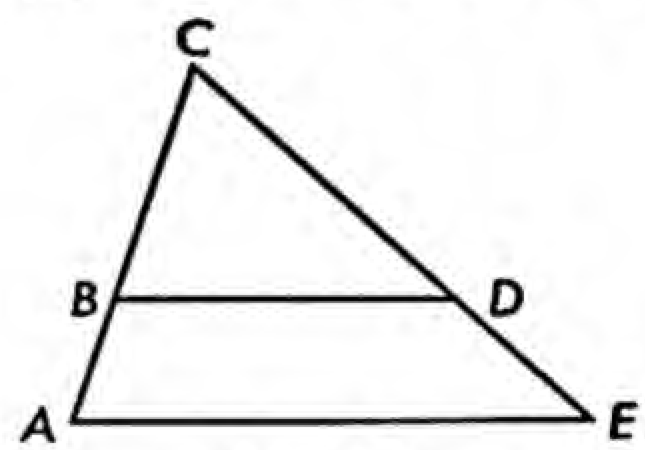


Fig. II–26

12. If $CB:CA = CD:CE$ (Fig. II–26), show that $\angle A$ is supplementary to $\angle ABD$.

13. If $\angle B$, $\angle D$, and $\angle ACE$ are right angles (Fig. II–27), show that $\angle A = \angle ECD$.

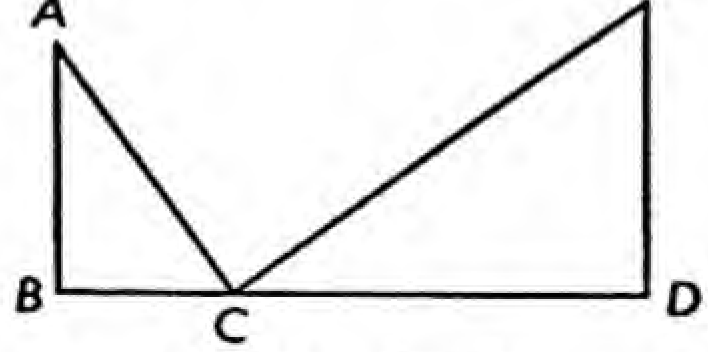


Fig. II–27

II–6 REVIEW

1. If at two different points of a given line perpendiculars are erected, the perpendiculars are "nonintersecting lines." Explain.

2. Suggest two other lines for exercise 1, other than perpendiculars, that would be nonintersecting.

3. a. Show that the line segments AC and DE in Fig. II–17(a) would be nonintersecting if extended.

b. State the conclusion as a general theorem.

4. a. Exercise 3 is suggestive of a more general case. State it and prove it.

b. What is the familiar name for *nonintersecting lines in a plane*?

5. A **regular polygon** is a polygon that is both equilateral and equiangular. Find the number of degrees in each exterior angle of a regular

a. pentagon b. hexagon c. decagon

6. What may be said of a hypothetical regular polygon if each exterior angle contains $m°$ and m does not divide 360 exactly?

7. Consider the four points: $A(2, 0)$, $B(0, 4)$, $C(0, 10)$, and $D(5, 0)$. What is the ratio of AB to CD?

8. Right triangle ABC is drawn with the altitude CD on the hypotenuse.

a. List all the pairs of complementary angles.

b. List all the pairs of equal angles.

9. Prove that if the diagonals of a quadrilateral bisect each other, the opposite sides of the quadrilateral are equal to each other.

10. The half-lines CA and BA actually form two angles. What convention or conventions have we been using tacitly to avoid ambiguity?

11. The point $P'(-a, b)$ is the *reflection* of the point $P(a, b)$ about the Y-axis. Describe this in words and illustrate.

12. What are the coordinates of a point that is the reflection of $P(a, b)$ about the X-axis?

13. By means of reflection find the coordinates of the vertices of two triangles that are each congruent to $\triangle ABC$, where $A(1, 2)$, $B(6, 4)$, and $C(3, 7)$.

14. Is it possible for a triangle to be its own *reflected image* about the Y-axis? Illustrate.

7. SIMILARITY CONTINUED

Let us return to the triangle for another important test for similarity. Considering any line AB and two fixed angles A and B (Fig. II–28), it appears that the shape of the triangle (but not the size) is uniquely determined. This is suggested by the observation that the directions of the two

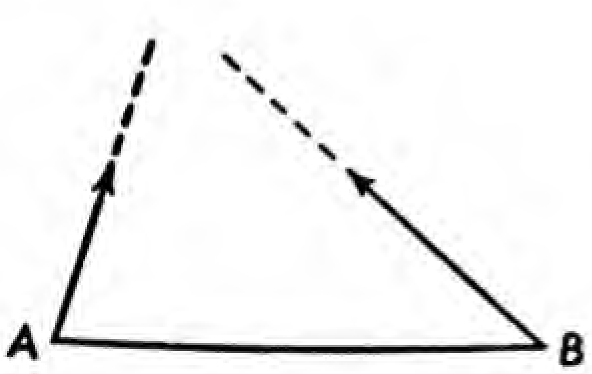


Fig. II–28

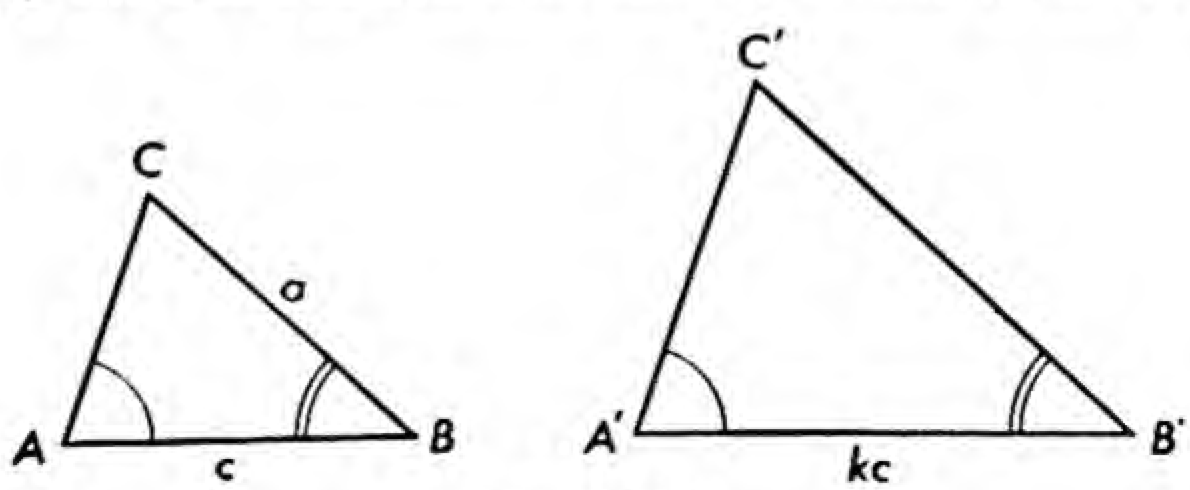


Fig. II–29

sides relative to AB are fixed, and so their intersection relative to AB is also fixed. This suggests the possibility that two triangles may be similar if they agree in two angles. This will be investigated presently.

Suppose that $\angle A = \angle A'$ and $\angle B = \angle B'$ (Fig. II–29). The lines AB and $A'B'$ have unique measures which are their lengths, and so they have some ratio with respect to each other. Taking k as the ratio of AB to $A'B'$, we set $AB = c$ and so $A'B' = kc$. Now, if $CB = a$, and if we can show that $C'B' = ka$, then indeed the triangles would be similar by the *sas* method.

We assume that $B'C' \neq ka$. Then there must be some point X on $B'C'$ (Fig. II–30), extended if necessary, so that $B'X = ka$. After joining A' to X, we note that $\triangle ABC \sim \triangle A'B'X$ by the *sas* method.

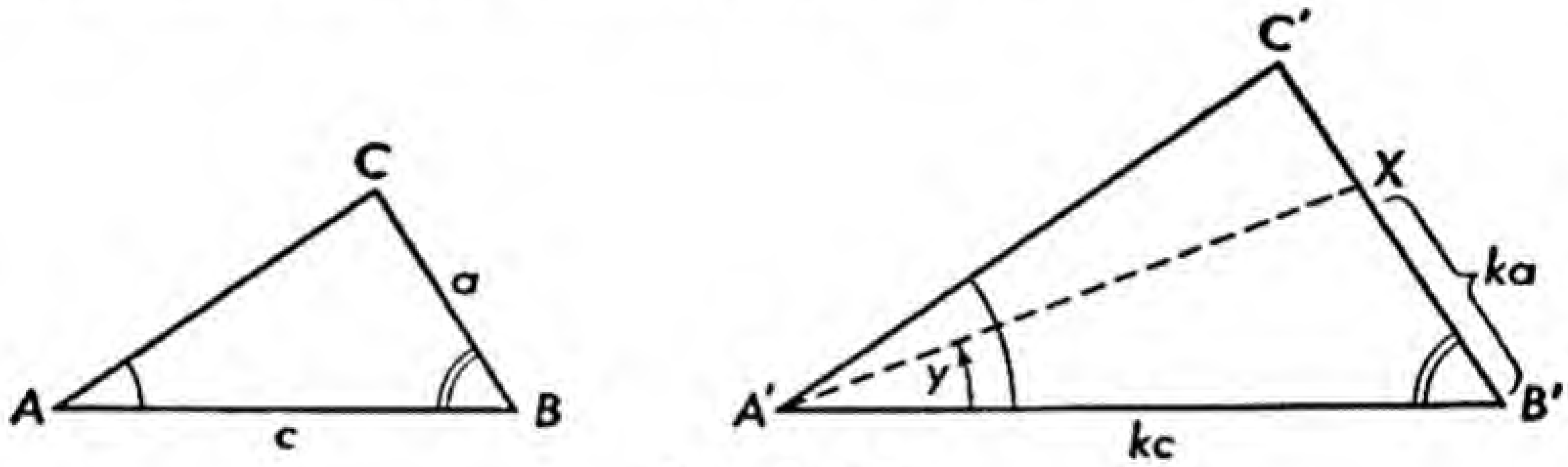


Fig. II–30

The similarity of the two triangles leads to $\angle A = \angle y$, and so $\angle y = \angle A'$. Now angles y and A' have the side $A'B'$ in common. This leads to the conclusion that $A'X$ lies along $A'C'$. The whereabouts of point X is now settled. Since it is on both $B'C'$ and $A'C'$, it must be at their intersection C'. Instead of three triangles, we have in reality only two, and these are similar. Consequently,

two triangles are similar when they agree in two angles.

This is known as the "*aa*" method.

If in addition to a pair of equal angles, the two triangles also have a pair of equal sides similarly placed, the triangles are congruent. This gives rise to two possibilities which are symbolized as "*asa*" and "*saa*" (Fig. II–31).

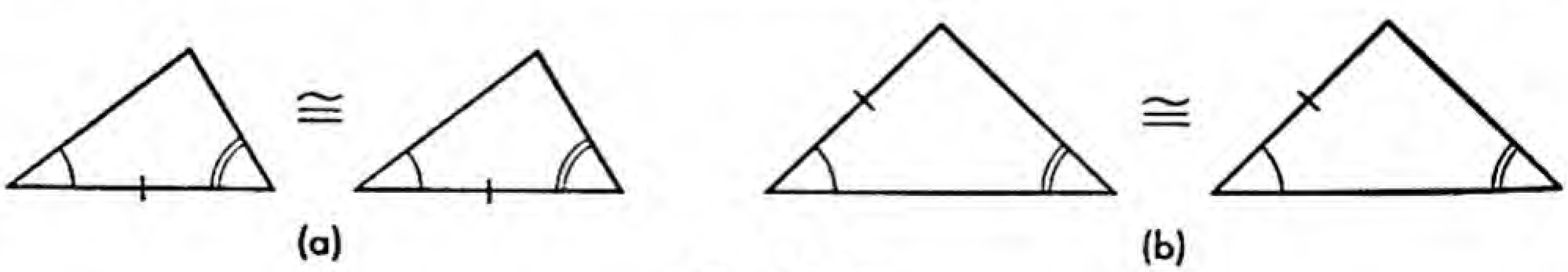


Fig. II–31

We learned earlier that an equilateral triangle is equiangular. The *converse* statement, that an equiangular triangle is equilateral, follows from the new finding on similar triangles. For, all equiangular triangles are

similar, and any side of one triangle corresponds to any side of any other because of the equality of all the angles. This is possible only if the three sides of any one triangle are equal to each other.

There is a class of triangles called **isosceles** (Fig. II–32) in which each one has two sides equal to each other. Any such triangle may be considered self-congruent by *sas:* $AB = BC, BC = AB,$ and $\angle B = \angle B$. Of course any triangle may be considered self-congruent. In ordinary cases this would lead us nowhere. Now, however, in this special case, as in the equilateral triangle, the viewpoint is fruitful of new conclusions.

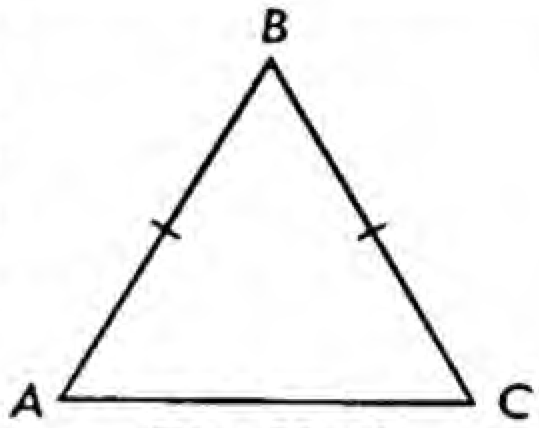


Fig. II–32

The $\angle A$ may be taken to correspond to itself and also to $\angle C$, since the two angles lie opposite equal sides. Thus we have the new conclusion that $\angle A = \angle C$; that is, two angles of an isosceles triangle are equal to each other. This is known generally as

the base angles of an isosceles triangle are equal to each other.

The equal sides of the isosceles triangle are known as the *arms* of the triangle. The third side is called the *base*, and the angles adjacent to the base are the *base angles*. The third angle, opposite the base, is the *vertex angle*.

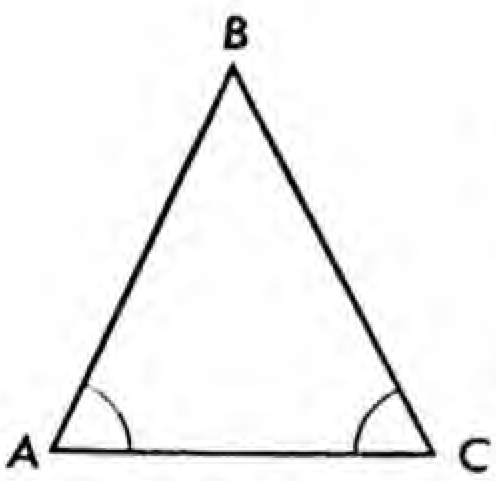


Fig. II–33

We have met one converse proposition which indicated that converse propositions mean *propositions in which the hypothetical parts and the conclusions are interchanged.* The occasion for this reference is the fact that the converse of the isosceles triangle theorem is also valid; that is,

if two angles of a triangle are equal to each other, the sides opposite these angles are equal.

If $\angle A = \angle C$, the triangle may be taken by *asa* to be similar to itself or even congruent to itself. Because of the equality of the angles, BA and BC correspond to each other, and so $BA = BC$ (Fig. II–33).

Earlier, we saw the possibility of the existence of the quadrilateral with four right angles (Fig. II–34), the rectangle. The familiar fact that the opposite sides of a rectangle are equal to each other can be seen from the congruent triangles obtained when a diagonal is drawn through the rectangle. Angles 1 and 2 are equal to each other because they are both complementary to $\angle 3$. This fact plus the equal right angles and the common diagonal make the triangles congruent by *saa.*

EXERCISES (II-7)

1. If angles 1 and 2 are equal and all the lines are straight, show that $AB:BC = EA:DC$ (Fig. II-35).

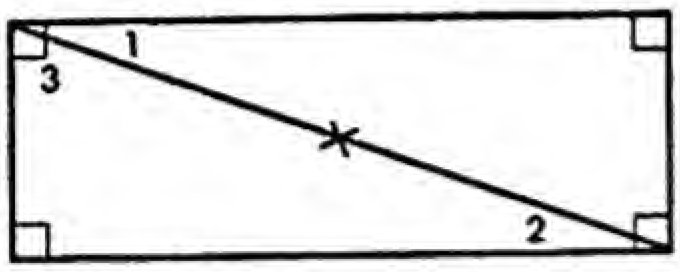


Fig. II-34

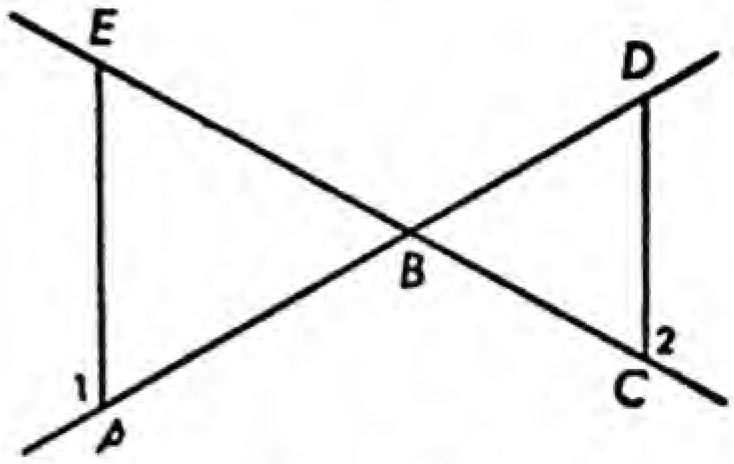


Fig. II-35

2. a. Show that two altitudes of a triangle are in the same ratio as two sides of the triangle.
   b. Consider the cases where the triangles are both acute, obtuse, and right.
   c. When would the altitudes be equal to each other?
3. a. *If a triangle is isosceles, then the base angles are equal to each other.* Show that this theorem is true by drawing a bisector of the vertex angle of the triangle.
   b. We note that the *if* part of a proposition is the hypothesis and that the *then* part is the conclusion. Prove the converse of the theorem in exercise 3(a) by means of the same angle bisector.

4. Prove that all isosceles triangles with equal vertex angles have equal base angles.

5. a. Prove that the altitudes to the arms of an isosceles triangle are equal.
   b. Carry this forward to the equilateral triangle.

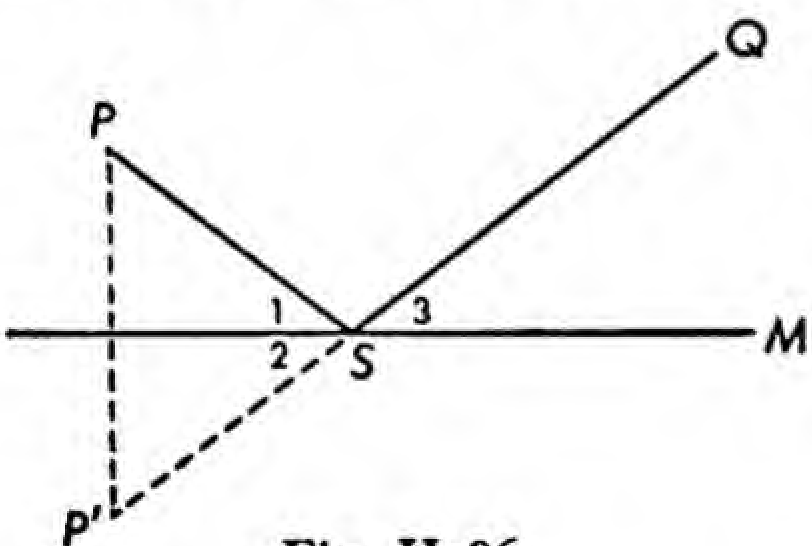


Fig. II-36

6. If M is a flat mirror and PP' is perpendicular to M and is bisected by M, then P' is the reflection of P in the mirror. An observer at Q would look directly at the reflection P' to see P (Fig. II-36).
   a. Prove that PSP' is an isosceles triangle and that $\angle 1 = \angle 2$.
   b. Prove that $\angle 1 = \angle 3$.
   c. If a light beam were directed from P toward S, it would be reflected to Q. The fact that $\angle 1 = \angle 3$ is equivalent to the well-known science rule that the angle of reflection equals the angle of incidence. More interesting, perhaps, is the fact that the path of a light ray from P to S to Q is the shortest path between P and Q touching M.

7. Under what minimal conditions concerning angles are two isosceles triangles similar? Prove your conclusion(s).

8. Given: $DB \perp AB$ and $DC \perp AC$ (Fig. II-37). Prove that $\angle BAC = \angle CDB$.

9. Prove that the diagonals of a rectangle are equal to each other.

10. $ABCD$ is a rectangle (Fig. II–38). $BF \perp AC$ and $FE \perp AB$.
    a. Prove that $\triangle ACD \sim \triangle EBF$, and so

$$AD:CD = EB:FE$$

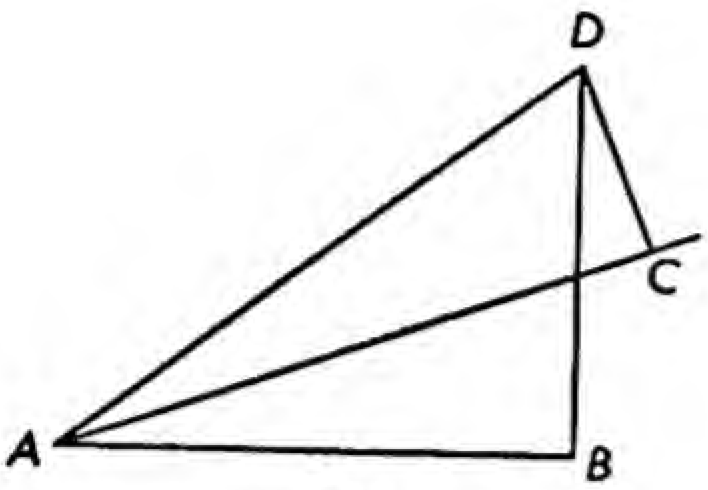


Fig. II–37

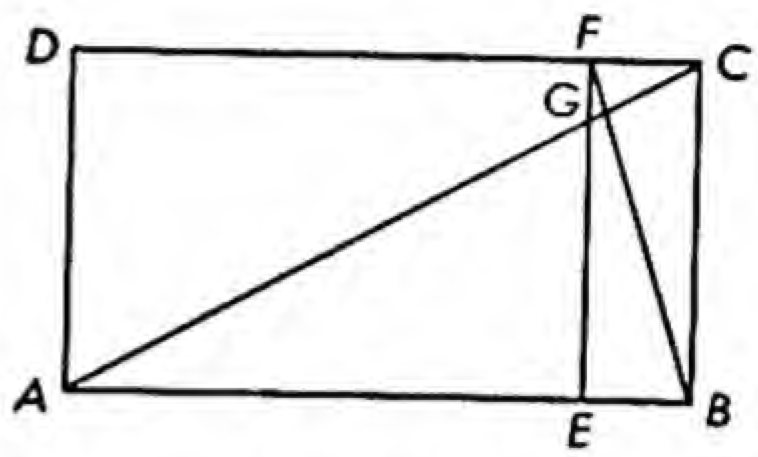


Fig. II–38

   b. Why are the rectangles $ABCD$ and $EBCF$ similar?
   c. If from G another perpendicular is drawn to BC, another rectangle will be formed similar to the first two. In this way, and by continuance of this procedure, a sequence of similar rectangles can be constructed. The subdivision of a rectangle area into similar figures is known in art as *dynamic symmetry* and was supposed to have been used by the Greeks, as well as other artists, to achieve beautiful proportions in paintings, sculpture, and architecture.

8. FACTS ABOUT PROPORTIONS

We have noted that *the cross products of a proportion are equal.* This earlier observation is so important that it deserves more formal attention. In

$$\frac{a}{b} = \frac{c}{d}$$

we may simplify the equation by multiplying both members by bd.

$$bd \cdot \frac{a}{b} = \frac{c}{d} \cdot bd$$

Consequently

$$ad = bc$$

The result substantiates the conclusion regarding cross-products.

When a proportion is written as

$$a:b = c:d$$

the terms a and d are called the *extremes* because of their position in the

proportion, and the middle terms b and c are called the *means*. With this terminology we say that

in a proportion the product of the extremes is equal to the product of the means

instead of the theorem concerning cross-products.

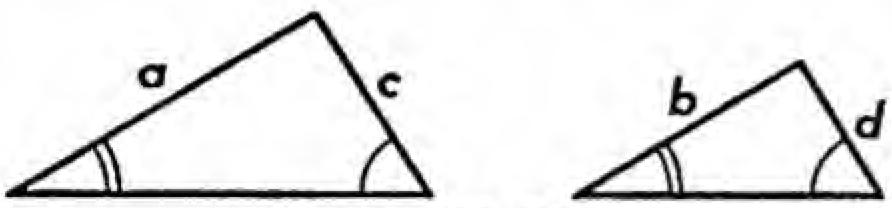


Fig. II–39

Suppose that the proportion $a:b = c:d$ is derived from a pair of similar triangles (Fig. II–39). This leads to the fact that

$$ad = bc$$

But this equation in turn may have come from any one of many proportions, such as

$$\frac{a}{c} = \frac{b}{d} \qquad \frac{d}{b} = \frac{c}{a}$$

and so on. If we refer these proportions to the diagram, we discover what, perhaps, might have been anticipated; that is, in writing a proportion for similar triangles, it is possible to start with any side and take the ratio of this side to any other side of the same triangle, or the corresponding side of the other, and then obtain the second ratio in like fashion.

EXERCISES (II–8)

1. Find the value of the unknown in each of the following proportions:

   a. $\dfrac{3}{4} = \dfrac{m}{6}$

   b. $\dfrac{7}{k} = \dfrac{21}{30}$

   c. $\dfrac{a+4}{6} = \dfrac{7}{8}$

   d. $3x:4 = 7:18$

   e. $\dfrac{x+4}{20} = \dfrac{3x}{14}$

   f. $8:x = x:5$

2. If $\angle 1 = \angle 2$, find a (Fig. II–40).

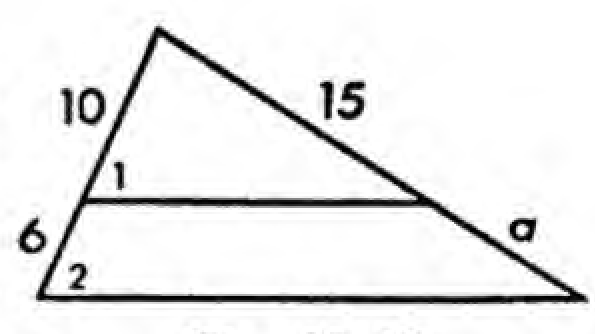


Fig. II–40

3. a. Prove that the corresponding altitudes of similar triangles are in the same ratio as the corresponding sides.

   b. Consider the same situation for corresponding angle bisectors instead of altitudes.

4. a. If CB and EA (Fig. II–41) are perpendicular to AB, and DE is perpendicular to AE, prove that $AB:DE = BC:AE$.

   b. Write three variations of the concluding proportion.

   c. If $AB = 12$, $AD = 10$, and $DC = 6$, find DE.

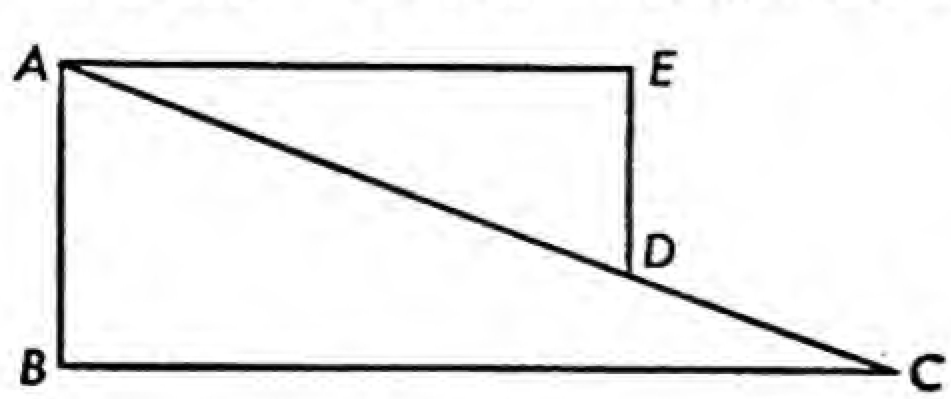


Fig. II–41

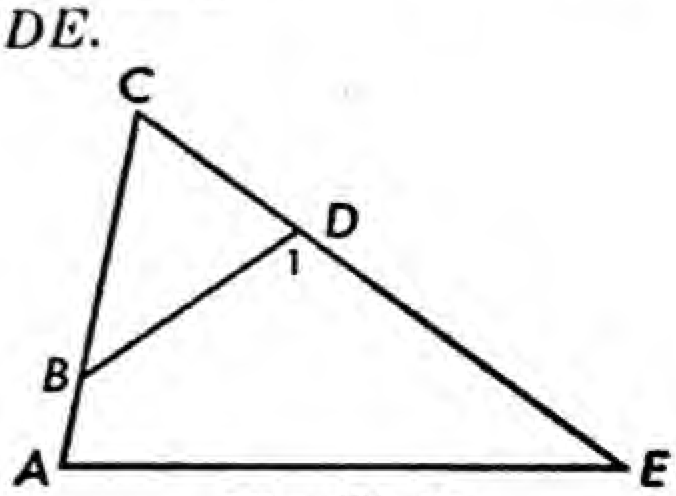


Fig. II–42

5. If $\angle A$ is the supplement to $\angle 1$ (Fig. II–42), prove that $BC \times AE = CE \times BD$.
6. The ratio a/b can be equal to c/d if and only if either one can be changed to the other by means of the fundamental principle of fractions. Use this as a point of departure to prove that if $a/b = c/d$ then $ad = bc$.

9. A SECOND DEGREE EQUATION

Consider a similar triangle situation (Fig. II–43) wherein the two sides denoted by y are equal to each other. Then

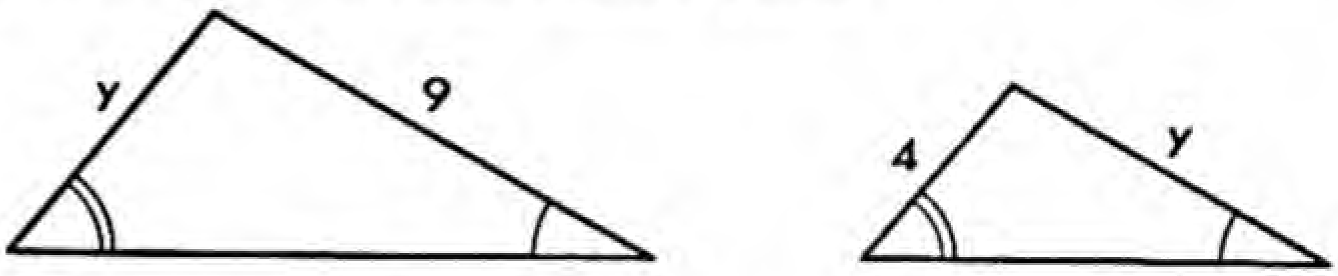


Fig. II–43

$$\frac{9}{y} = \frac{y}{4}$$

and

$$y^2 = 36$$

$$y = \sqrt{36} = 6$$

The cross-products introduce us to a new equation which is described an *equation of the second degree with one unknown*. Since $y^2 = yy$, the value of y must be the square root of the value of y^2. If a number multiplied by itself is 36, the number must be the square root of 36. Now, both $+\sqrt{36}$ and $-\sqrt{36}$ squared will yield 36. So, y could be either of these;

that is,

$$y = \pm\sqrt{36}$$

However, since the length y represents an undirected line segment, we disregard a possible negative answer. In general, if x is a *mean proportion* between a and b, we have

$$\frac{a}{x} = \frac{x}{b}$$

$$x^2 = ab$$

$$x = \pm\sqrt{ab}$$

EXERCISES (II–9)

1. Given $\angle 1 = \angle 2$ in $\triangle ABC$ (see Fig. II–44). Prove:
   a. $\triangle BCD \sim \triangle ABC$.
   b. BC is a common side of the two triangles. Draw the implication of whether BC does or does not correspond to itself.
   c. Prove that $DC:BC = BC:AC$.
   d. If $AD = 4$ and $DC = 12$, find BC.
   e. If $BC = 8$ and $DC = 6$, find AC.

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2. If $CA = CD$ and $AD = BD$ (Fig. II–45), prove that $AD = \sqrt{BA \times CA}$.
3. Find the value of the unknown:

   a. $\frac{x}{3} = \frac{12}{x}$ c. $3a:1 = 1:a$

   b. $\frac{4}{x} = \frac{x}{7}$ d. $12:5a = 2a:7$
4. Find the mean proportion between 6 and 24; 5 and 8.

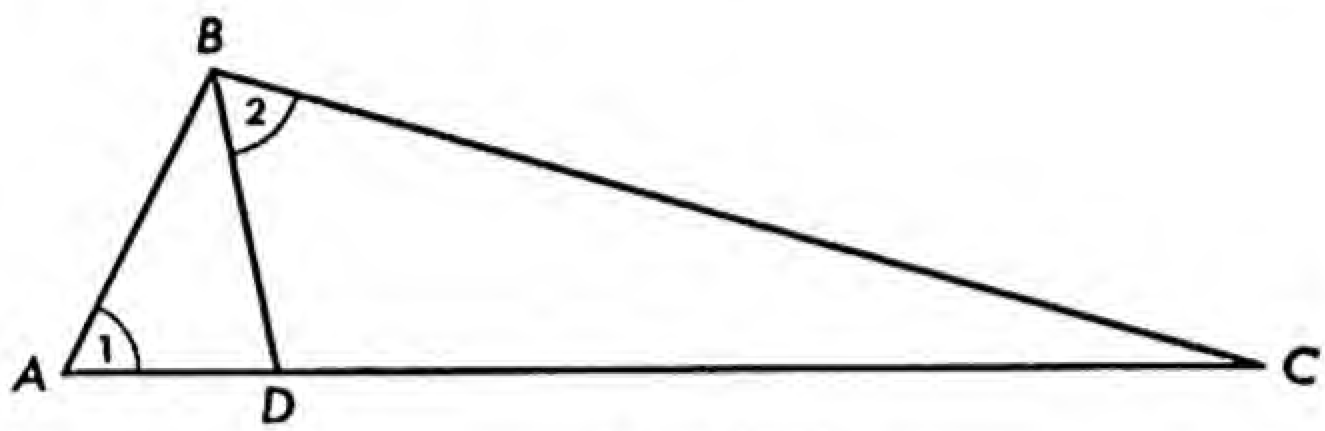


Fig. II–44

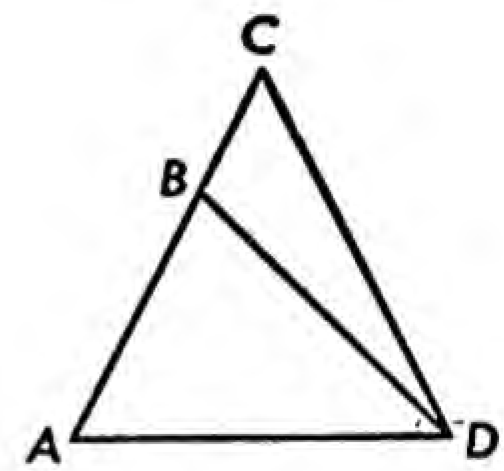


Fig. II–45

10. THE PYTHAGOREAN THEOREM

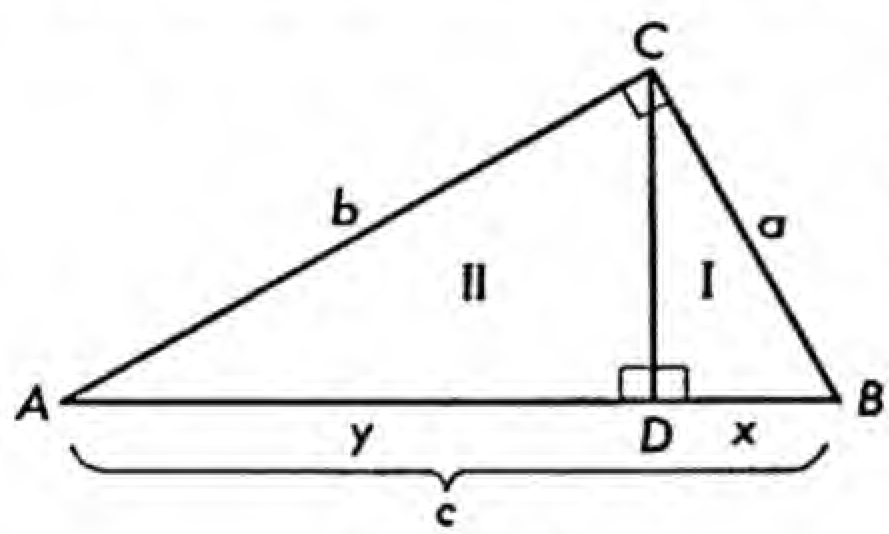


Fig. II–46

The right triangle ABC, with the altitude CD, introduces a figure with three right triangles: $\triangle$I, $\triangle$II and $\triangle ABC$ itself (Fig. II–46).

a. $\triangle\text{I} \sim \triangle ABC$ because $\angle B = \angle B$ and both have equal right angles.

b. $\triangle\text{II} \sim \triangle\text{ABC}$ because $\angle A = \angle A$ and both have equal right angles.

From (a) and (b) we get, respectively:

$$\frac{x}{a} = \frac{a}{c} \quad \text{so that} \quad x = \frac{a^2}{c}$$

and

$$\frac{y}{b} = \frac{b}{c} \quad \text{so that} \quad y = \frac{b^2}{c}$$

We are in a position now to derive one of the most famous and significant relations in mathematics. Adding the values of x and y, and noting that $x + y = c$, we get

$$c = \frac{a^2}{c} + \frac{b^2}{c}$$

or

$$\mathbf{a^2 + b^2 = c^2}.$$

The side opposite the right angle in the triangle is called the *hypotenuse*, and the other two sides are called the *arms*. Thus,

> *the sum of the squares of the arms of a right triangle is equal to the square of the hypotenuse.*

The converse of this theorem is also valid, for, suppose we had a triangle (Fig. II–47) in which $d^2 + e^2 = f^2$. If it is thought that F is not a right angle, then we could construct a right triangle XYC with d and e as arms. For this right triangle the Pythagorean theorem holds, and so $d^2 + e^2 = y^2$. Comparing the two equations, we conclude that $y^2 = f^2$, and so $y = f$. The two triangles are now congruent by *sss*, and consequently $\angle F = \angle Y =$ a right angle.

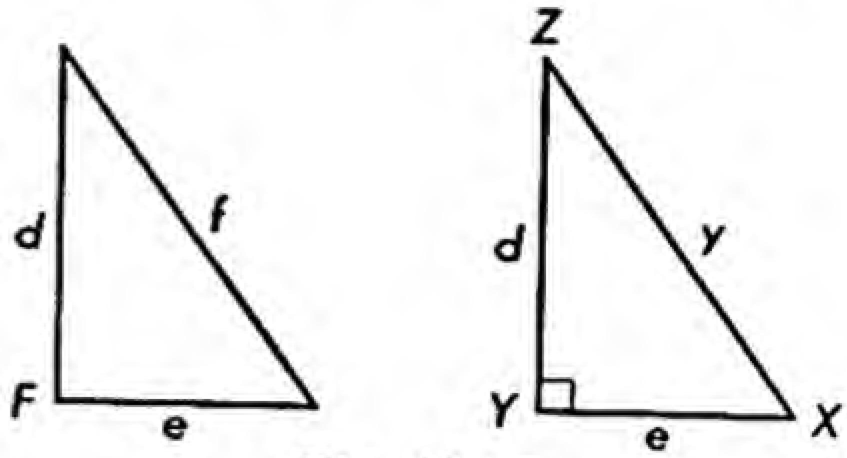


Fig. II–47

Long before the Pythagorean theorem was formulated, certain combinations of sides were known to form right triangles. The Egyptians knew the 3, 4, 5 combination ($3^2 + 4^2 = 5^2$; $9 + 16 = 25$). This knowledge provided the early Egyptian surveyors with their most important tool in resurveying their lands along the Nile after the annual floods. By placing knots in a rope, spaced in the ratio 3:4:5, the rope could be formed into a right triangle. The surveyor was literally a *rope stretcher*.

The knowledge of *Pythagorean triples*, such as 3, 4, 5; 5, 12, 13; 8, 15, 17; 7, 24, 25 and many others, gave rise to the question as to whether integers could be found that would satisfy the more general condition of

$$a^n + b^n = c^n$$

for integral values of $n > 2$. Although Fermat, seventeenth century, claimed to have proved that the equation had no solutions, no one since then has been able to substantiate his claim even though the efforts at one time were spurred on by a sizable reward. However, his claim has been validated for n up to 4002.

With regard to the Pythagorean theorem, it must be noted that this is the first formulated prediction of a linear kind. The length of a hypotenuse is now a logical prediction, a prediction based on the postulational system developed so far, with its undefined terms, postulates, definitions, and theorems.

We glance now at a couple of applications of the new formula. Of course numerous occasions will arise for its use as we go along. (See Fig. II–48 and Fig. II–49).

a. $x^2 = 12^2 + 4^2$

$x^2 = 160$

$x = \sqrt{160} = 4\sqrt{10}$

$x \approx 12.6$

b. $x^2 + 4^2 = 10^2$

$x^2 + 16 = 100$

$x^2 = 84$

$x = \sqrt{84} = 2\sqrt{21}$

$x \approx 9.2$

or

EXERCISES (II–10)

1. Using the similar triangles developed in the text, find a; b; and $\sqrt{a^2 + b^2}$. Compare the last with c. (See Fig. II-50.)

2. If a, b, and c form the sides of a right triangle, why must ka, kb, and kc, with $k > 0$ be the sides of a right triangle too. Use this fact to list a few additional Pythagorean triples.

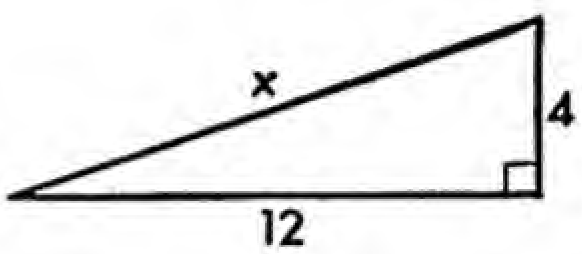


Fig. II–48

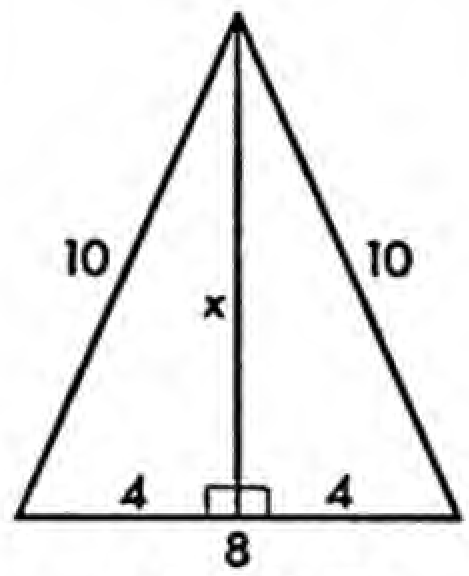


Fig. II–49

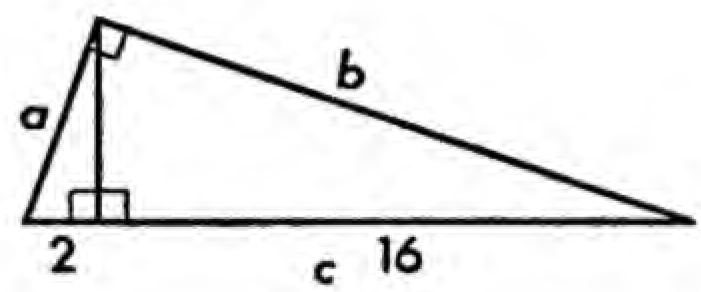


Fig. II–50

3. A *primitive Pythagorean triple* is one wherein the three values have no common factor. If the positive numbers p and q are not both odd, have no common factor, and $p > q$, then the following values will yield primitives: $a = p^2 - q^2$, $b = 2pq$ and $c = p^2 + q^2$. For example, if $p = 3$ and $q = 2$, we get $a = 5$, $b = 12$, and $c = 13$.
   a. Find other primitives.
   b. Prove that the values given for a, b, and c do necessarily lead to the sides of a right triangle.

4. Find the altitude of an equilateral triangle whose side is s.

5. The dimensions of a rectangle are 5 and 8 inches. Find the length of the diagonal.

6. Two sides of a triangle are 4.1 and 6.4. The altitude to the third side is 3.7. Find the third side, if possible.

7. In Chapter II, Art. 7, exercise 6, it was indicated that PRQ is the shortest path of a light ray from P to Q reflected by the mirror M. (See Fig. II–51). By way

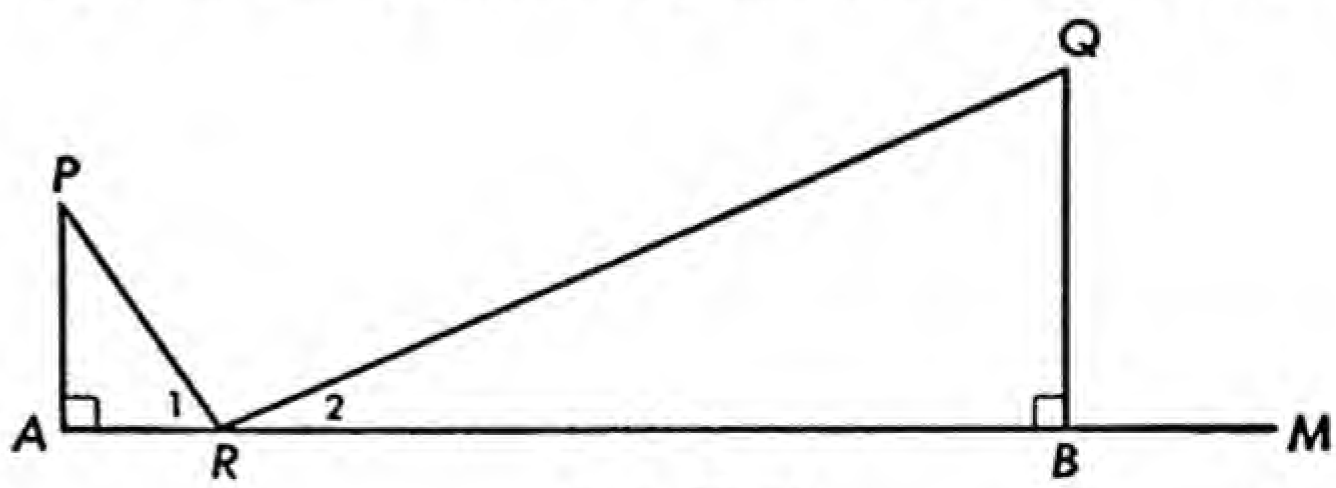


Fig. II–51

of illustration, take $PA = 8$ inches, $QB = 12$ inches, and $AB = 25$ inches. Recall that $\angle 1 = \angle 2$, and as a result the triangles are similar.
   a. Find AR and RB.
   b. Find PR, QR, and $PR + QR$.

c. Let R be any other point between A and B. Take any length for AR and RB, providing that their sum is 25. Keep $PA = 8$ and $QB = 12$. Find $PR + QR$ now, and compare with result in (b). Vary the position of R again and follow through for another comparison.

8. Find the diagonal of a square whose side is 10 inches; s inches.
9. The diagonal of a square is d. Find the length of a side.
10. The ratio of the arms of a right triangle is 1:2 and the hypotenuse is 15. Find the arms.
11. One arm of a right triangle is 15 inches. The hypotenuse is 9 inches more than the other arm. Find the hypotenuse.
12. Prove that the sides of a right triangle could not be in the ratio of 5:6:7.
13. Prove that the altitude on the hypotenuse of a right triangle is a mean proportional between the segments on the hypotenuse.

11. INEQUALITY FACTS AND CONSEQUENCES

Let us turn aside from the pursuit of triangle relationships and return briefly to the real number system. Our number system is characterized by the fact that if one number is larger than another, it is larger by a positive amount. Symbolically this is:

$$\text{If } b > a, \qquad \text{then } b = a + k$$

where $k > 0$.

The converse of this is equally true. This follows from our knowledge of numbers and equations; that is,

$$\text{If } b = a + k, \qquad \text{then } b > a$$

where $k > 0$.

Similarly,

$$\text{If } c = d + m, \qquad \text{then } c > d$$

where $m > 0$.

By applying our postulate of addition of equalities, we get

$$(b + c) = (a + d) + (k + m)$$

Now $k + m > 0$, since the sum of two positive numbers is a positive number. Applying this fact to the last equation, we must conclude that

$$(b + c) > (a + d)$$

By comparing this result with the earlier conclusions that

$$b > a$$

and

$$c > d$$

we see that we have a conclusion concerning the addition of *inequalities:*

> *The sum of inequalities in the same sense is an inequality in that sense.*

By the *same sense* we mean that both left-hand members of the inequalities are the larger elements, or both are the smaller elements. Where it is otherwise, we say that the inequalities are in the *opposite sense,* as in $x > y$ and $z < w$.

Again, from

$$b = a + k \qquad \text{or} \qquad b > a$$

and

$$c = d + m \qquad \text{or} \qquad c > d$$

we get, by our postulate of multiplication with respect to equalities:

$$bc = ad + (am + dk + km)$$

We are in a position to compare bc and ad if the quantity in the parentheses is positive. Since k and m are positive, the simplest approach is to require that a and d also be positive. Of course this in turn means that b and c are positive. So, the simplest conclusion that we can reach concerning the product of inequalities involves positive numbers only. We have, from the last equation,

$$bc > ad$$

and we can then conclude:

> *For positive terms, the product of inequalities in the same sense is an inequality in that sense.*

This leads to an interesting special case where a and b are still positive, for if $b > a$, then also $b > a$, and $b > a$, and $b > a$, and so forth repetitively. By the preceding multiplication theorem, we get

$$b^n > a^n$$

for any positive integral value of n; therefore

> *An inequality remains in the same sense if both members are raised to the same positive integral power.*

By the same token, we have

$$b^{1/n} > a^{1/n} \qquad \text{or} \qquad \sqrt[n]{b} > \sqrt[n]{a}$$

for, if we assumed otherwise, we would run into a contradiction after we raised both members to the nth power.

The diversion in inequalities may be put to use immediately, together with the Pythagorean theorem. Suppose that from P we drop a perpendicular to the line m and then draw any other line c (Fig. II–52). We have immediately the following deductions:

$c^2 = a^2 + b^2$	by the Pythagorean theorem
$c^2 > a^2$	since b^2 is a positive quantity
$c > a$	by the last inequality theorem

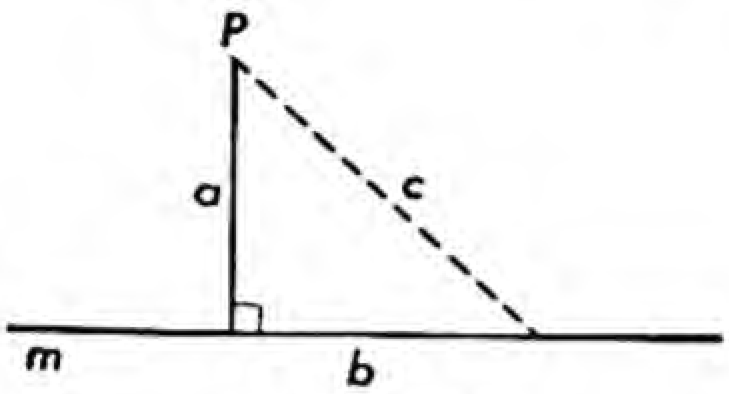


Fig. II–52

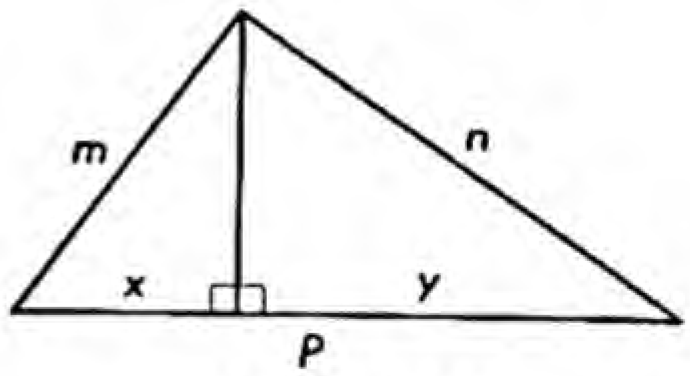


Fig. II–53

Thus

the perpendicular distance is the shortest distance from a point to a line.

We note that the preceding deductions could be changed slightly to lead also to the fact that $c > b$ in the same diagram. With that, we return to a familiar diagram (Fig. II–53) and apply a few of the recent findings:

$m > x$	by the last deduction
$n > y$	by the same
$m + n > x + y$	by addition of inequalities
$m + n > p$	by substitution

This means that

the sum of two sides of a triangle is greater than the third.

or

the straight line is the shortest distance between two points.

The last statement goes somewhat beyond what we have shown. One would need to go on to a polygon to help justify the breadth of that conclusion. Indeed, one would even have to consider, for greater generality, the possibility of a curved path between two points in a plane in contrast to the straight line between two points. This case, however, would have to await an analysis, indeed a definition, of *distance along a curve*. It will be seen (Fig. II–54) that the curved line will have to be considered as a limit of a sum of an infinite number of straight-line segments, at which point we shall find that our conclusion does indeed hold completely.

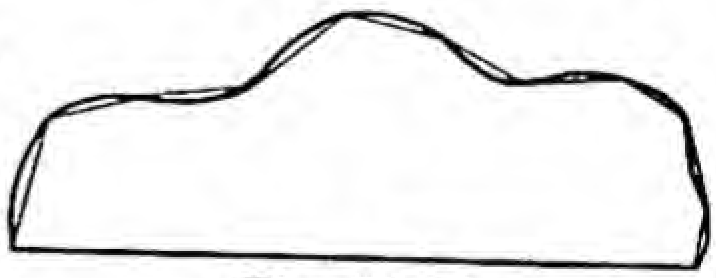


Fig. II–54

EXERCISES (II–11)

1. If $a > b$ and $c > d$, then $a + c > b + d$.
   a. Give three illustrations of this, using positive as well as negative numbers.
   b. Rephrase the preceding inequality theorem, using "less than," and provide three numerical illustrations.
2. a. Consider the two inequalities $a > b$ and $c > d$ with respect to subtraction. What is the best possible generalization that you can make?
   b. Illustrate your conclusion.
3. If $c > d$ show that $c + b > d + b$ for any real values of the letters involved.
4. If $b > a$ and c is positive, prove that $bc > ac$.
5. a. If $b > a$ and c is negative, prove that $bc < ac$.
   b. Note the special case when $c = -1$ by stating the conclusion in words.
6. Prove each of the following (all letters represent positive values):
   a. If $a > b$, then $a - b > 0$ and $b - a < 0$.
   b. If $a + b > c$, then $a > c - b$.
   c. If $a - b > c$, then $a > c + b$.
   d. If $a > b$, then $\frac{a}{c} > \frac{b}{c}$.
   e. If $a > b$ and $b > c$, then $a > c$.
7. Find the real values of x that satisfy the inequalities:

a. $x + 5 > 7$	g. $5x - 2 > 0$
b. $3x + 4 > 10$	h. $\frac{x}{3} > 2$
c. $6 < x + 2 < 10$	i. $\lvert 3x + 1 \rvert < 4$
d. $3 \leq 2x + 1 \leq 11$	j. $\frac{3x}{4} < 5$
e. $2x - 8 < 12$	k. $5 - x > 10$
f. $\lvert x \rvert > 3$	l. $12 - 2x > 3$
	m. $x^2 > 9$

8. If $a > b$ and $b > 0$, show that $a^{m/n} > b^{m/n}$ for $m > 0$ and $n > 0$.
9. Using a quadrilateral figure, prove that the straight line is the shortest distance between two points.
10. Prove that right triangles are congruent if they agree in hypotenuse and arm.
11. Prove that the difference between two sides of a triangle is less than the third side.
12. If oblique lines (in contrast to perpendicular lines) are drawn from a point to a line, the ones farthest from the foot of the perpendicular are the longer ones. Prove.
13. If the hypotenuse and arm of one right triangle are in the same ratio as a corresponding pair of another right triangle, then the triangles are similar. Prove.
14. On two previous occasions we have referred to a reflection property of the mirror. The path where the angle of incidence is equal to the angle of reflection, it was mentioned, is the shortest path from P to Q via the mirror m. We are in a position to prove this now.

We recall that $PS = P'S$ and $PP' \perp m$. QRP' is a straight line (Fig. II–55). Let

U be any point on RT. We wish to show that $a + b < x + y$. Find the lengths of $P'R$ and $P'U$ in terms of the letters in the diagram. Complete the proof by a single observation on $\triangle P'UQ$.

Show that the same conclusion holds if U is on SR.

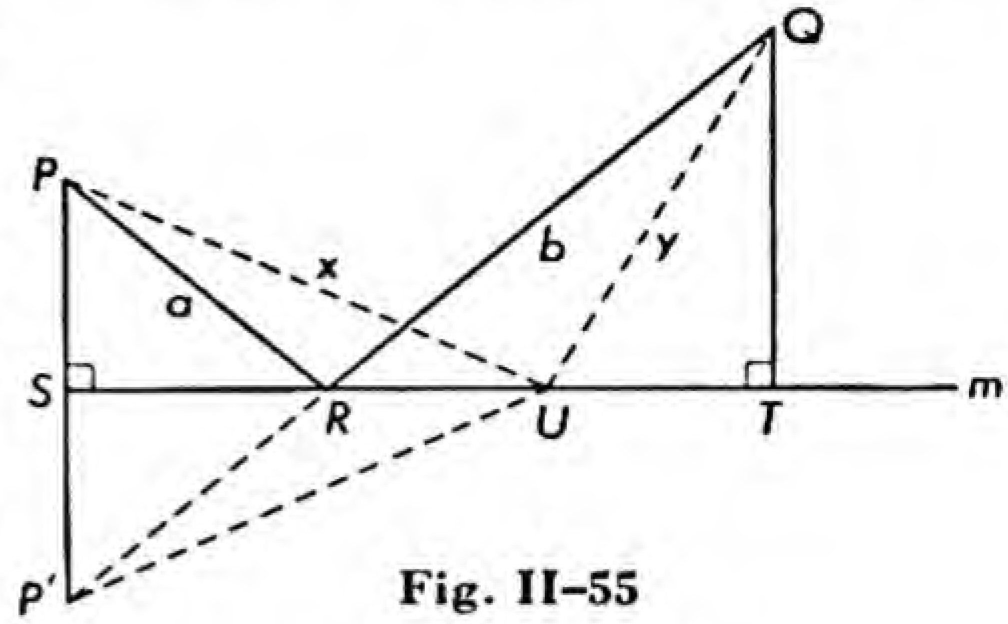


Fig. II–55

15. Show that the line formed by joining any point in the base of an isosceles triangle with the vertex is less than an arm.

16. If AB, an arm of an isosceles triangle ABC, is extended through the vertex B to any point D, then $AD > DC$. Prove.

17. Which of the following sets of data do not determine a triangle?

a. 7, 9, 14   b. 15, 23, 7   c. 9, 5, 3

II–11 REVIEW

1. In acute triangle ABC, the altitudes BD and CE intersect in F. Prove that $BF \cdot FD = CF \cdot FE$.

2. In $\triangle ABC$, point D is located between A and C so that $AD = AB$. Prove that $BC > CD$.

3. Solve the following equations. Irrational answers may be left in simplest radical form.

a. $\frac{5}{y} = \frac{y}{20}$ c. $3a^2 = 24$

b. $2:w = w:18$ d. $5a^2 - 3 = 3a^2 + 17$

4. Find the real values that satisfy the following:

a. $x^2 - 9 > 0$ c. $\frac{1}{x^2} < \frac{1}{8}$

b. $5x^2 - 8 \leq 2x^2 + 4$ d. $\frac{3}{x} > \frac{x}{12}$

5. The dimensions of a rectangular room are as follows: length $= l$, width $= w$, and height $= h$.

a. Show that the length of the diagonal distance across the floor is $\sqrt{l^2 + w^2}$.

b. Show that the diagonal distance across the room, from a corner on the ceiling to the opposite corner on the floor, is given by $\sqrt{l^2 + w^2 + h^2}$.

6. A rectangular box is 30x20x18 inches. Will a straight and thin 40 inch stick fit into the box?

7. A room is 20x13x12 feet. An insect crawls from a corner on the floor to the opposite corner on the ceiling. Find the length of the shortest path the insect could take.

8. a. If triangles ABC and $A'B'C'$ are similar, show that $hc/h'c' = a^2/a'^2$, where h and h' are corresponding altitudes.
   b. We shall see later that "hc" is proportional to the area of the triangle if h is the altitude to c. Interpret the proportion in (a) with this additional information.

9. The vertex angle C of an isosceles $\triangle ABC$ is one-half a base angle. The bisector of a base angle meets the opposite side in D. Prove that $\triangle ABD \sim \triangle ABC$.

10. The base of an isosceles triangle is 10, and the altitude to an arm is 8. Find the length of the arm.

11. If x and y are positive quantities, show that

$$\tfrac{1}{2}(x + y) \geq \sqrt{xy}$$

12. If m and n are two distinct, positive rationals, not 0 or 1, show that $\sqrt{mn}$ is between m and n.

13. a. Find three rational numbers between $\sqrt{2}$ and $\sqrt{3}$.
    b. Find three irrationals between $\frac{11}{16}$ and $\frac{3}{4}$.

14. a. The arms of a right triangle are each 1. What is the length of the hypotenuse?
    b. Another right triangle is constructed on this hypotenuse, using the hypotenuse as an arm, and the other arm is again taken as 1. The two triangles do not overlap. What is the length of the new hypotenuse?
    c. This procedure may be continued again and again. Write a few terms of the sequence of irrationals that results from this procedure.

12. PARALLEL LINES

Thus far little mention has been made of nonintersecting lines. We refer to **parallel** lines, which are "lines in a plane that have no point in common."

Consider a point P and a line M (Fig. II–56). If we draw a perpendicular from P to M and another perpendicular LP to PA, we can be sure that the line LP will not intersect line M. For, if it did, we would have a triangle with more than 180°, quite contrary to our findings. The lines P and M are parallel to each other.

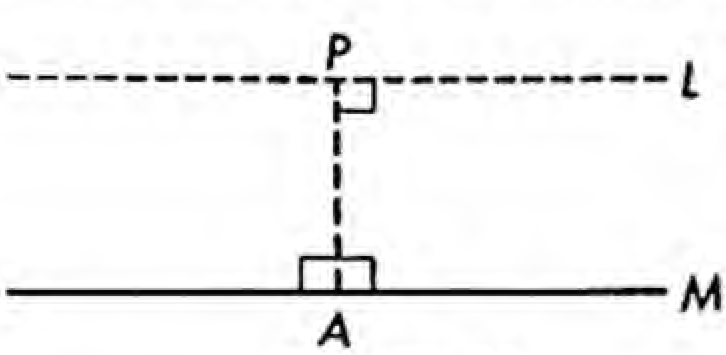


Fig. II–56

Since on AP, and AP extended, perpendiculars may be erected at each and every point, we can have an infinite number of lines parallel to any one line and parallel to each other. None of these will intersect, for the same reason that the first two cannot intersect. Any set of lines with a common characteristic property is also described as a *family of lines* or a *system of lines*. We have just described a set, family, or system of parallel lines.

The x and y lines, the basis of the network in the coordinate system, constitute two families of parallel lines that are mutually perpendicular.

Some reflection on the first two parallels that we have encountered here will reveal the fact that the right-angled condition of parallelism is only a special case. If, instead of right angles, we have any supplementary angles, the lines L and M (Fig. II–57) do not meet for precisely the same reason, i.e., the limitation of 180° to the triangle.

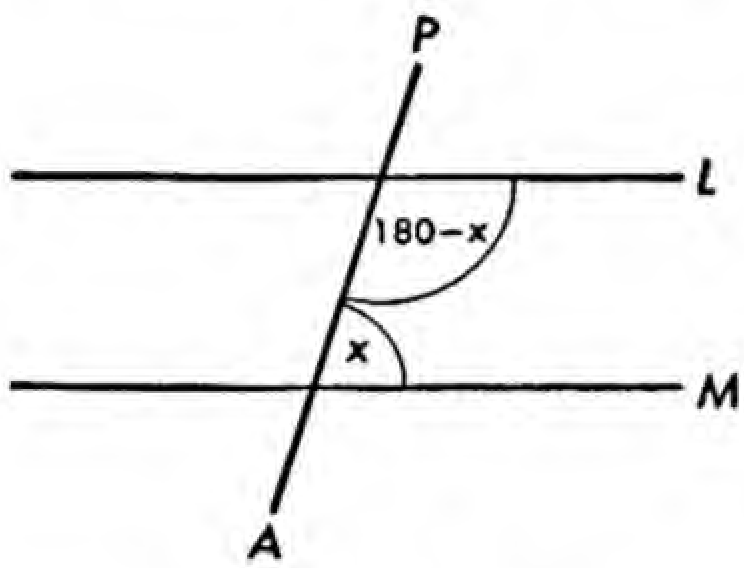


Fig. II–57

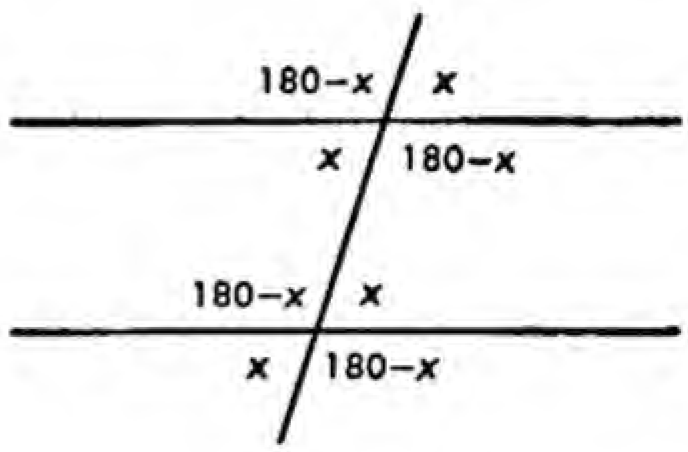


Fig. II–58

A line that intersects other lines is referred to generally as a *transversal.* We have the fact that

> *If a transversal meets two lines so that two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

By virtue of the many relations of supplements and of vertical angles in the last figure, it is possible to find all the angles in the drawing, as is done in Fig. II–58. The fact is revealed that all the angles (and there are eight of them) are either equal to x or to $180° - x$. This means that we have many pairs of supplements as well as pairs of equal angles. For future reference it is wise to single out a few.

As always, classification is a boon to thought. Angle pairs such as a and a' (Figs. II–59 and II–60) are called, descriptively, *alternate interior*, and

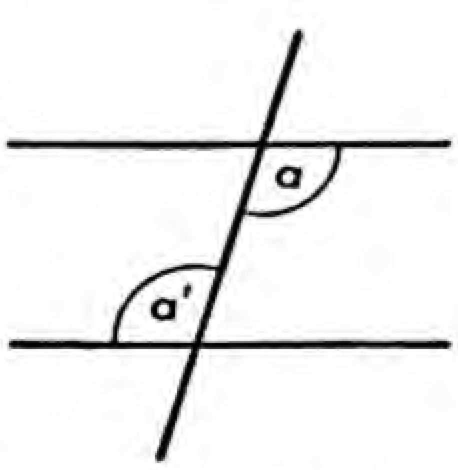


Fig. II–59

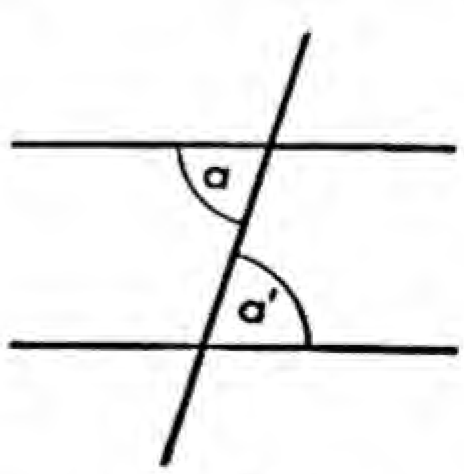


Fig. II–60

the pairs c and c' (Fig. II–61) are called *corresponding*. We have, then, a new theorem as a corollary to the theorem on parallel lines:

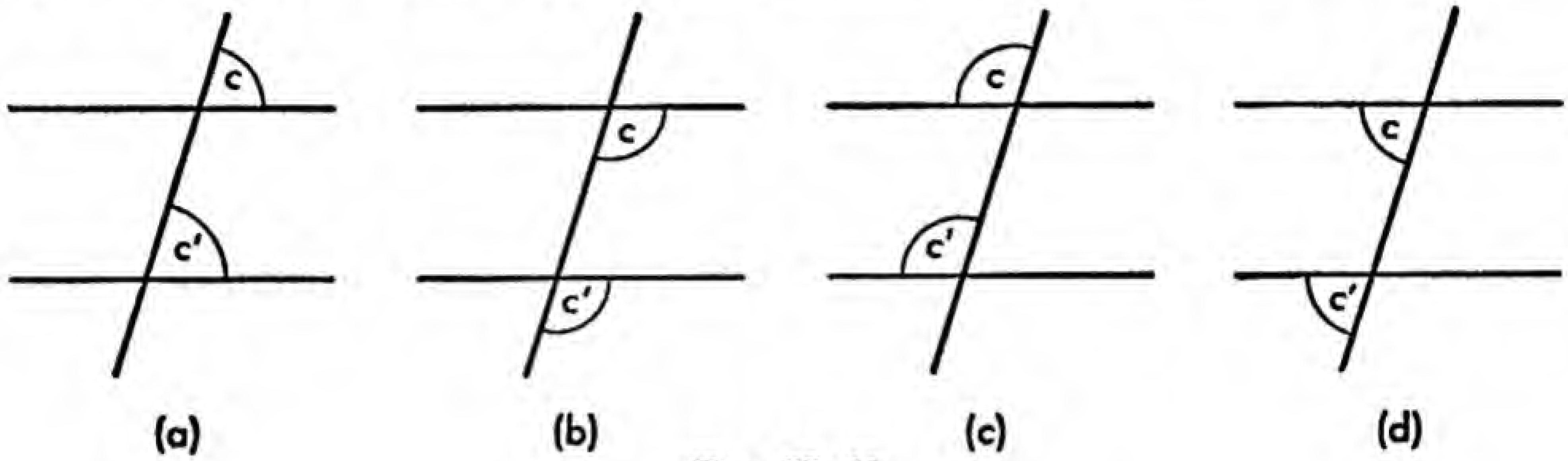


Fig. II–61

If a transversal forms equal alternate interior or corresponding angles with two lines, the lines are parallel.

Sooner or later we must face a very important question regarding parallel lines. We have seen (Fig. II–56) that there exists a parallel to a line through a point outside that line. Are there additional parallels through that same point?

Well, let us suppose that there is another parallel, L′ (Fig. II–62). The line L′ cannot also be perpendicular to PA, for at P there is only one perpendicular to PA. Thus L′, if it exists, forms a pair of supplementary angles with AP, one of which is acute and the other obtuse. Let us assume that $\angle 1$ is acute.

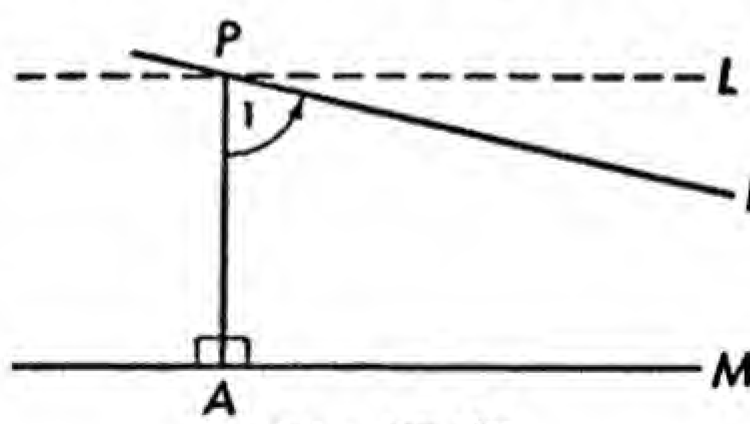


Fig. II–62

It is now possible to choose any point B, other than P, on L′ from which to drop BC perpendicular to PA (Fig. II–63). The lines PC and PA bear some ratio k to each other, and so we may refer to their lengths as b and kb, respectively. If we let $BC = p$ (Fig. II–64), then there exists a length kp which we may measure off on line M. Let $AD = kp$. Now, we draw PD.

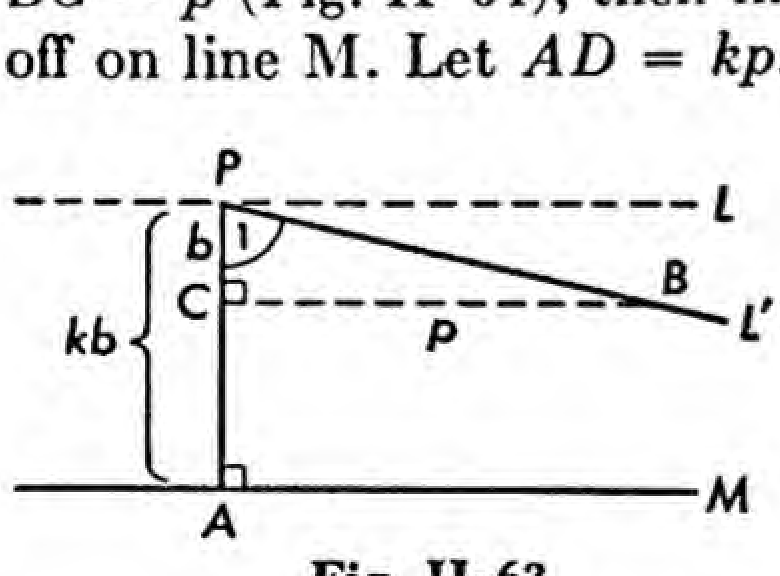


Fig. II–63

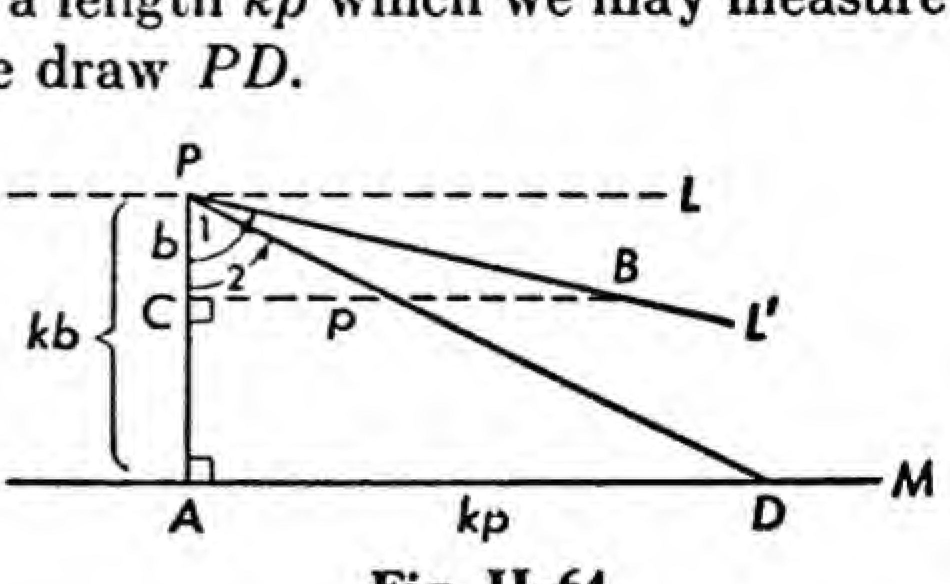


Fig. II–64

If we compare triangles PCB and PAD, we find that they are similar by *sas*, since the two pairs of sides have the same ratio k, and the included angles are right angles. As a result of this, we have the astonishing fact that

$\angle 1 = \angle 2$. The line PA, being a common side of these two angles, forces the sides PD and PB to be one and the same line. But this is preposterous because L′ is supposed to be a parallel to M. This assumption must be false, and so we have:

Through a point outside a line, there is one and only one parallel to that line.

We have seen earlier that a quadrilateral can contain four right angles. Because any pair of right angles are supplementary, it follows that the opposite sides of this figure (which is, of course, a rectangle) are parallel. These opposite sides were proved earlier to be equal.

In connection with the development of the theorem concerning the sum of the angles of a triangle, we have presented the theorem that the line joining the midpoints of two sides of a triangle cuts off a triangle similar to the original. If B and D are such points (Fig. II–65), then BD itself is one-half of AE. Now, because $\angle B = \angle A$, we can go further and say that $BD \parallel AE$, or *BD is parallel to AE*. The symbol for parallelism in writing is shown ($\parallel$), and the symbol in diagrams is indicated by arrows along the lines.

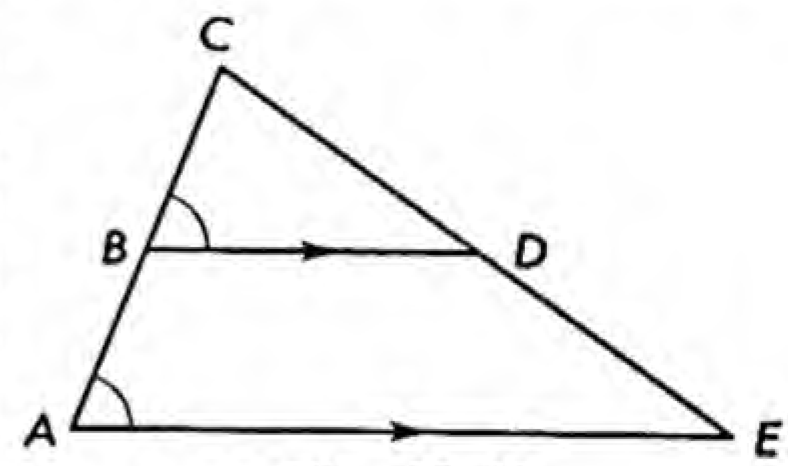


Fig. II–65

The triangles BCD and ACE will remain similar if BD cuts off any proportional segments along the sides in Fig. II–65. The ratio of 1:2, which is the case when we are dealing with midpoints, is only a special case. Consequently BD will remain parallel to AE no matter what the ratio may be. In general,

If a line divides two sides of a triangle proportionally, it is parallel to the third side.

So far we have started with relationships between angles and have deduced (under certain conditions, of course) the parallelism of lines. What of the converses? Does the parallelism of lines imply the angle relationships?

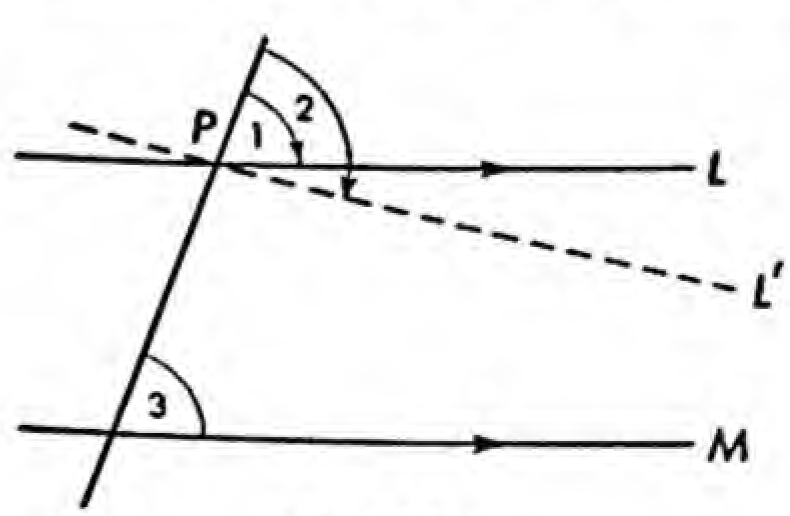


Fig. II–66

Consider the parallel lines L and M (Fig. II–66). If, for example, a transversal does not make $\angle 1 = \angle 3$, then it is possible to draw another line L' so that $\angle 2 = \angle 3$. ($\angle 2$ can be assumed to be less than $\angle 1$ with a slight, but inconsequential, change in the diagram.) The equality of the corresponding angles leads to the conclusion that $L' \parallel M$. Now we have both L and L' parallel to M. This is impossible, since through a point outside a line, only one parallel is possible to the

line. So L and L′ are indeed one and the same line, making $\angle 1 = \angle 3$.

We have seen earlier that when the corresponding angles are equal, the alternate interior angles are also equal, and the interior angles on the same side of the transversal are supplementary. Thus,

If two lines are parallel, then a transversal forms equal corresponding angles, equal alternate interior angles, and supplementary interior angles on the same side of the transversal.

In Fig. II–67 we recognize a familiar situation but now appearing as a converse. The line that is parallel to one side of a triangle, creating equal

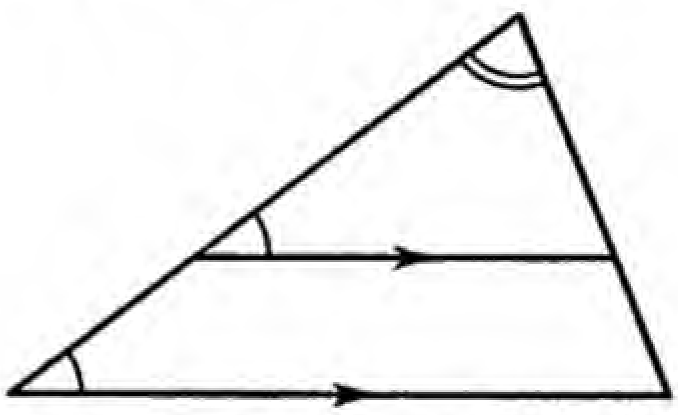


Fig. II–67

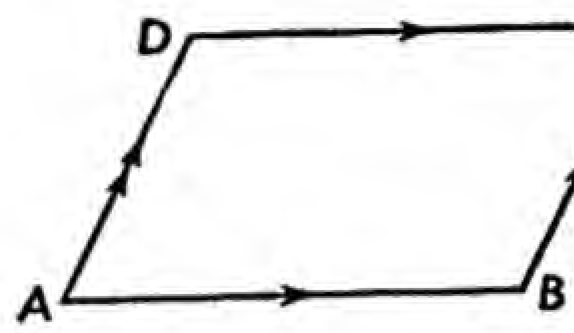


Fig. II–68

corresponding angles, cuts off similar triangles. Thus the two sides are divided proportionally. The existence of parallel lines permits us to define a special quadrilateral (Fig. II–68), one whose opposite sides are parallel. This is called a **parallelogram**, whose symbol is a small image of it, ▱.

The parallel lines make both angles A and C supplementary to $\angle B$. Consequently $\angle A = \angle C$. Similarly, $\angle B = \angle D$. This is familiarly known as

The opposite angles of a parallelogram are equal.

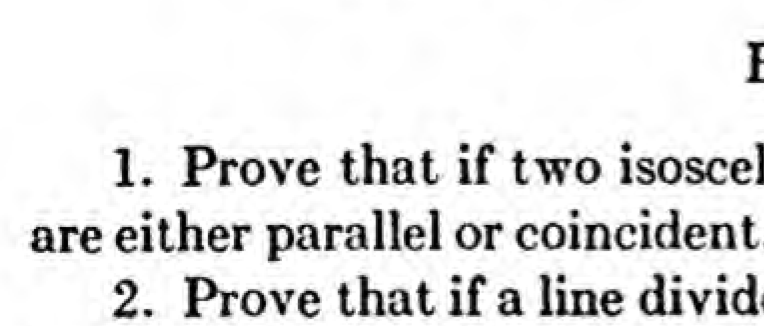


Fig. II–69

The line DB in Fig. II–69 is called a *diagonal*. If the quadrilateral is a parallelogram, we see that the diagonal forms congruent triangles. Now we find that

The opposite sides of a parallelogram are equal.

EXERCISES (II–12)

1. Prove that if two isosceles triangles have a common vertex angle, the bases are either parallel or coincident.

2. Prove that if a line divides two sides of a triangle proportionally, it is parallel to the third side.

3. Prove that the diagonals of any parallelogram bisect each other.

4. A *rhombus* is a parallelogram with two adjacent sides equal. Why does this definition make the figure equilateral? Is it necessarily equiangular? Prove that the diagonals bisect the angles in a rhombus.

5. A *trapezoid* is a quadrilateral with only two sides parallel. The parallel sides are called the *bases*. Prove that the diagonals divide each other proportionally.

6. In an isosceles trapezoid the nonparallel sides are equal. Prove that the base angles are equal. (The base angles are either pair adjacent to either base.)

7. Prove that each of the following conditions in a quadrilateral determines a parallelogram:

a. Opposite sides equal.
b. One pair of sides equal and parallel.
c. Opposite angles equal.
d. Diagonals bisect each other.
e. Adjacent angles supplementary.

8. If the midpoints of adjacent sides of a quadrilateral are joined successively, a parallelogram is formed.

9. If a diagonal of a parallelogram bisects the angles, the figure is a rhombus.

10. Prove that the diagonals of a rhombus are perpendicular to each other.

11. Prove that the diagonals of an isosceles trapezoid are equal to each other.

12. A square may be defined as a rectangle with a pair of adjacent sides equal or as a rhombus with a right angle. For either definition, list the consequent properties of the square.

13. ANOTHER BASIC CONCEPT—AREAS

The matter of areas is so commonplace that, although not as yet mentioned here, it surely has occurred to the reader in connection with many of the figures with which we have dealt.

Intuitively the concept arises from contact with closed figures, figures that enclose something. This "something" appears more or less in various contexts. Its quantitative aspect in practical historical terms was of no less importance than it is today, for among other things it meant land for farming and land for building. Of course the concept we refer to is "area" or, more completely, the "area of an enclosed surface," or "area of an enclosed figure."

Once a quantitative aspect is recognized, it is a matter of urgency to invent and define a unit of measure for it, one which often has its own historical evolution. In the development of the unit, the area of a unit square was generally taken and was called 1 square unit. If the unit on the side of the square is 1 inch, 1 mile, or 1 centimeter, then the area is called 1 square inch, 1 square mile, or 1 square centimeter, respectively. The unit areas are also written as 1 in.^{2}, 1 mi.^{2} or 1 cm.^{2} to indicate the nature of the operation which we shall consider momentarily.

The definition of area is adequate to determine the area of a rectangle (Fig. II–70) with rational dimensions by means of subdivisions of the sides as indicated. The result is

$$A = lw$$

The *area of a rectangle is equal to the length multiplied by the width.* However, in accepting this theorem for irrational dimensions, one would have to indulge in the limits and methods of procedure that were discussed in connection with irrational numbers. We shall assume instead that the formula holds for all real values of the dimensions.

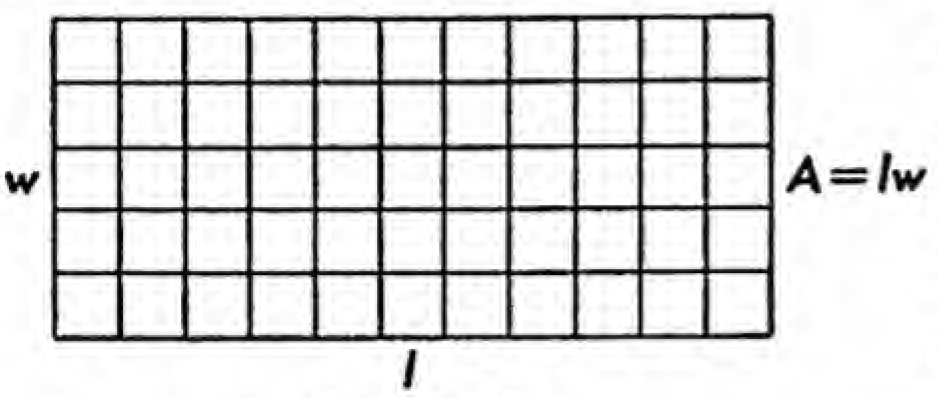


Fig. II–70

Other formulas follow quickly, once this groundwork has been laid. The area of the square, for example, is only a special case of the rectangle. If s is the side of the square, then the area is

$$A = s^2$$

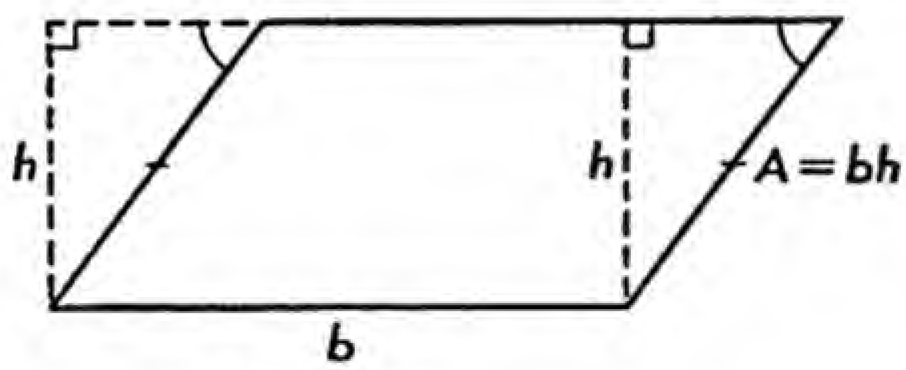


Fig. II–71

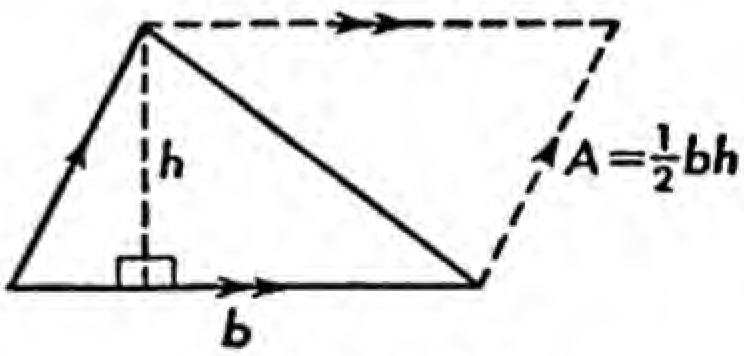


Fig. II–72

The parallelogram is easily related to the rectangle, as indicated in Fig. II–71. Because of the congruent triangles and the common quadrilateral, both the rectangle and the parallelogram have the same area. Thus

$$A = bh$$

The triangle, in turn, is shown to be one-half of a parallelogram, as indicated in Fig. II–72, with the resulting one-half in the formula

$$A = \tfrac{1}{2}bh$$

EXERCISES (II–13)

1. a. Find the area of a square whose diagonal is 8 inches.
   b. If d is the diagonal of a square, show that its area is $\frac{1}{2}d^2$.

2. The hypotenuse of a right triangle is 18 inches and an arm is 14 inches. Find the area of the triangle.

3. Prove that the median of a triangle divides it into two equivalent parts (equal in area).

4. Prove that the line that joins the midpoints of two sides of a triangle cuts off a triangle that is one-fourth of the original. (Use may be made of exercise 3 or of an earlier conclusion regarding corresponding altitudes of similar triangles.)

5. Derive the formula for the area of a trapezoid:

$$K = \tfrac{1}{2}h(B + b)$$

where B and b are the lengths of the bases, and h is the distance between them, called the *height*. (The trapezoid must be referred to previous cases by subdividing or even building up the figure. There are a few possible ways you may explore.)

6. If d and D are the diagonals of a rhombus, show that $K = \frac{1}{2}dD$.

7. Show that the diagonals divide a trapezoid into three pairs of equivalent (equal in area) triangles.

8. Prove that corresponding altitudes in similar triangles are in the same ratio as are the corresponding sides.

9. Prove that the area of similar triangles are to each other as are the squares of any corresponding parts. (Use exercise 8.)

10. The nonparallel sides of an isosceles trapezoid are each 5, and the bases are 7 and 15. Find the area.

11. In Fig. II–73 F and G are the midpoints of the sides of $\triangle ABC$, $AE \perp ED$ and $DB \perp ED$.

a. Prove that $ABDE$ is a rectangle.

b. Prove that $\triangle ABC = ABDE$. Drop $\perp$ from C. (The equal sign alone means equal in area, or equivalent.)

c. This diagram will be of utmost importance shortly in *new geometries*, where (a) will not hold but (b) will.

d. Without using (a) or any specific number of degrees, show that $\angle EAB + \angle DBA$ = the sum of the angles of $\triangle CAB$.

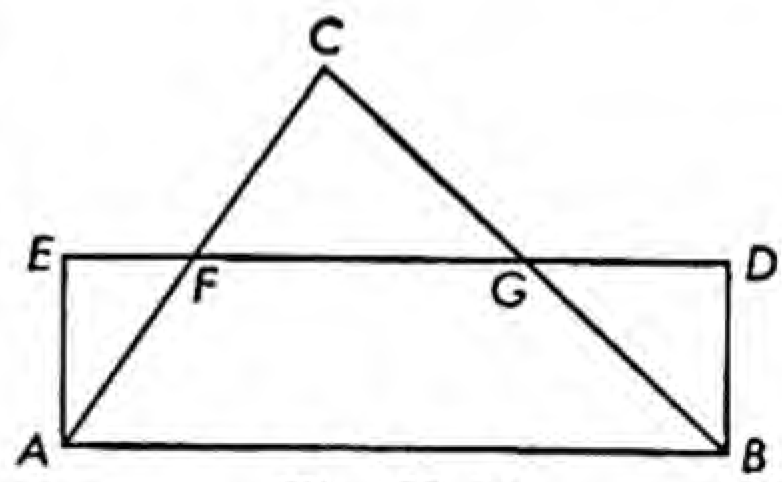


Fig. II–73

II–13 REVIEW

1. Prove that the figure formed by joining successively the midpoints of the sides of a rectangle is a rhombus.

2. Prove that the figure formed by joining successively the midpoints of the sides of a rhombus is a rectangle.

3. If two opposite angles of a parallelogram are bisected, the bisectors and a pair of opposite sides form another parallelogram.

4. a. If P, Q, and R are the midpoints of the sides AB, BC, and CA, respectively, of $\triangle ABC$, prove that $PQRA$ is a parallelogram.

b. What additional condition on $\triangle ABC$ would make $PQRA$ a rhombus?

c. What additional condition to those in (a) would make the figure a rectangle?

d. What additional conditions on the original triangle would result in a square?

5. Find the area of a rhombus whose diagonals are $3k$ and $4k$.

6. The bases of a trapezoid are 6 inches and 14 inches. Find the area if the nonparallel sides are each 5 inches.

7. Express the formula for the area of a trapezoid in terms of

a. the altitude b. the lower base c. the upper base

8. The arms of an isosceles triangle are each 10 inches, and the base is 6 inches. Find the area.

9. a. Find the area of an equilateral triangle whose side is 8 inches.
b. Show that the area of an equilateral triangle is given by

$$K = \frac{s^2}{4}\sqrt{3}$$

where s represents the length of a side.

10. How can one transform a given triangle into an equivalent

a. right triangle? b. isosceles triangle?

11. The corresponding sides of two similar triangles are 10 and 15 centimeters.
a. What is ratio of corresponding altitudes?
b. What is the ratio of their perimeters?
c. What is the ratio of their areas?
d. If the area of the smaller triangle is 25 square centimeters, what is the area of the other?

12. What is the area of a rectangle whose diagonal is 40 inches and whose length is 25 inches?

13. Find the area of a square with a 14-foot diagonal.

14. The area of a trapezoid is 96 square inches. Its height is 8 inches and one base is 18 inches. Find the other base.

III — TRIGONOMETRIC FUNCTIONS

1. THE BEGINNINGS

We have achieved two landmarks in geometry. One is the prediction of 180° for the sum of the angles of every triangle. The other is the Pythagorean theorem, $a^2 + b^2 = c^2$.

The first imposes a logical constant on every triangle as conceived within our postulational framework. The other provides the means of predicting the length of any side of a right triangle when we know the lengths of the other two sides. In the latter case this is possible because a right triangle is uniquely determined by two sides, which gives either an *sas* or a hypotenuse arm combination of data. With such information, one and only one triangle is possible unless, of course, one of the postulates or theorems is violated. For example, it would be a violation to conceive of a right triangle where the hypotenuse was less than a side. Should we conclude that we are ready for other predictions in the triangle?

Barring violations, the principle of congruence teaches us that a triangle is uniquely determined by any of the following sets of data: *sss*, *sas*, *asa*, *saa*, and hypotenuse arm. To say, for example, that all triangles with the same *sas* are congruent is to say that that combination of *sas* data determines one and only one triangle. This means that the other parts of the triangle are each uniquely determined; each is some constant. That being so, we are hopeful of being able to determine those parts, as has already been partially done in connection with the right triangle.

Strangely enough, the *aa* case of similarity applied to the right triangle provides the means of a breakthrough.

Suppose that $\angle A$ (Fig. III–1) is fixed for the right triangle ABC. All right triangles with an angle equal to A are similar to this triangle. All the infinite pairs of sides of these similar triangles are proportional. This means only one thing, and that is that the ratio of any two sides of this triangle is constant, and that not the sides but the ratios are unique.

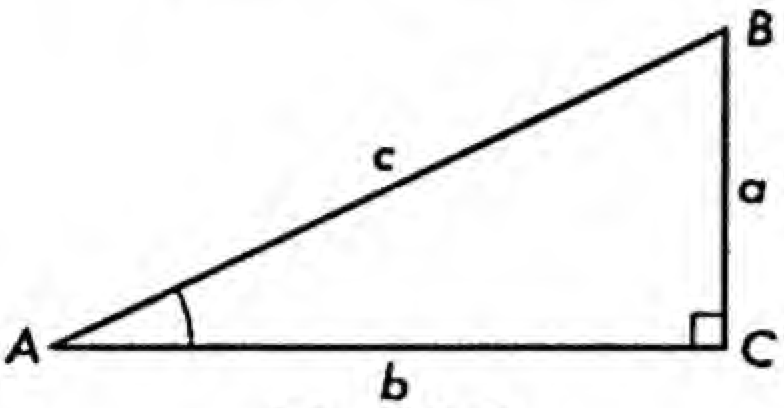


Fig. III–1

There are six possible ratios of the sides of this triangle. Because of the eventual significance of these ratios they are assigned names for purposes of communication.

$$\text{sine } A = \frac{a}{c} \qquad \text{cosine } A = \frac{b}{c}$$

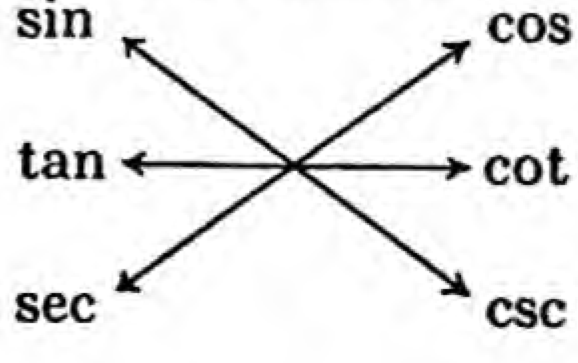
$$\text{tangent } A = \frac{a}{b} \qquad \text{cotangent } A = \frac{b}{a}$$

$$\text{secant } A = \frac{c}{b} \qquad \text{cosecant } A = \frac{c}{a}$$

The values of these ratios are dependent on $\angle A$ since $\angle C$ is a right angle and will remain one, at least for a while. Each definition represents a ratio that is dependent on some acute angle. We shall see that each value of A will lead to a single value of **sine** A or any one of the others. This creates a *set of ordered pairs* of values, which can be indicated generally by $\{A, \text{sine } A\}$ or $\{A, a/c\}$. The same can be expressed for the other ratios. *A set of paired values is known as a function.* This is only the beginning of our contact with functions.

Because of the fact that this subject has its origin in *triangle measurement*, it is called by the Greek equivalent, **trigonometry.** Consequently the definitions given here are referred to as **trigonometric functions.** The sine function is defined by the equation sine $A = a/c$ or by the paired values $\{A, \text{sine } A\}$. This will be made more concrete shortly.

We recognize in the listing of the trigonometric functions a number of reciprocal functions. The sine and cosecant are reciprocals, since a/c and c/a are reciprocals. Likewise the tangent and cotangent and the cosine and secant are reciprocal functions. These facts are conveniently summarized in the accompanying table wherein the arrows connect the reciprocal functions. Note too that the first three letters of each name is used as the abbreviation of the function.



(some use ctn for cot)

Our knowledge of similarity has brought us to the realization that for every permissible value of $\angle A$ the value of each function is specifically determined. However, we have no inkling as yet of how the values of the ratios are found. We turn to this now.

Suppose that $\angle A = 30°$ in the right $\triangle ABC$ (Fig. III–2). Then, $\angle B = 60°$. Suppose too that $AB = 12$. If BC is extended its own length to B' and AB' is drawn, two congruent triangles are formed. Then $\angle B' = 60°$ too, and consequently $\angle BAB' = 60°$. Thus $\triangle BAB'$ is equiangular and, by an earlier theorem, equilateral. This yields the facts that $BB' = 12$ and $BC = 6$. The length of AC may be found now by means of the Pythagorean theorem:

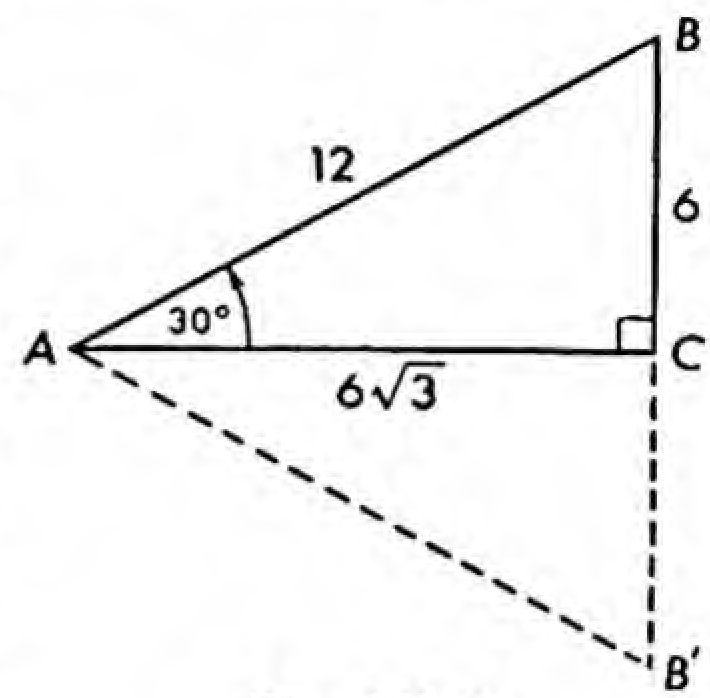


Fig. III–2

$$\begin{aligned} x^2 + 6^2 &= 12^2 \\ x^2 + 36 &= 144 \\ x^2 &= 108 \\ x &= \sqrt{108} = \sqrt{36}\sqrt{3} = 6\sqrt{3} \end{aligned}$$

The knowledge of the three sides of the triangle permits us now to list the values of the functions of 30° in the light of their definitions.

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot 30° = \sqrt{3}$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \csc 30° = 2$$

We can use the same triangle to determine the ratios for $\angle B$. All we need do is to substitute b for a in the original definitions. Actually, after some acquaintance with the ratios, one gets to sense them without reference to the literal definitions. Using $\angle B$, which is 60°, we get:

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

$$\tan 60° = \sqrt{3} \qquad \cot 60° = \frac{\sqrt{3}}{3}$$

$$\sec 60° = 2 \qquad \csc 60° = \frac{2\sqrt{3}}{3}$$

The fact that we took a particular 30°–60° triangle with the hypotenuse 12 should be of no concern, since all 30°–60° triangles are similar and the

corresponding sides are in the same ratios. Or, as we emphasized earlier, the ratios are constant.

Similarly, by taking an isosceles right triangle (Fig. III–3), we get all the functions of 45°. This time we take the equal arms, each 1.

$$x^2 = 1 + 1$$
$$x = \sqrt{2}$$

The functions of 45° may be read directly from the triangle. Indeed it is preferable to visualize the triangle than to memorize lists of values.

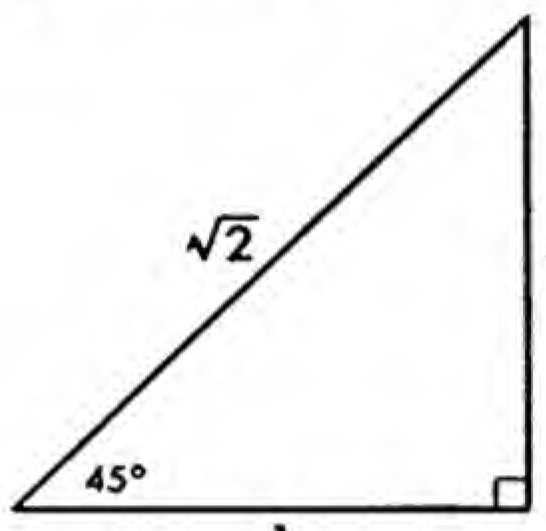


Fig. III–3

It is becoming apparent that our trigonometric values are largely irrational. In fact the rational values are the exception. With the development of additional geometric knowledge and trigonometric formulas, many other values of the functions can be calculated, but this would be an almost insuperable task to determine enough for contemporary needs. It was not until the development of the calculus that the theoretical means were provided for calculating any trigonometric functional value. We shall see this later. It is remarkable, though, that our ancestors, long before the calculus, developed highly accurate tables which they needed and used in astronomy and navigation.

We have gone far enough to undertake a simple application which will underline the method of *indirect measurement*, or (better) *indirect determination*. This is a powerful tool in science as well as in mathematics. We have already seen this method applied in a few places. Thus an angle of a triangle is uniquely determined by the other two angles. A side of a right triangle is uniquely determined by the other two sides. And now, a side of a right triangle is uniquely determined by a side and an acute angle.

Suppose that BC (Fig. III–4) represents some depth or distance whose length is desired but which it is impossible or very difficult to obtain directly. If this is the case, it may be possible to involve BC as a side in a triangle, a right triangle at this point, in which another side and an angle are amenable to direct measurement. The result is the data shown in the diagram. Because of *saa* we know that the triangle is uniquely determined, and therefore BC is determined too. The only question is whether our theory has advanced far enough for the occasion. Indeed it has.

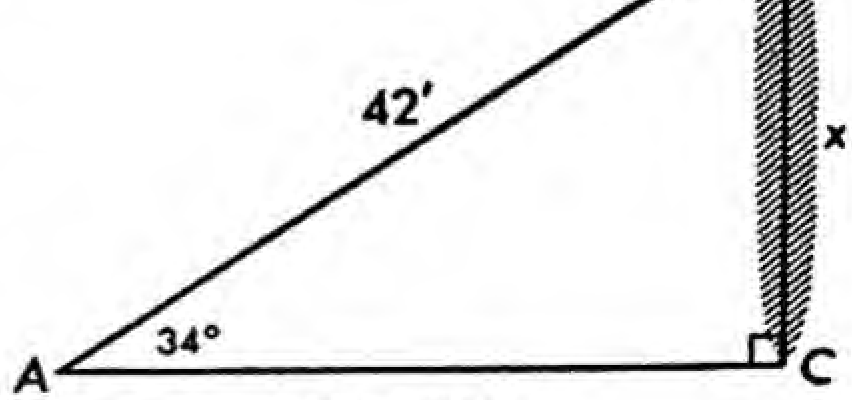


Fig. III–4

Which of the six functions shall we employ? There is little choice left to us at this point. The known facts, 34° and 42′, are to be related to the un-

known side which is labeled x. Only two functions utilize these three parts, and they are the sine and the cosecant. We choose the sine and get the following solution:

$$\frac{x}{42'} = \sin 34°$$
$$x = 42' (\sin 34°)$$
$$x = 42' (0.5592) = 42' (0.559)$$
$$x = 23.48 \approx 23$$

Our tables give the trigonometric function values, correct to four places. However, the 0.5592 was cut down to 0.559. We are dealing with measurements which, by their very nature, are approximate. The line $AB = 42'$ has been measured to two significant figures. That is, we have knowledge only of the 4 and the 2 in the 42'. Had this line been measured more accurately, meaning usually with finer instruments, we could have additional figures. In the absence of this information, it would hardly be reasonable to expect an answer with a greater degree of accuracy. Since the 42' has only two significant figures, the answer has been curtailed to two significant figures. During the calculations it is of little avail to carry more than one figure beyond those needed in the answer. For this reason the tabular value of 0.5592 has been cut down to 0.559.

Consider, by way of another illustration, the case where the length of DC is to be found (Fig. III–5). It may be that BC is also inaccessible, in which case the indicated measurements can be made. As a result of these measurements, $\triangle ABD$ is determined by *asa*, which includes the supplement of the 43° angle. If a triangle is uniquely determined, so are all its parts, including altitudes. We have, from the two right triangles.

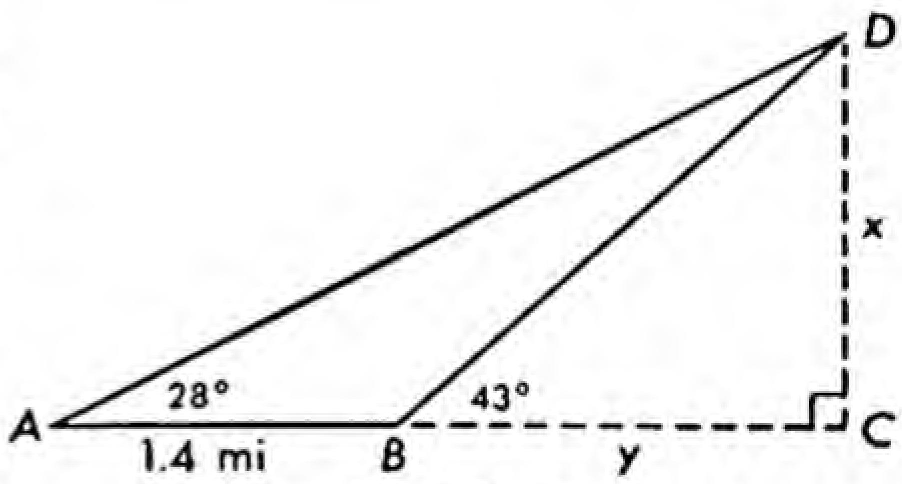


Fig. III–5

$$\frac{y}{x} = \cot 43° \qquad \frac{y + 1.4}{x} = \cot 28°$$
$$y = x \cot 43° \qquad y + 1.4 = x \cot 28°$$

Substituting the y value from the first equation into the second, we get

$$x \cot 43° + 1.4 = x \cot 28°$$
$$1.07x + 1.4 = 1.88x$$
$$1.4 = 0.81x$$
$$x = \frac{1.4}{0.81} \approx 1.7 \text{ mi.}$$

It is well to note that the length of a line has been just calculated from data situated at a considerable distance from the line. In a very small way this is indicative of the procedure in surveying, navigation, and astronomy.

EXERCISES (III–1)

1. Name some specific instances, from fields other than mathematics, of indirect determination.

2. As mentioned in the text, it is best to visualize the functions. This is suggested by the accompanying diagram (Fig. III–6). The shaded angle is the angle of reference for the indicated functions. Since the cosecant is the reciprocal of the sine, one need only reverse the arrow for the sine to get the cosecant ratio. This can also be done for the other reciprocals. Find the numerical values of all the functions of the shaded angle in each case (see Fig. III–7).

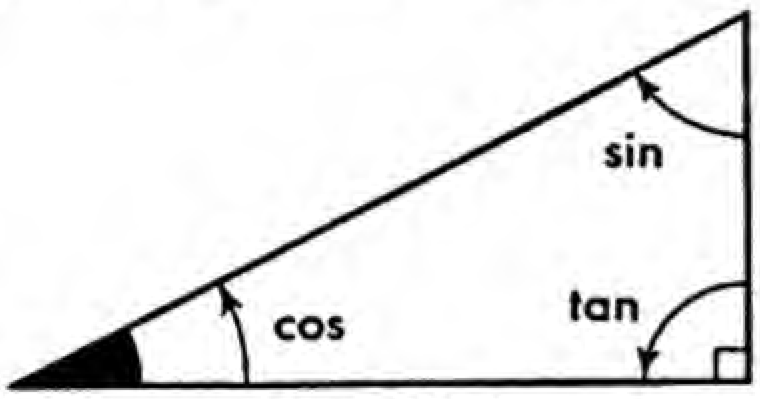


Fig. III–6

3. The trigonometric function refers to a ratio of sides of a right triangle (Fig. III–8). For any particular angle the value is not disturbed if one of the sides is arbitrarily selected. By assigning the value 1 to one

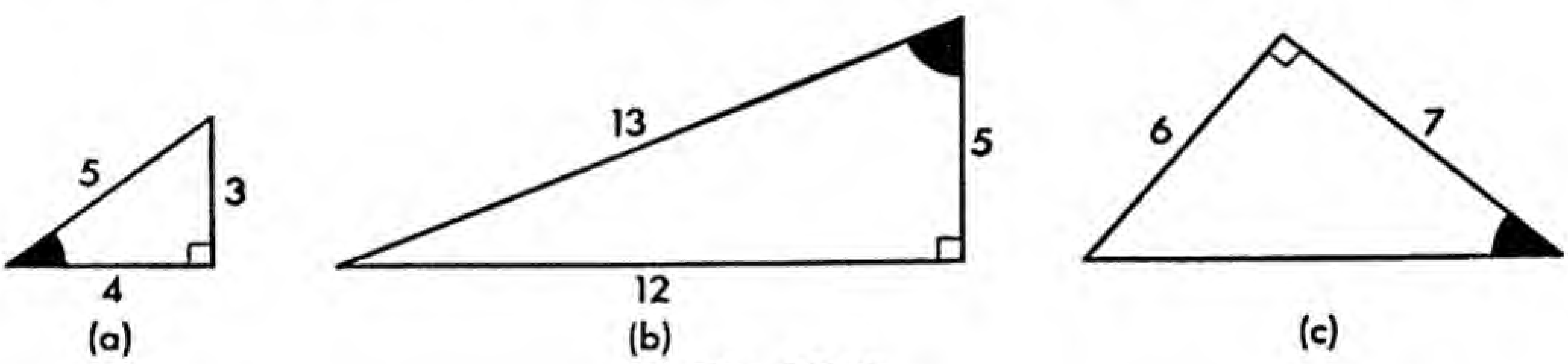


Fig. III–7

of the sides, certain of the functions are directly expressible as function values of one side only. For example,

$$\sin P = \frac{QR}{1}$$

$$\cos P = \frac{PQ}{1}$$

Derive the expressions for the values of Fig. III–9.

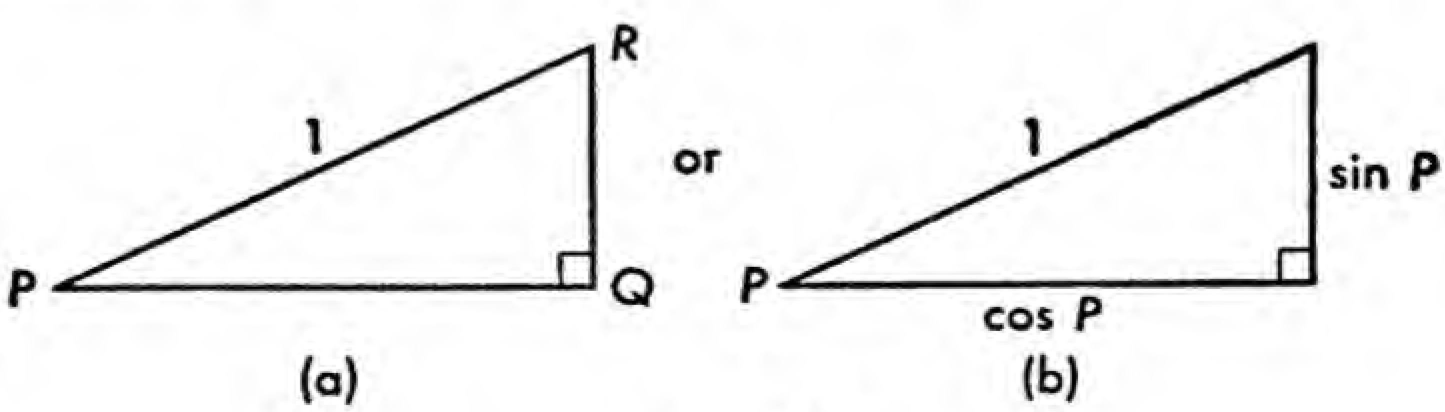


Fig. III–8

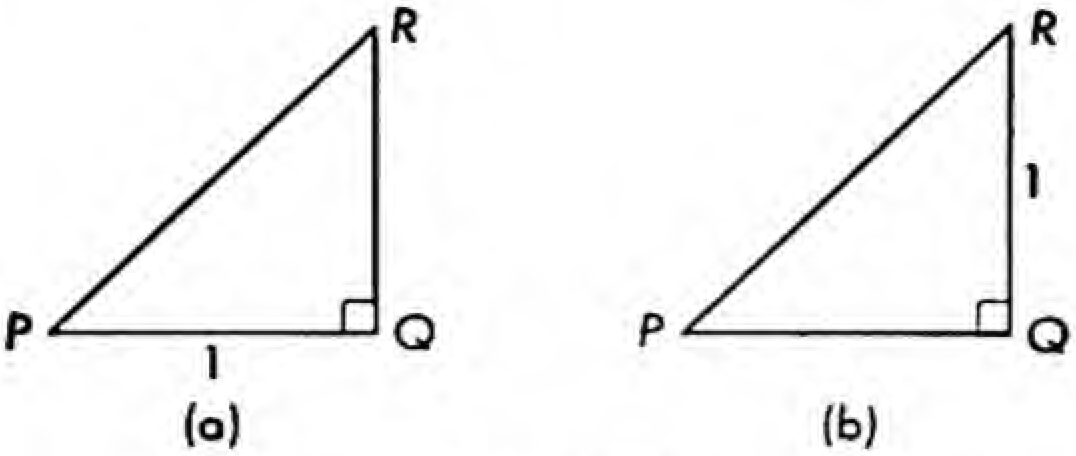


Fig. III–9

4. The conclusions in exercise 3 express the trigonometric functions as function values of the sides. However, the three sides in any one of the triangles are intimately related by the Pythagorean theorem. Use the theorem to write three new formulas, one for each of the cases in exercise 3. A slight problem in notation will arise here. Choose your own notation, which you believe will avoid ambiguity.

5. List the function values for 45°.

6. It would be particularly informative and revealing to compute some function values from careful drawnings. A protractor and ruler are required. Try this for 38° and compare results with the tabular values. Why would more than one drawing be desirable?

7. The term **significant figures** refers to those digits in a measured quantity that reflect the degree of accuracy of the measurement. Thus, to speak of the speed of light as being 186,000 miles per second implies, unless otherwise known or indicated, that the measurement is expressed in *thousands of miles per second* as the unit. It would be more informative to write that the speed of light is 186 thousand miles per second, thereby making explicit the accuracy implied. The measure has only three significant figures: 1, 8, and 6. The actual speed probably lies between one-half the unit of measure, more or less. One-half of a thousand miles per second is 500 miles per second. So, the speed of light is somewhere between 185,500 and 186,500. This is conveniently summarized by the legend 186,000 $\pm$ 500 miles per second.

Similarly a measurement of 0.0048 inch really refers to a measure of 48 ten-thousandths of an inch and may be off as much as $\frac{1}{2}$ ten-thousandths, which is $\frac{1}{2}(0.0001) = 0.00005$. So, this measure has two significant figures, 4 and 8, and is more accurately described as 0.0048 ± 0.00005 inch.

The two illustrations indicate briefly that the zeros at the right and the zeros at the left of a measured number are not significant.

a. Consider the product 5.2×3.7. Taking both these numbers as approximate (measured), find the product of their lowest values and also of their highest values. Compare the results with the usual result.

b. Do the same for 0.08×0.0046.

c. In both (a) and (b) note the advisability of *rounding off*, as suggested in the text.

8. How many significant figures are there in each of the following measured numbers?

a. 1200 b. 8.007 c. 700.3 d. 0.00406

9. From a point on the ground 142 feet from the foot of a vertical tower (Fig. III–10a), the *angle of elevation* of the top is 19°. Find the height of the tower.

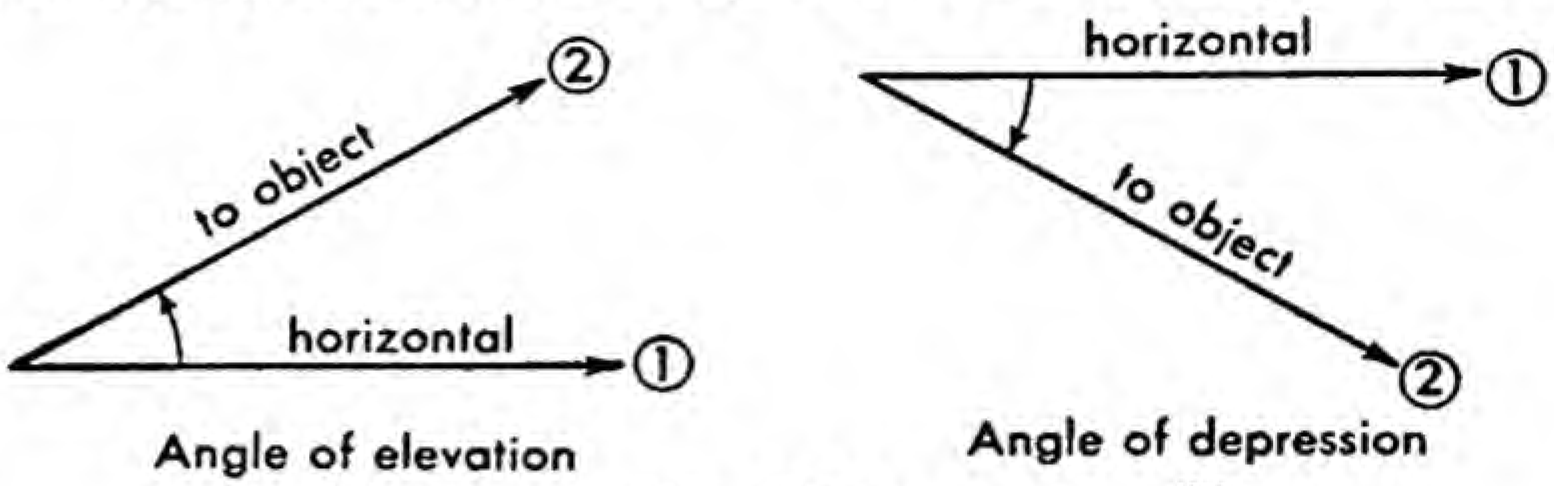


Fig. III–10

10. From a point P 22 feet above the ground (Fig. III–10b), the *angle of depression* of a marker is 21°. (a) How far is the marker from the point directly beneath P? (b) How far is the marker from P?

11. The dimensions of a rectangle are 17x24 inches. Find the angle that the diagonal makes with the larger side. (Note: If, for example, $\cos M = 14/19 \approx 0.7368$, then by reference to the tables, $M \approx 43°$.)

12. Find the *modulus* and angle of the complex number (5, 3). (The modulus is the absolute value of the length of the vector, and the angle, or argument, is that formed by the vector and the positive part of the real number axis.)

13. Find the value of x in Figs. III–11(a) and III–11(b).

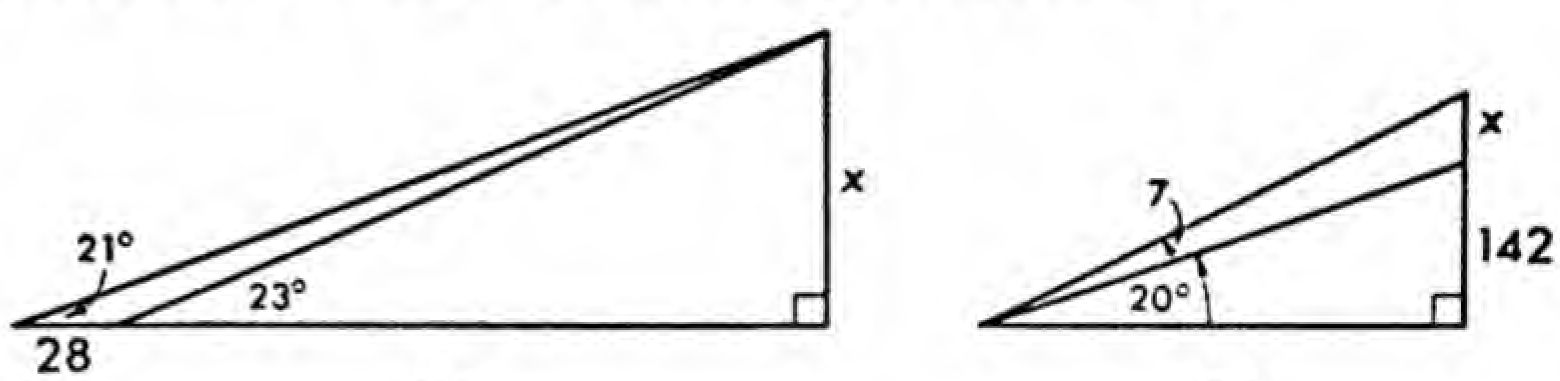


Fig. III–11

14. The vector concept discussed in the first chapter in connection with complex numbers is applicable to all other vectors such as velocity, acceleration, and force. A projectile is fired with an initial velocity of 840 feet per second in a direction inclined 32° with the horizontal. Find the horizontal and vertical components of the initial velocity vector.

15. The take-off velocity of a plane is 72 miles per hour at an 8° angle to the ground. Find the horizontal and vertical components of this velocity vector and describe the meaning of the two results.

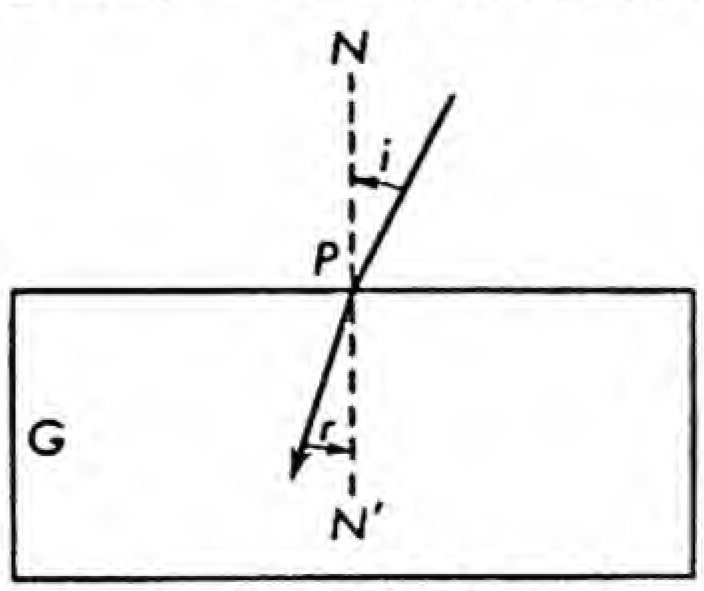


Fig. III–12

16. The change of velocity of light as it passes from one medium to another is utilized effectively in scientific studies. Suppose that a ray of light enters a glass medium from air at a point P (Fig. III–12). G represents the glass medium. NN' is the *normal* perpendicular to the surface of the glass, i is the *angle of incidence*, and r the *angle of refraction*. The ratio $\sin i : \sin r$ is defined as the *index of refraction*. If $i = 20°$ and $r = 13°$, find the index of refraction.

2. THE LAW OF SINES

As a result of previous experiences in arithmetic and algebra, one gets to anticipate efforts at removal of initial limitations wherever they arise. One restriction that we have imposed in trigonometry is the right triangle. After all, triangles other than the right triangle are also uniquely determined and by the same sets of data.

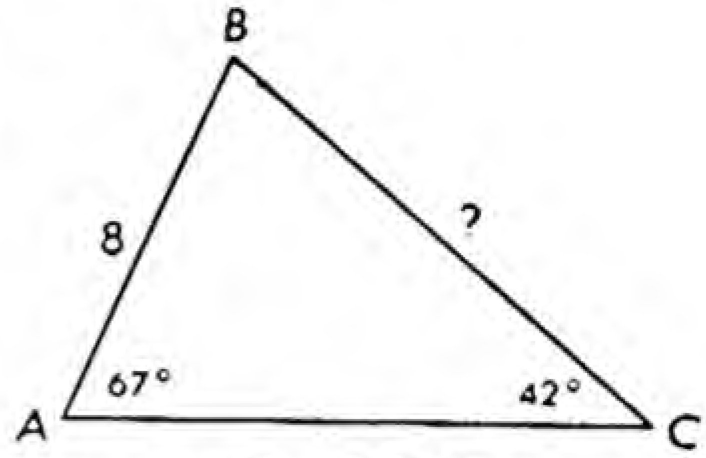


Fig. III–13

Let us begin by considering a "nonright" triangle *saa* case (Fig. III–13). We know now that each unknown part of this triangle has one and only one value under given circumstances. In $\triangle ABC$, with facts given, BC must have a unique value. Let us attempt to find this value.

For one thing, because of our knowledge of the right triangle, a reasonable first step is to draw the altitude BD (Fig. III–14). Of the two right triangles formed, we see that $\triangle ABD$ is determined by *saa*, and so BD can be computed. Should this be done, then $\triangle BCD$ will be determined too by *saa*, thereby making it possible to find BC. The calculations follow:

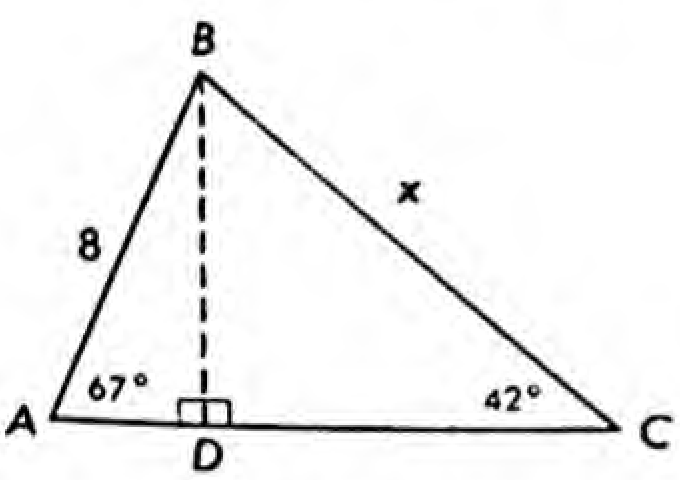


Fig. III–14

$$\frac{BD}{8} = \sin 67°$$

$$BD = 8 \sin 67° = 8(0.92) = 7.4$$

$$\frac{7.4}{x} = \sin 42°$$

$$7.4 = x(0.67)$$

$$x = \frac{7.4}{0.67} \approx 11 = 10 \qquad \text{(to one significant figure)}$$

Would it not be a great waste of energy if, in every *saa* case, we proceeded to draw the altitude and to solve two triangles? Cannot one altitude be drawn in one triangle that will serve as a universal model? This is possible if we proceed exactly as in the numerical case, but on a higher level of abstraction, by means of algebraic representation. If we do this, instead of coming up with a numerical answer, we shall arrive at a literal, a general answer; in brief, a formula.

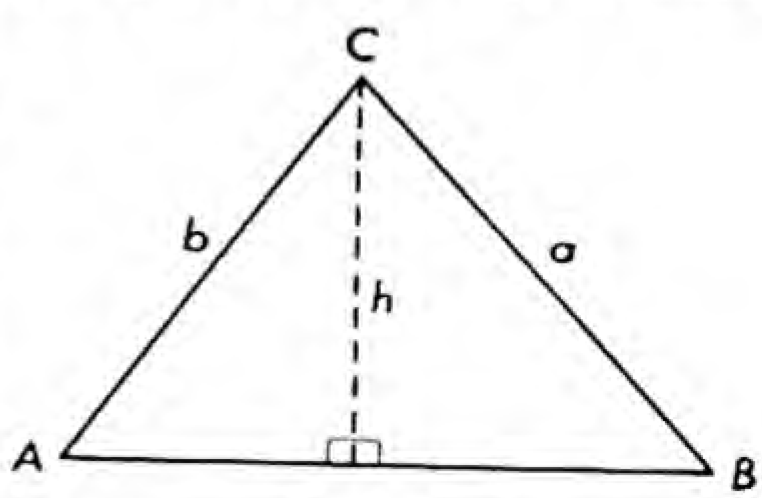


Fig. III–15

In Fig. III–15,

$$\frac{h}{b} = \sin A \qquad \frac{h}{a} = \sin B$$

$$h = b \sin A \qquad h = a \sin B$$

$$b \sin A = a \sin B \qquad \text{(by substitution)}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Had we drawn the altitude to CA instead, we would have gotten

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Thus

$$\frac{\mathbf{a}}{\mathbf{sin\ A}} = \frac{\mathbf{b}}{\mathbf{sin\ B}} = \frac{\mathbf{c}}{\mathbf{sin\ C}}$$

which is known as the **Law of Sines.**

We note that the ratio of any side of a triangle to the sine of the angle opposite is equal to the ratio of any other side to the sine of its opposite angle. In brief, the ratio of a side to the sine of the angle opposite is a constant for any triangle. Constants are intriguing things. In science, as well as in mathematics, their occurrence precipitates an almost immediate search for an interpretation. In this case (we shall be able to see later) the constant happens to be the diameter of the circle circumscribing the triangle.

We can now illustrate the use of this formula. In Fig. III–16,

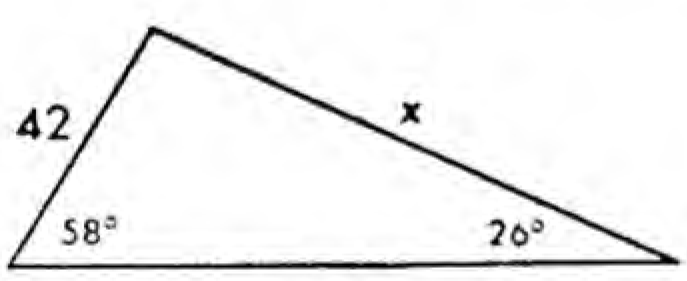


Fig. III–16

$$\frac{x}{\sin 58} = \frac{42}{\sin 26}$$

$$x = \frac{42 \sin 58}{\sin 26} = \frac{42(0.848)}{(0.438)} = 81.3$$

$$x \approx 81$$

Although the Law of Sines was developed for the *saa* case, it is immediately applicable to the *asa* case. This is so because any *asa* case can be converted to *saa* by the simple act of finding the third angle of the triangle.

EXERCISES (III–2)

1. With respect to the illustration of the use of the Law of Sines in the text:

   a. Why were the tabular values cut down to three significant figures?
   b. Why was the answer given to two significant figures?

2. Find the value of m in Figs. III–17(a) and III–17(b).

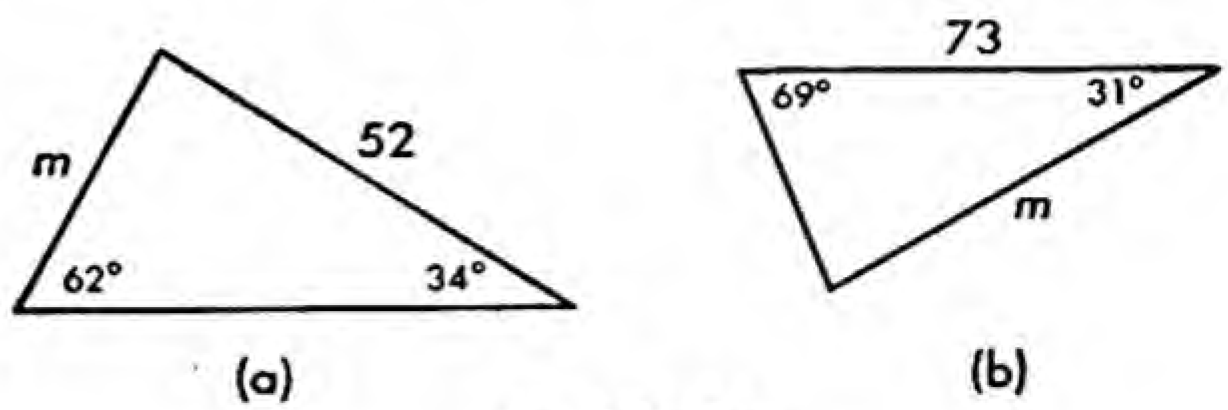


Fig. III–17

3. Find the third side in each of the cases in exercise 2.

4. At a particular moment a balloon was in the same vertical plane as two observation posts on the ground that were 850 feet apart and on opposite sides of the balloon. The angles of elevation of the balloon were found to be 59° and 34°. Find the altitude of the balloon.

5. Prove by means of the law of Sines that the base angles of an isosceles triangle are equal.

6. Prove that

$$h_c = \frac{c}{\cot A + \cot B}$$

where h_c is the altitude to side c in $\triangle ABC$.

7. In Fig. III–18, prove that $m = AB(\tan y - \tan x)$.

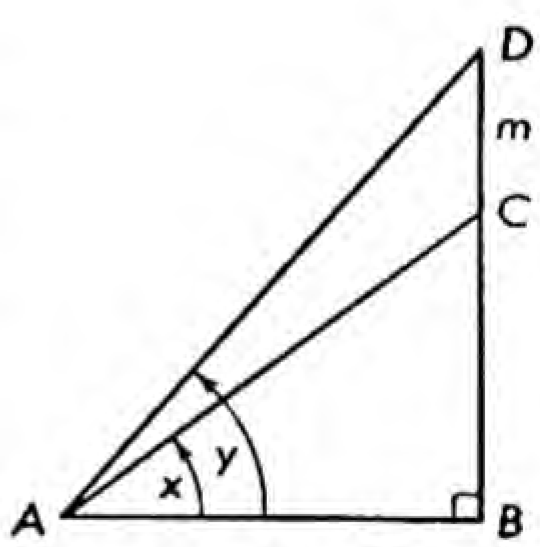


Fig. III–18

8. By means of illustrations of your own choosing, show that if $\sin M > \sin P$, then $M > P$, where M and P are acute.

Using this as a fact, show by means of the Law of Sines that of two sides in an acute triangle, the larger side lies opposite the larger angle.

9. a. Prove that, in any proportion, if $m/p = q/r$, then $(m + q)/(p + r) = m/p$.
   b. Using this fact, show that $p \sin A = a(\sin A + \sin B + \sin C)$, where p is the perimeter of the $\triangle ABC$.

10. Fermat's law of refraction bears an interesting resemblance to the Law of Sines. M and M' (Fig. III–19) represent two homogeneous media such as air and water. A light ray, taking the direction AB and forming an angle θ (theta) with a perpendicular to a line separating the media, is refracted, and the path of the ray is BC, which makes an angle θ' with the perpendicular. The velocities of the rays in the different media are indicated by v and v'. Then, according to an empirical law by Snell,

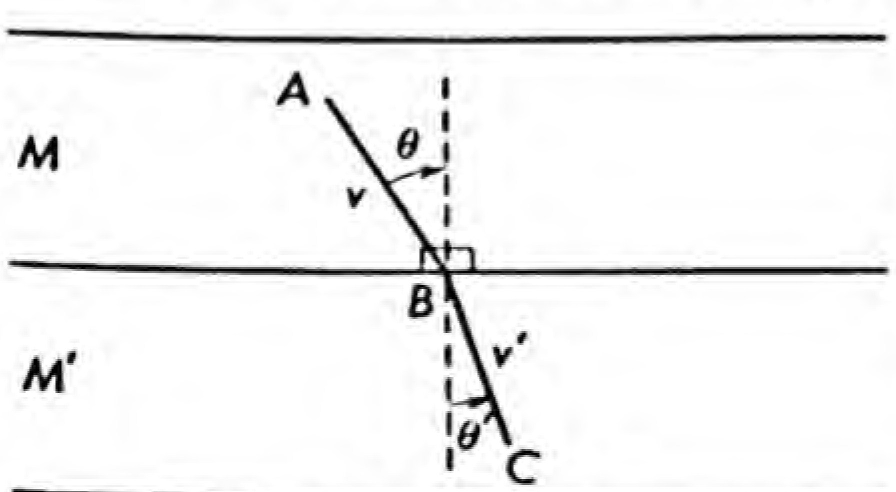


Fig. III–19

$$\frac{v}{\sin \theta} = \frac{v'}{\sin \theta'}$$

Fermat continued to prove that the path ABC is the minimum possible path between A and C. This has an interesting resemblance to the minimum path of reflection for a mirror, a principle which we met earlier.

Show that

$$\frac{v}{v'} = \frac{\sin \theta}{\sin \theta'} = i$$

(See exercise 16, Art. 1).

3. THE LAW OF COSINES

We have now established the trigonometry of the right triangle and of the acute triangle for the *saa* and *asa* cases. There remains the need to resolve the *sas* and *sss* cases. In the *sas* case in Fig. III–20, our only recourse in finding side c is to try an altitude again. We note that $\triangle BCD$ has been determined. This means that we can find BD and also CD. The latter will permit us to find immediately the value of AD. Now, in $\triangle ABD$ we have *sas*, and therefore this triangle is solvable too.

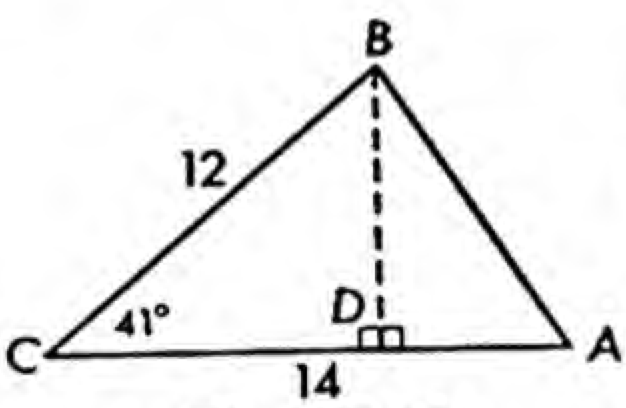


Fig. III–20

If we understand clearly the plan of action, our experience may already be sufficient for us to pass right along to a generalized procedure by means of which we should be able to develop another formula (see Fig. III-21).

From I: $a^2 = h^2 + x^2$
From II: $c^2 = h^2 + (b - x)^2$

$$c^2 = h^2 + b^2 - 2bx + x^2 \qquad (\text{sub: } a^2 = h^2 + x^2)$$
$$c^2 = a^2 + b^2 - 2bx \qquad (\text{sub: } x/a = \cos C$$
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C} \qquad x = a \cos C)$$

This is known as the **Law of Cosines.** We can illustrate (Fig. III–22) its use in connection with the earlier numerical illustration.

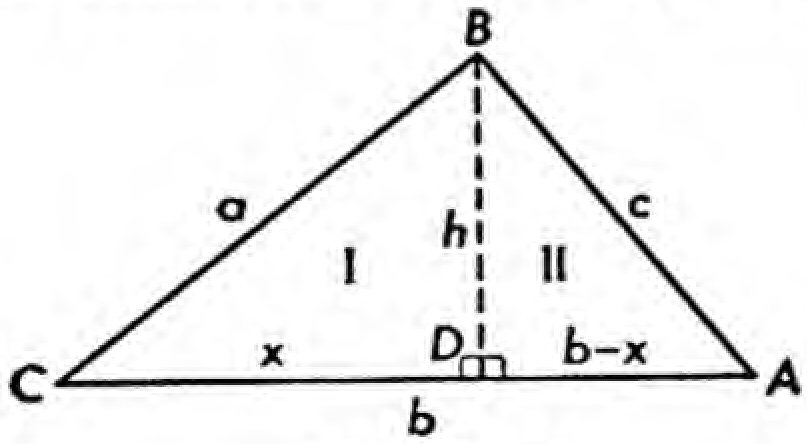


Fig. III–21

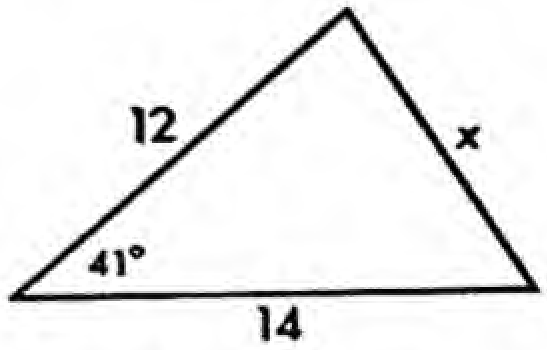


Fig. III–22

$$x^2 = 12^2 + 14^2 - 2(12)(14)\cos 41°$$
$$x^2 = 144 + 196 - 336(0.755)$$
$$x^2 = 340 - 254 = 86$$
$$x = \sqrt{86} = 9.3$$

Fortunately the same formula is adequate for the *sss* case, for if a, c, and b are known, then $\angle C$ in the formula is the only unknown and can be found. The following illustrates (see Fig. III–23) the case:

$$4^2 = 7^2 + 8^2 - 2(7)(8)\cos x$$
$$16 = 113 - 112\cos x$$
$$112\cos x = 97$$
$$\cos x = \frac{97}{112} \approx 0.866$$
$$x \approx 30°$$

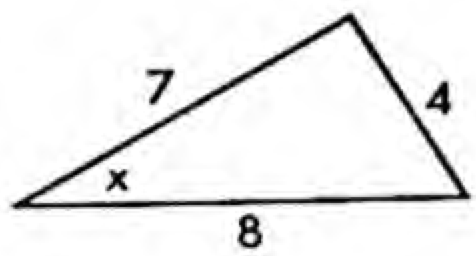


Fig. III–23

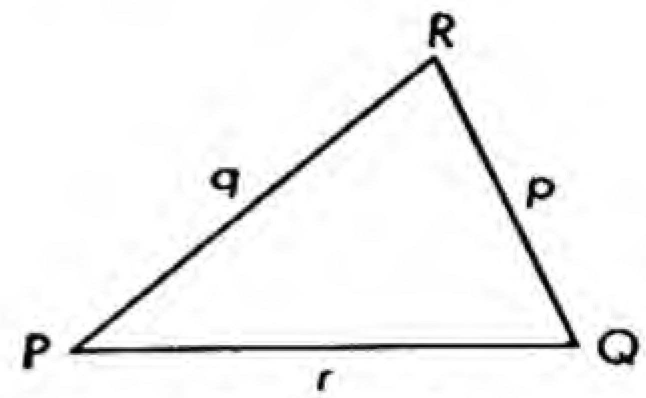


Fig. III–24

As with the Law of Sines, it is desirable to become familiar with the sense of the formula rather than with a memorized sequence of letters. To this end it is desirable to be able to write the formulas, no matter what the letters are, so that the emphasis is on the relationship of the lines and angles of the triangle (Fig. III–24).

$$\frac{p}{\sin P} = \frac{q}{\sin Q}$$
$$p^2 = q^2 + r^2 - 2qr\cos P$$

EXERCISES (III–3)

1. In order to determine an inaccessible distance from A to C, a position B was located from which the following measurements were made: $B = 61°$, $BA = 150$ yards, and $BC = 170$ yards. Find AC.
2. Solve as indicated:
   a. For c, if $a = 17$, $b = 12$, $C = 57°$.
   b. For a, if $b = 23$, $c = 16$, $A = 42°$.
   c. For b, if $a = 6$, $c = 7$, $B = 16°$.
   d. For A, if $a = 7$, $b = 8$, $c = 9$.
   e. For B, if $a = 1.4$, $b = 3.2$, $c = 2.7$.
3. Express the value of $\cos C$ in terms of the sides of the triangle

III–3 REVIEW

1. The velocity of an airplane is 95 miles per hour at take-off and at an angle of 8° to the ground. Find the horizontal and vertical velocities at take-off.
2. A motor boat is continually steered at 90° to the current of a river which is flowing at the rate of 4 miles per hour. The boat speed is 6 miles per hour. (a) Find the actual distance the boat will travel in half an hour. (b) What direction does the actual course of the boat take with respect to the current?

3. From a point above the ground and 56 feet from a building, the angles of elevation and depression of the top and bottom of the building are 23° and 28°, respectively. Find the height of the building.

4. When the angle of elevation of the sun is 47°, find the length of the shadow of a 6 foot 2 inch man. (The angle of elevation of the sun is the angle made by its light rays with the ground in any immediate vicinity.)

5. A projectile is fired at a 37° angle to the ground and with an initial velocity of 115 miles per hour. Find the vertical and horizontal components of the velocity.

6. A ship S is observed from points on shore, A and B, which are 865 feet apart. The angles ABS and BAS are found to be 67° and 53°, respectively. Find (a) the distance of the ship from A and B, and (b) the distance of the ship from the shore.

7. Two planes leave an airport at the same time. One flies N 25° W and the other N 18° E. After some time the navigator in the latter plane, which is now 250 miles from the airport, places the other plane in the direction of S 74° W from itself. How far apart are the planes?

8. Express the value of cos A in terms of the three sides of $\triangle ABC$.

9. A point A is 51 feet from B and 46 feet from C. If $\angle A = 37°$, find BC.

10. Find the smallest angle of a triangle whose sides are 11, 12, and 13.

11. Use the definitions of the trigonometric functions to express each of the following as a single function:

a. $\sin A \cot A$ c. $\sin A \sec A$

b. $\dfrac{\sin A}{\cos A}$ d. $\csc A \cos A$

12. a. Find the moduli and vector angles for $4 + i$ and $1 + i$.
   b. Find the algebraic product of $4 + i$ and $1 + i$.
   c Find the modulus and vector angle of the product in (b).
   d. Compare the results in (c) with those in (a) and state a hypothesis that they suggest.
   e. Test your hypothesis on $(3 + i)(2 + 3i)$.

13. Consider $a + bi$, where a and b are both positive, θ is the vector angle (argument), and r is the modulus.
   a. Show that $a = r \cos \theta$ and $b = r \sin \theta$.
   b. Use the Pythagorean theorem to show that $\sin^2 \theta + \cos^2 \theta = 1$.

14. Take the acute $\triangle ABC$ and draw the altitude AD.
   a. Express the values of CD, AD, and BD in terms of b, C, and a.
   b. By using the Pythagorean formula in $\triangle ADB$, and results of the preceding exercise, develop the Law of Cosines.

4. REMOVING ANGLE BARRIERS

The reader may be somewhat concerned about the limitation of the trigonometric functions with regard to acute angles. All our conclusions are derived essentially from the acute angles of a right triangle. We have seen that the development of mathematics is marked by the continual broadening of its scope. We witnessed this in our extension of the natural numbers to positive and negative numbers, to rational numbers, to irrational numbers, and to complex numbers. Our original intuitive definitions of positive integral exponents were similarly extended to include all rational

numbers. Now we must raise the issue of trigonometric functions for angles other than the acute, for positive as well as negative angles.

With this objective in mind, let us formalize a bit our original definitions by relating the right triangle to a coordinate framework. The expansion of the trigonometric system will be guided by the same principles of consistency that have marked the other cases of extension. No new definition will violate any previously determined conclusion.

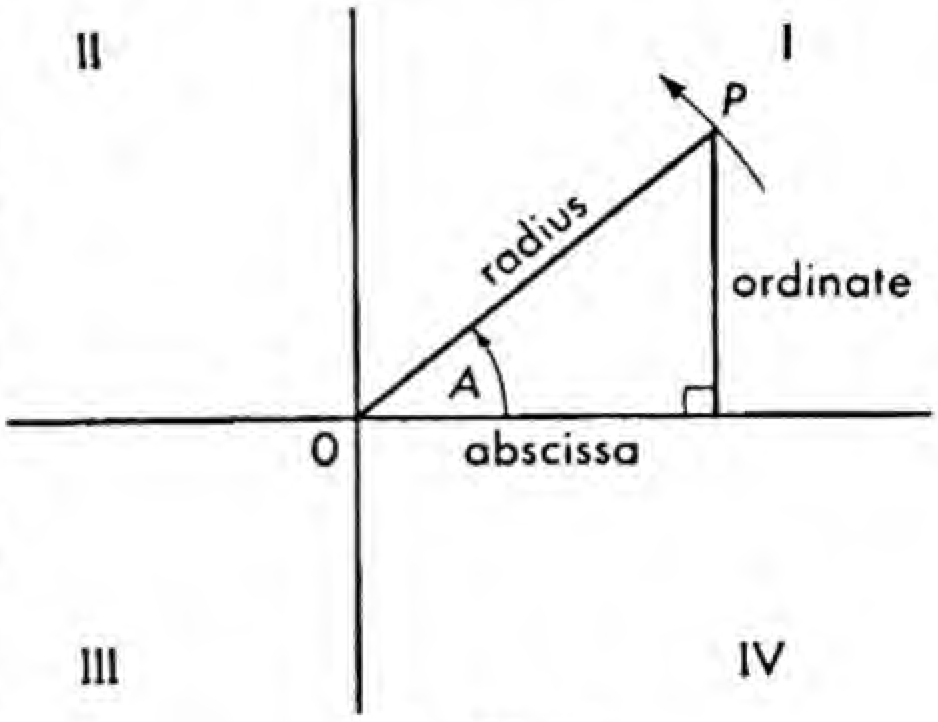


Fig. III–25

We shall refer to the four parts of the plane, formed by the axes, as the four *quadrants*, numbered counterclockwise in Roman numerals as indicated. As with the natural numbers, after the introduction of the signed numbers, we associate a positive aspect to all the quantities in the original triangle in the first quadrant, Fig. III–25. Indeed the abscissa and ordinate are already positive by previous commitment. The line OP will be called the *radius*. The direction of increase of angle A will be in the order of the quadrants, and thus a *positive angle* will have a counterclockwise orientation from the X-axis.

In this new context, we reformulate our earlier definitions:

$$\sin A = \frac{\text{ordinate}}{\text{radius}} \qquad \cos A = \frac{\text{abscissa}}{\text{radius}} \qquad \tan A = \frac{\text{ordinate}}{\text{abscissa}}$$

and so forth.

Let OP be rotated so that P is in the second quadrant, Fig. III–26. In the process of rotation, the radius undergoes no change in length. The ordinate increases somewhat and then begins to decrease. The abscissa decreases to 0 and then increases in the opposite direction. Consequently, in conformity with past decisions in quadrant two as well as in view of the concept of the negative quantity, we must consider the abscissa negative. The other two lines remain as positive.

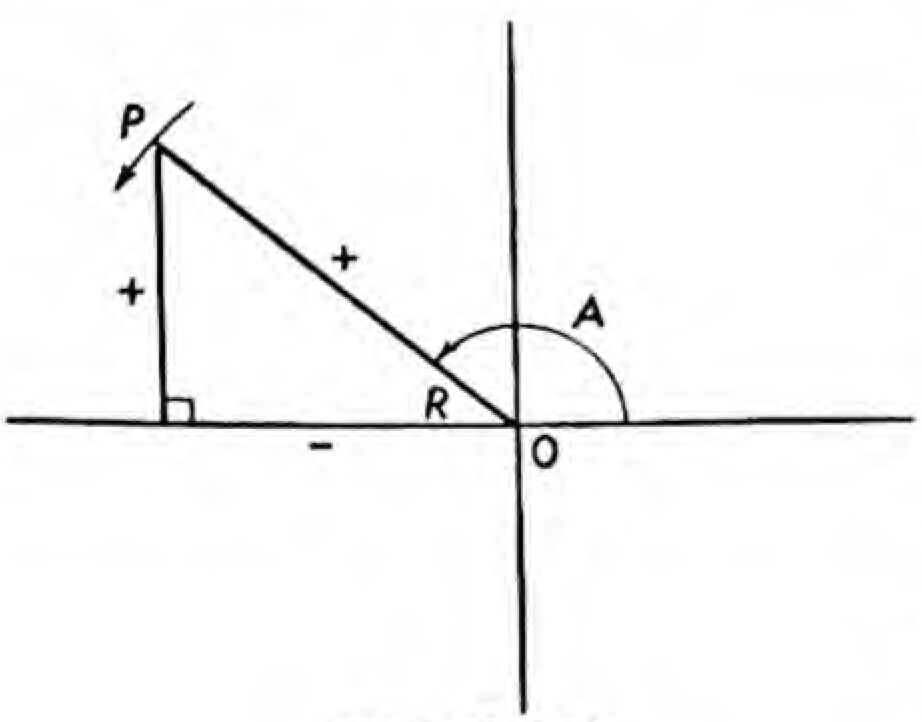


Fig. III–26

The $\angle A$ is no longer an angle of the triangle. Rather it is the supplement of $\angle R$ which lies adjacent to it and which is in the triangle. $\angle R$ is called the **reference angle,** and we know that $\angle R = 180° - \angle A$. Now we must take the bold step. The sin A, as well as all the other functions of A, will have the same definition as those in the first quadrant. It will be (for sin A) ordinate/radius, even though $\angle A$ lies outside the triangle. This is the kind

of mathematical stubbornness that has been so rewarding in the past and is likely to continue to be rewarding.

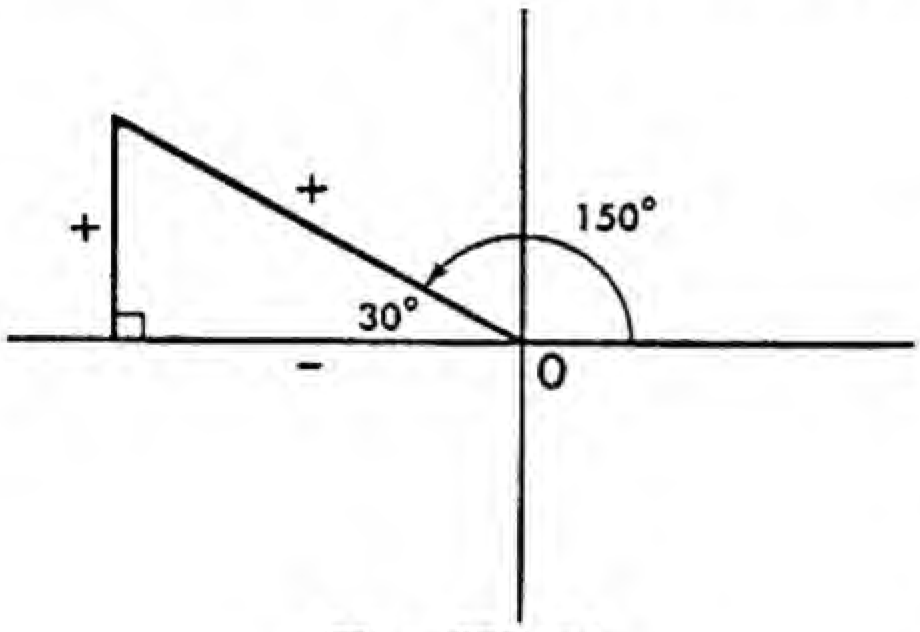


Fig. III–27

In effect this means that the functions of $\angle A$ are the same as those for $\angle R$, which is acute and which we have taken care of earlier. The only difference is that some function values will be negative. The cosine, the tangent, and their reciprocals are negative because of the negative abscissa involved. The other function values of $\angle A$ are positive. We say then that

Trigonometric function of $\angle A$ = $\pm$ trigonometric function of $\angle R$

or $\quad |\text{Trigonometric function } \angle A| = |\text{trigonometric function } \angle R|$

Thus, illustratively (see Fig. III–27),

$$\sin 150^\circ = \sin 30^\circ$$
$$\tan 150^\circ = -\tan 30^\circ$$

If the rotation is continued into the third quadrant (Fig. III–28), the ordinate this time decreases to 0 and then reappears on the opposite side of the X-axis. The ordinate will have to be taken as a negative line. The abscissa, while varying in length, stays in the same place and so remains negative. The radius length does not change at all and is kept positive. The reference triangle is now situated as shown, and the reference angle R represents $\angle A$'s excess over 180°; $R = A - 180°$. It is seen that only the tangent and cotangent are positive in this quadrant. We define again:

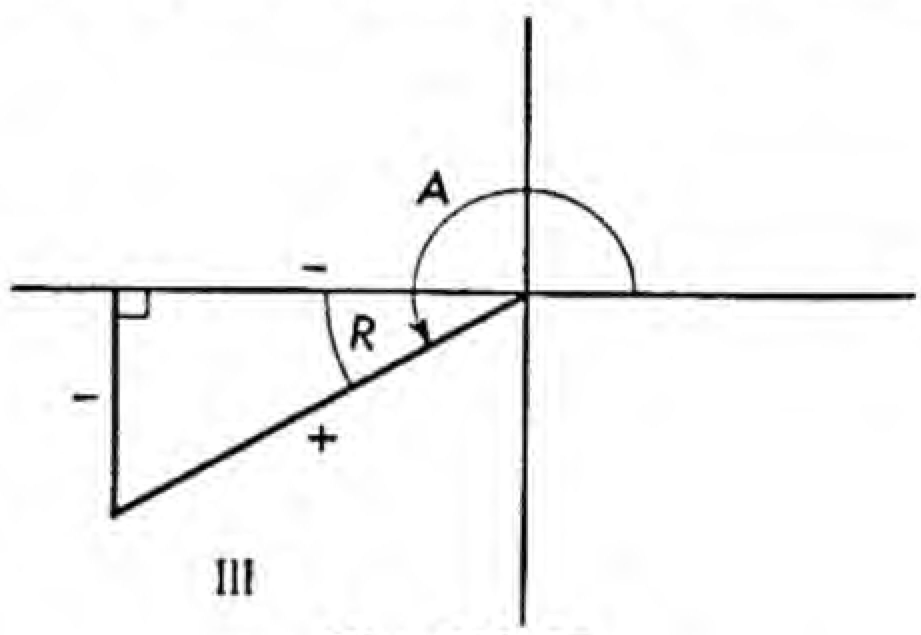


Fig. III–28

$$|\text{Trigonometric function } \angle A| = |\text{trigonometric function } \angle R|$$

Thus

$\sin 220 = -\sin 40 \quad (220 - 180 = 40)$
$\tan 220 = \tan 40$
$\sec 220 = -\sec 40 \quad$ and so forth

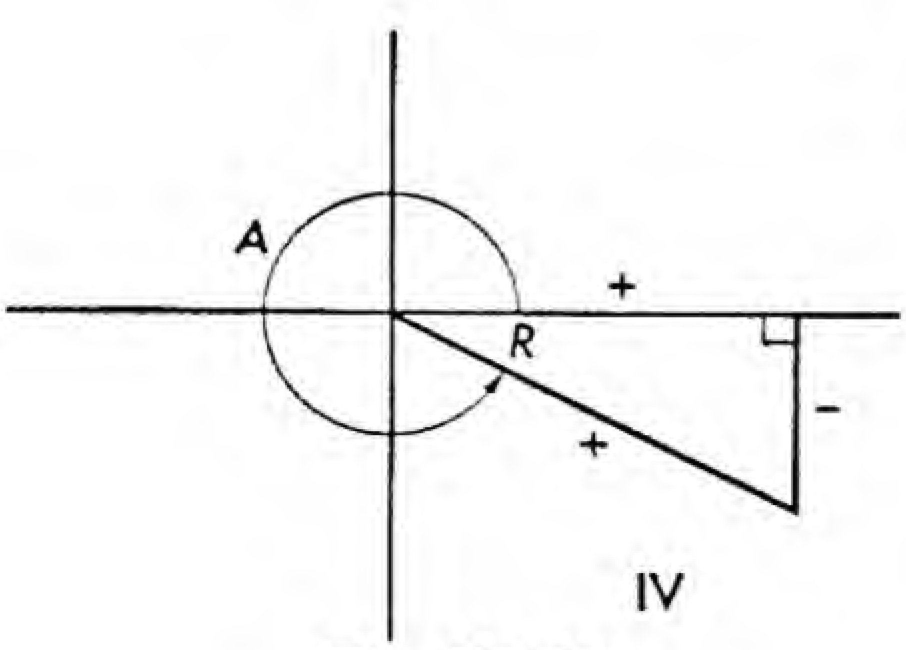


Fig. III–29

Finally, when we arrive in the fourth quadrant (Fig. III–29) by continued rotation, we see that $\angle R = 360° - \angle A$ and that only the cosine and secant are positive. Again, the same definition is found to be descriptive of our intentions:

$$|\text{Trigonometric function } \angle A| = |\text{trigonometric function } \angle R|$$

$$\begin{aligned} \cos 310 &= \cos 50 \\ \cot 310 &= -\cot 50 \\ \sin 310 &= -\sin 50 \end{aligned}$$

Rotations beyond 360° return the radius to the same quadrants. Consistency demands that the definitions be maintained irrespective of the number of rotations about O. Thus,

$$\begin{aligned} \sin 420° &= \sin 60° \qquad (420° \text{ is in I}; R = 60°) \\ \cos 515° &= -\cos 25° \qquad (515° \text{ is in II}; R = 25°) \end{aligned}$$

The negative angle, consistency demands, has the opposite sense, an opposite direction of rotation, of the positive angle. That is, the negative angle starts as 0° at the positive end of the X-axis, as before, and is formed by clockwise rotation. We enter the quadrants in reverse order. But, the definitions for each quadrant have been set, and they are maintained irrespective of the mode of arrival. To do otherwise would be to harbor ambiguity and inconsistency. Thus an angle of $-130°$ yields the same reference angle and the same reference triangle as the angle of 230°, or of 590° for that matter. In each case, we are in quadrant III (Fig. III–30), with $R = 50°$. So,

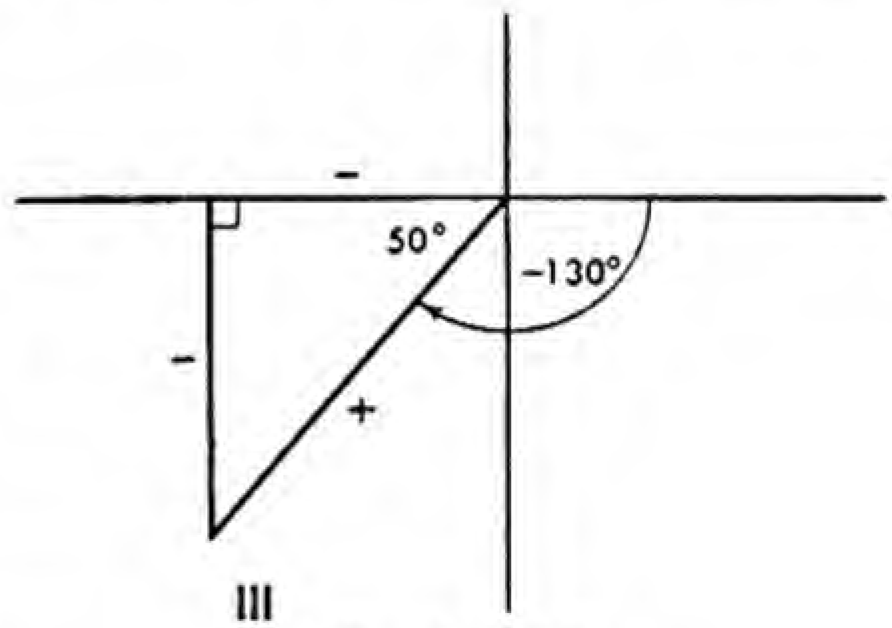


Fig. III–30

$$\begin{aligned} \sin(-130) &= -\sin 50 \\ \tan(-130) &= \tan 50 \end{aligned}$$

Similarly

$$\begin{aligned} \cos(-315) &= \cos 45 \\ \cot(-190) &= -\cot 10 \end{aligned}$$

It should be emphasized that the reference angle is always positive because it is identified with an acute angle in the original right triangle, where all quantities were taken as positive at the start.

We have not quite extended the trigonometric functions to all possible real values. Unfortunately there are periodic gaps at every 90° interval, starting with 0°. At 0°, 90°, 180°, 270°, 360°, and so forth, our right triangles actually disappear. We are left with the choice of allowing the gaps (leaving us with **discontinuous** functions) or making additional definitions, if possible, to close the gaps and thereby create **continuous functions.**

Needless to say, the latter alternative is the one we have been following and will continue to follow. However, we shall need for this some notions

about **limits**, which we can treat only informally at this point (see Fig. III–31).

Suppose that $\angle A$ is permitted to *approach* 0°, getting closer and closer to 0° and ultimately closer to 0° than any previously assigned value, no

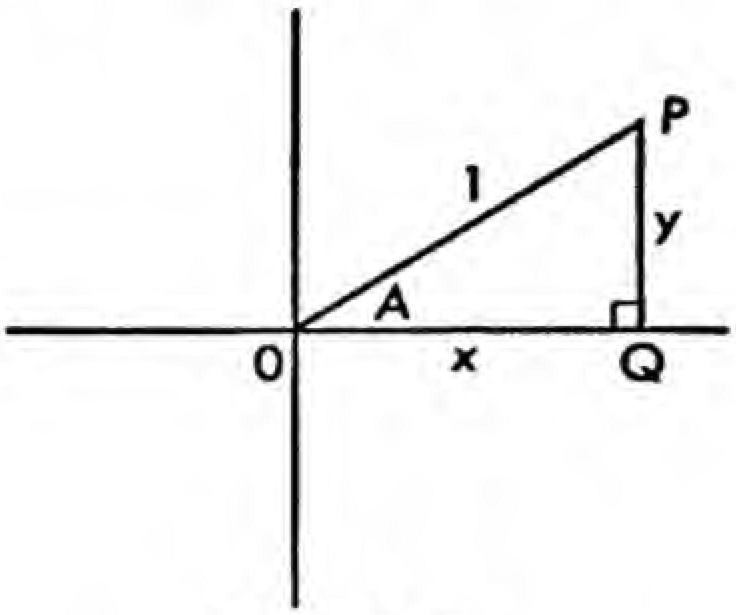


Fig. III–31

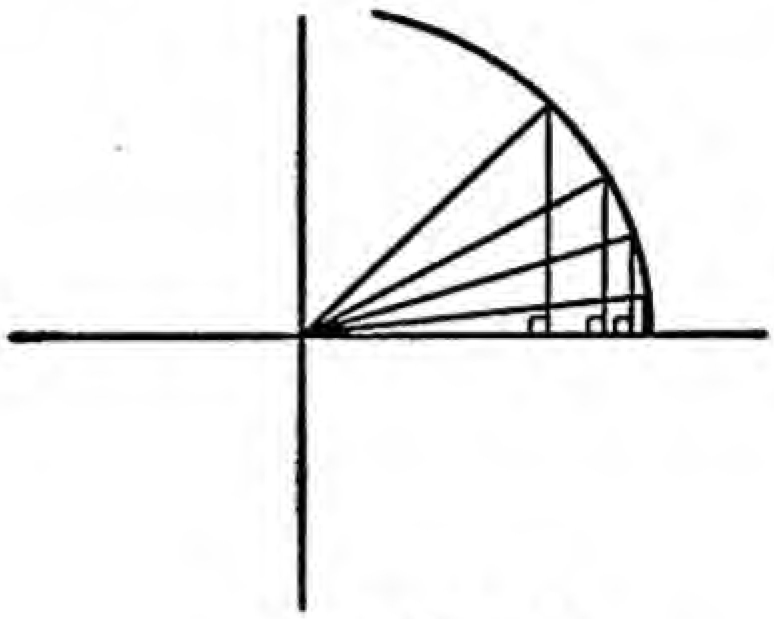


Fig. III–32

matter how small. If we let $OP = 1$, merely for convenience, x will approach the value 1 and get closer and closer to it as $\angle A$ approaches 0° (Fig. III–32). The value of x, ultimately, will differ from 1 also by less than any previously assigned value, no matter how small. Similarly the value of y will approach 0. These thoughts are effectively symbolized as follows:

$$\lim_{A\to 0} x = 1 \qquad \text{and} \qquad \lim_{A\to 0} y = 0$$

The first is read as *the limit of x is 1 as A approaches zero.*

These notions may be conveniently summarized in a diagram (Fig. III–33) with the understanding, of course, that the triangle is far from having reached its limiting position. Thus,

$$\lim_{A\to 0} \sin A = \lim_{A\to 0} \frac{y}{1} = \frac{0}{1} = 0$$

For brevity, we merely write the *definition* that

$$\sin 0 = 0$$

Similarly, $\cos 0° = 1$, $\tan 0 = 0$ and $\sec 0 = 1$. These are read from the diagrammatic aid in the same way as for any other angle. There is but one hitch which arises when we seek cot 0 and csc 0, both of which lead to $1/0$. We know that this is meaningless because of the division by zero. However, what we are concerned with is not actually $1/y$ when $y = 0$, but rather the limit of $1/y$ *as* y *approaches zero.*

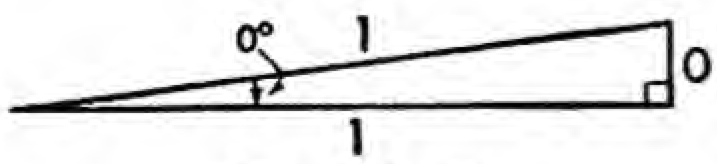


Fig. III–33

Let us consider a number of values for y that are successively closer to 0. We may begin with 0.1, 0.001, 0.000001, and so forth. Then,

$$\frac{1}{0.1} = 10 \qquad \text{or} \qquad \frac{1}{10^{-1}} = 10$$

$$\frac{1}{0.001} = 1{,}000 \qquad \text{or} \qquad \frac{1}{10^{-3}} = 10^3$$

$$\frac{1}{0.000001} = 1{,}000{,}000 \qquad \text{or} \qquad \frac{1}{10^{-6}} = 10^6$$

$$\frac{1}{0.000000001} = 1{,}000{,}000{,}000 \qquad \text{or} \qquad \frac{1}{10^{-9}} = 10^9$$

In general, we picture the above as

$$\frac{1}{10^{-n}} = 10^n$$

As n gets increasingly larger, 10^{-n} gets continually smaller, and 10^n gets larger and larger. *It is possible to find a value n' so that for all n larger than n', 10^n becomes and remains larger than any previously assigned number* N, *no matter how large.* We shall take this remark in quotes to be, in essence, the definition of the *mathematical infinite.* Attention must be called to the fact that the definition of infinity (∞) describes a condition and does not assign a numerical value. Infinity is not a number and cannot be treated as we treat numbers. Returning to the initial problems, we now write

$$\lim_{y \to 0} \frac{1}{y} = \infty$$

With this we complete the list of definitions of the functions of 0° by adding **csc 0° = ∞** and **cot 0° = ∞**.

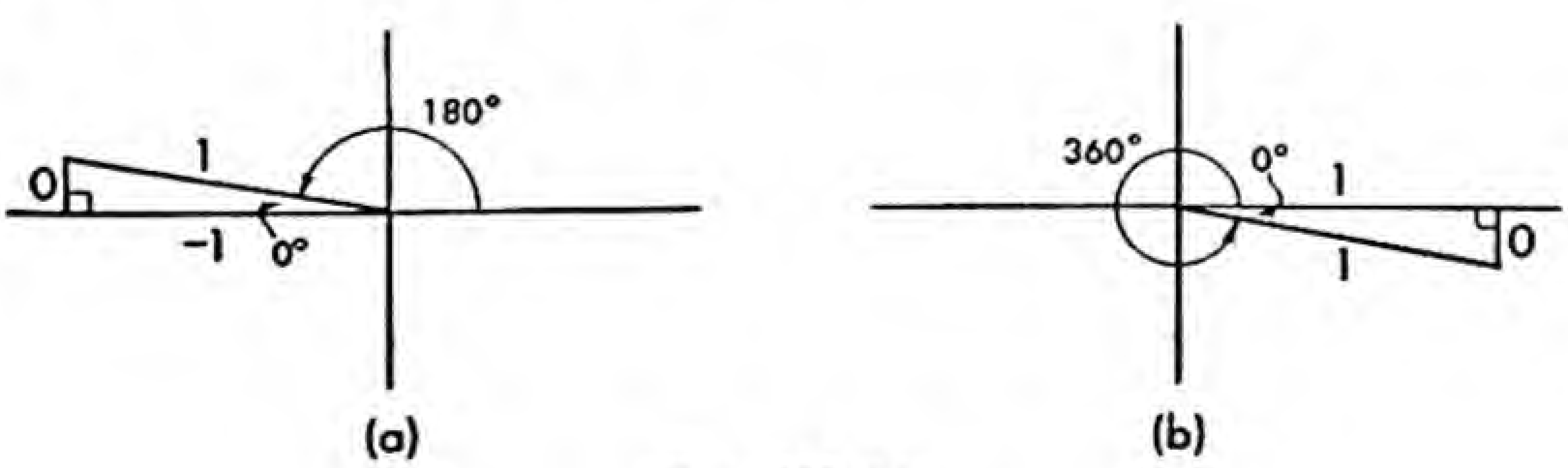


Fig. III–34

With due attention to signs, these definitions embrace those for 180°

and 360°, whose references angles are 0° (see Fig. III–34). We illustrate these cases:

$$\begin{aligned} \tan 180 &= 0 \\ \cos 180 &= -1 \\ \sin 360 &= 0 \\ \sec 360 &= 1 \end{aligned}$$

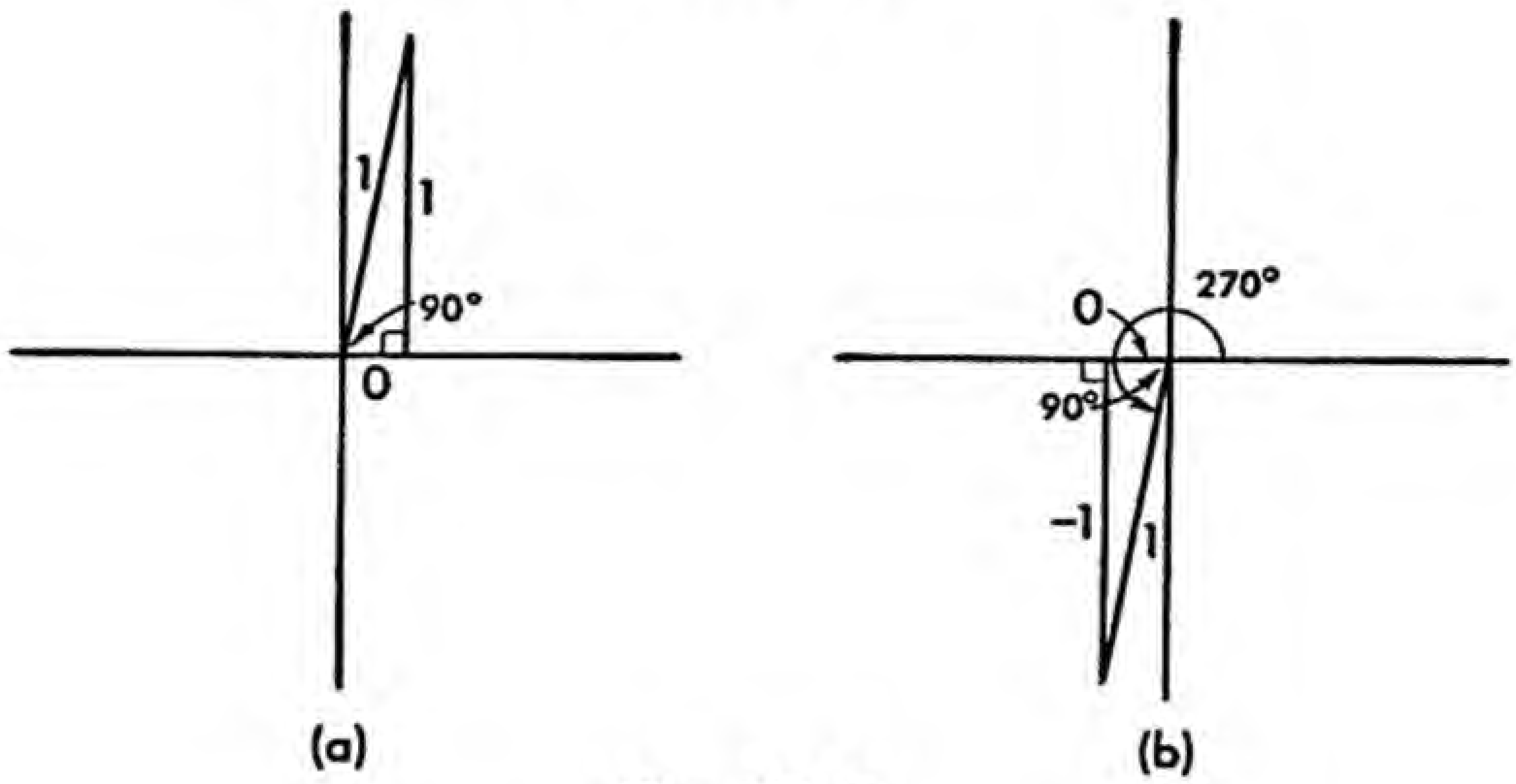


Fig. III–35

Precisely the same kind of thinking is brought to bear with regard to the definitions for the 90° and 270° functions. The diagrams (Fig. III–35) indicate the limiting conditions. We read

$$\begin{aligned} \sin 90 &= 1 \\ \cos 90 &= 0 \\ \tan 90 &= \infty \\ \sin 270 &= -1 \\ \cos 270 &= 0 \end{aligned}$$

In connection with graphs, we will shortly take another look at these infinities, giving attention to signs.

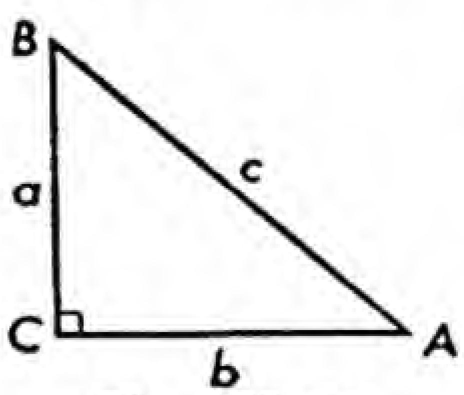


Fig. III–36

If consistency and logic have been maintained successfully throughout this extension, we should expect relevant conclusions to hold up under the new circumstances. At this present moment we have two laws, the Law of Sines and the Law of Cosines, both derived for the acute angle triangle. What is the status of these laws if the triangle is a right triangle (Fig. III–36) or an obtuse triangle?

Consider

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

When $C = 90°$, $\sin C$ becomes 1, and we have

$$\frac{a}{\sin A} = \frac{c}{1}$$

or

$$\sin A = \frac{a}{c}$$

which is in agreement with the original definition for the right triangle.

Consider too, $c^2 = a^2 + b^2 - 2ab \cos C$. If $\angle C = 90°$, $\cos 90° = 0$, and so we have $c^2 = a^2 + b^2$, which is, of course, the Pythagorean formula for the right triangle.

Similarly, if the triangle is obtuse (Fig. III-37), both laws retain their form and validity. For the Law of Sines, $\sin C$ is still h/a, and so the development of the law is identical with what took place earlier. For the Law of Cosines, $AD = b + x$, where it was $b - x$ earlier. This change of sign is rectified when we use the $\cos C$, which is negative now, $-x/c$.

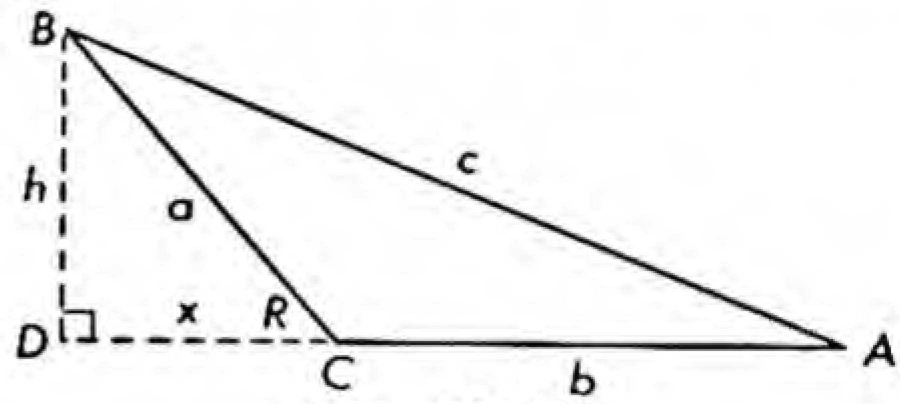


Fig. III-37

EXERCISES (III-4)

1. Express the following as functions of the reference angle:

a. sin 168	f. cot 475
b. tan 247	g. cos (−42)
c. sec 153	h. tan (−164)
d. cos 310	i. cot (−205)
e. tan 284	j. csc (−312)

2. Express the following in terms of the reference angle:

a. $\sin x$ if $90 < x < 180$
b. $\cot x$ if $270 < x < 360$
c. $\cos x$ if $180 < x < 270$

3. Find the numerical values (answers may be left in radical form):

a. cos 120	d. cot 270	g. csc (−150)
b. tan 180	e. sin 330	h. tan (−270)
c. sec 135	f. sin. (−90)	i. sin (−45)

4. Complete the proof of the Law of Sines for the obtuse triangle.
5. Complete the proof of the Law of Cosines for the obtuse case.

6. If in $\triangle ABC$, $C = 142°$, $A = 21°$, and $b = 31'$, find c.
7. Find the largest angle of the triangle whose sides are 4, 7, and 10.
8. From a point 84 feet above the ground, the angles of depression to two objects on the ground are 29° and 34°. The point of observation and the two points A and B are all in the same vertical plane. Find the distance between A and B (two cases).
9. Find the modulus and the argument of the vector given by the complex number:

a. $-2 + 3i$ b. $5 - 4i$

5. TRIGONOMETRIC IDENTITIES

It was mentioned earlier that the six trigonometric functions are interrelated. This must be true because all six refer ultimately to only three sides of a right triangle.

Since $\sin A = a/c$, $\cos A = b/c$, and $\tan A = a/b$ in the right triangle ABC, and since

$$\frac{a}{b} = \frac{a}{c} \div \frac{b}{c}$$

we have, by substitution,

$$\tan A = \frac{\sin A}{\cos A}$$

Similarly it can be shown that

$$\cot A = \frac{\cos A}{\sin A}$$

Suppose now that A is obtuse, and so $A = 180° - R$, where R, as usual, is the reference angle. The $\tan A$ relation becomes

$$-\tan R = \frac{\sin R}{-\cos R}$$

and so

$$\tan R = \frac{\sin R}{\cos R}$$

In similar fashion we can show that the formula holds for any real value of A. In this sense the angle A is a place holder, and we can indicate the great generality of the formula by using an asterisk in the last formula.

$$\tan * = \frac{\sin *}{\cos *}$$

For this reason, this and the other relations are referred to as "identities," connoting an equality that is true for all permissible real values of the

variable. Ultimately even the complex number is included in the trigonometric fold.

Returning to the right triangle (Fig. III–38), we recall that it is possible to use angle B for the angle of our functions. We have $\sin B = b/c$, $\cos B = a/c$, $\tan B = b/a$, and so forth. Again, since the same sides of the triangle are being used, we can expect further identities. Comparison of results for A and B indicates that

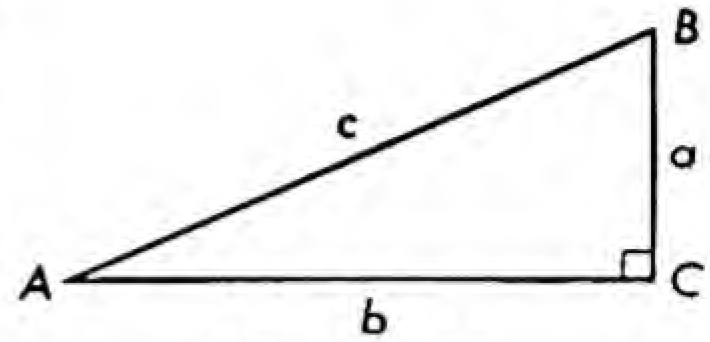


Fig. III–38

$$\sin A = \cos B = \frac{a}{c}$$

$$\tan A = \cot B = \frac{a}{b}$$

$$\sec A = \csc B = \frac{c}{b}$$

If we note that angles A and B are complementary and that cos, cot, and csc are abbreviations for *complementary sine*, *complementary tangent*, and *complementary secant*, we see the reason for the assignment of the names in the manner indicated.

We observe, in brief, that *any function of one acute angle is equal to the cofunction of the complementary angle*. Concretely, this means that $\sin 27 = \cos 63$, $\cot 46 = \tan 44$, and so forth.

Other relations among the functions may be derived from the Pythagorean formula, $x^2 + y^2 = r^2$, where x, y, and r represent the abscissa, ordinate, and modulus in any quadrant. This is a formula about the sides, taken separately. If this statement can be recast so that instead it contains ratios of the sides, it will be possible to convert the algebraic formula to a trigonometric formula.

This objective may be realized with ease, for we can divide both sides of an equation by any quantity other than 0. In fact this can be done here in three ways. All we need do is to divide by r^2, y^2, or x^2, one at a time. From

$$x^2 + y^2 = r^2$$

we get $\quad \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1 \quad$ or $\quad \sin^2 A + \cos^2 A = 1$

also $\quad \dfrac{x^2}{y^2} + 1 = \dfrac{r^2}{y^2} \quad$ or $\quad 1 + \tan^2 A = \sec^2 A$

and $\quad 1 + \dfrac{y^2}{x^2} = \dfrac{r^2}{x^2} \quad$ or $\quad 1 + \cot^2 A = \csc^2 A$

We note the introduction of a new convention, which the reader may have developed for himself earlier in connection with a query in exercise 4,

Art. 1. (In fact, all three identities should have been discovered then.) This is that $\sin^2 A = (\sin A)^2$.

It may be anticipated that because of the interrelations of the functions, any two of the three new formulas may be derived from the third. Rather than do that, let us take a single function and show how, by valid substitutions, one can develop identities and even chains of identities.

a. $\tan x = \dfrac{\sin x}{\cos x} = \sin x\left(\dfrac{1}{\cos x}\right) = \dfrac{1}{\csc x}(\sec x) = \dfrac{\sec x}{\csc x}$

b. $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{\pm\sqrt{\cos^2 x}} = \dfrac{1}{\pm\sqrt{1 - \sin^2 x}}$

c. $\sin x = \cos x \cdot \dfrac{\sin x}{\cos x} = \dfrac{1}{\sec x} \cdot \dfrac{1}{\cot x} = \dfrac{1}{\sec x \cot x}$

It is not always possible, even in the simpler cases of (a), (b), and (c), to recognize a trigonometric identity as such. When such situations arise or are suspected, a fairly definite technique of decision is available. Consider the following proof of an identity:

$$\frac{1 + \sec x}{\csc x} = \sin x + \tan x \qquad (\csc x \neq 0)$$

$$\frac{1}{\csc x} + \frac{\sec x}{\csc x} = \sin x + \tan x \qquad \text{(distributive postulate)}$$

$$\sin x + \frac{1/\cos x}{1/\sin x} = \sin x + \tan x \qquad \text{(substitution of reciprocals)}$$

$$\sin x + \frac{\sin x}{\cos x} = \sin x + \tan x \qquad \text{(simplify fraction)}$$

$$\sin x + \tan x = \sin x + \tan x \qquad (\text{substitution, } \cos x \neq 0)$$

That an identity exists is demonstrated by the reduction of either member, or both, to the same quantity. In achieving this aim, substitutions of known identities can be made. One may also use permissible operations on expressions and fractions. In the illustration the left-hand member was written as two terms, since the right-hand member had that many. This was followed by substitutions that brought out the same function on each side. In the process, one may have to rule out certain values because division by zero is meaningless.

More frequently, the trigonometric equation is likely to be a conditional equation rather than an unconditional one, which is the identity. For example, in $2 \sin x = \csc x$, if an effort is made to show that this is an identity, results should be forthcoming quickly that will indicate otherwise.

Rather than do this, we may be able to find some values for x, for which the equation is true. Thus solving, we have

$$\begin{aligned} 2 \sin x &= \csc x \\ 2 \sin x &= \frac{1}{\sin x} \\ 2 \sin^2 x &= 1 \\ \sin^2 x &= \frac{1}{2} \\ \sin x &= \pm \frac{1}{\sqrt{2}} \end{aligned}$$

If $\sin x = \dfrac{1}{\sqrt{2}}$, $x = 45, 135°$, and if $\sin x = -\dfrac{1}{\sqrt{2}}$, $x = 225°, 315°$.

Thus the equation is true for only specified values of x. As a rule we do not bother with negative angles or angles beyond 360°. Otherwise, many other answers may be indicated. Indeed all we need do in this instance is to describe a 45° reference angle in any of the quadrants.

EXERCISES (III-5)

1. Show that $\cot A = \cos A/\sin A$ for any value of A in the third quadrant.
2. Starting with $\tan A = \sin A/\cos A$, with A acute, replace A by its complementary angle B and derive an identity in terms of B.
3. Express each of the following as functions of angles less than 45°:

a. $\tan 67$	c. $\sin 108$	e. $\csc(-71)$
b. $\cos 84$	d. $\sec 253$	f. $\cot(-306)$

4. Derive the identity $1 + \tan^2 x = \sec^2 x$ from $\sin^2 x + \cos^2 x = 1$.
5. Express each of the following in simplest form, using only the sine and cosine functions:

a. $\cos \theta \tan \theta$	d. $\cot y\sqrt{1 - \cos^2 y}$
b. $\dfrac{\cot x}{\csc x}$	e. $\dfrac{1 + \cot t}{\csc t}$
c. $\sin^2 y(\sec^2 y - 1)$	f. $\sec^2 M + \csc^2 M$

6. Prove the following identities and indicate non-permissible values where they occur:

a. $\dfrac{1 - \cos^2 x}{\sin x} = \dfrac{1}{\csc x}$

b. $\dfrac{\cos \theta}{\sin \theta} + \dfrac{\sin \theta}{1 + \cos \theta} = \dfrac{1 + \sec \theta}{\tan \theta(1 + \cos \theta)}$

c. $\dfrac{\sin^3 y + \sin y \cos^2 y}{\cos y} = \dfrac{1}{\cot y}$

d. $(\sin A + \cos A)^2 = \dfrac{\sec A \csc A + 2}{\sec A \csc A}$

7. Solve the following equations for positive values between 0° and 360° inclusive:

a. $2 \tan x + 3 = \tan x + 4$

b. $\dfrac{3}{\sin x} + 2 = \dfrac{2}{\sin x}$

c. $\tan p = \cot p$

d. $4 \cos h = 3 \cos h$

e. $3 \tan u + 4 = \tan u + 5$

f. $\tan \theta = 5 \cot \theta$

6. GRAPHS OF TRIGONOMETRIC FUNCTIONS

Each trigonometric function has now been defined for every positive and negative real angle. As the angle increases beyond 360°, the same values recur periodically for any function, since we return to the same quadrants. The same will hold if the angles decrease through negative values. The many values and repetitions may be illustrated effectively through **graphs** of the functions.

We start with the sin x. To each value of x there corresponds a specific value of sin x which, for convenience, we call y. Thus we shall be concerned with a graph of

$$y = \sin x$$

The simultaneous occurrences of x and y suggests the utilization of the coordinate system and its geometric image for number pairs (x, y). For convenience, we select our x values in intervals of 30°, and our y values may be rounded off to tenths.

x	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \sin x$	0	0.5	0.9	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0

As we go beyond 360° or below 0° in intervals of 30°, the same values of y appear. The table then suggests a graph of an infinity of discrete points. We know that other points, an infinity of other points, may be interpolated. However, we can get a pretty good idea of the *flow* of values by connecting the points with a smooth curve. To be sure, this creates a number of profound problems which we shall tackle in due course.

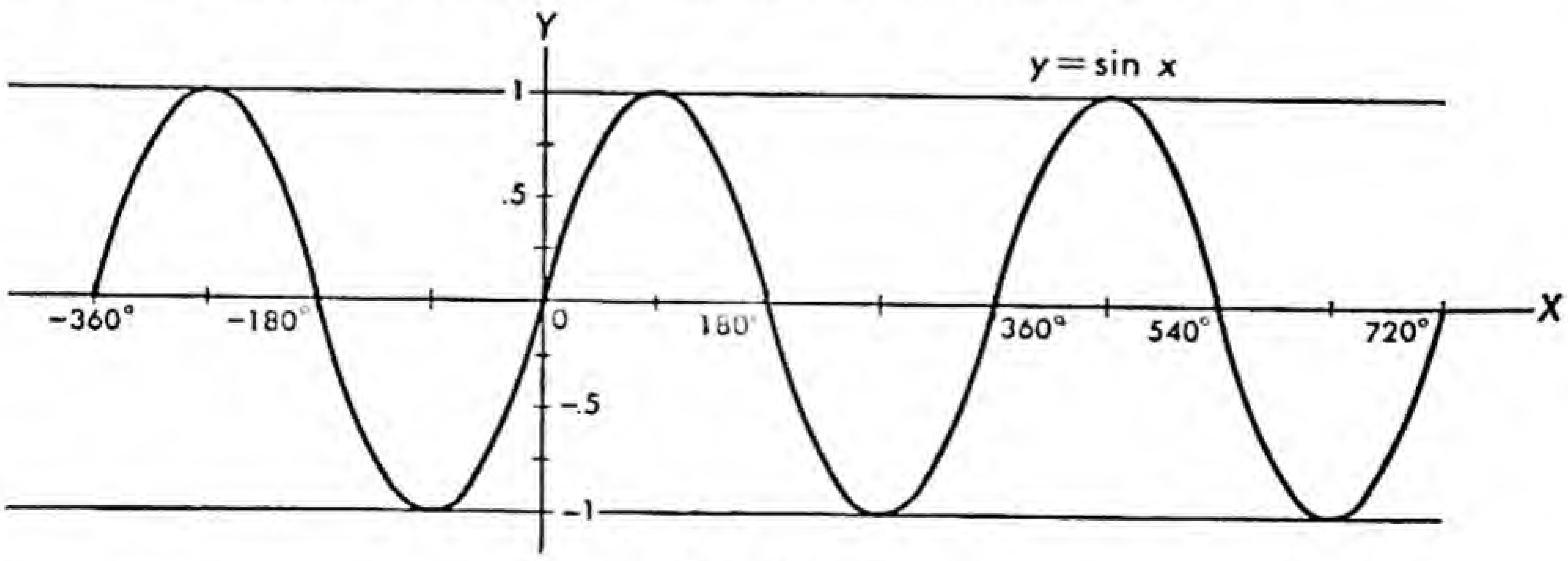


Fig. III–39

The graph (Fig. III–39) demonstrates in visible clarity the *periodicity* of the *sine curve*. The curve fluctuates between a high of $+1$ and a low of -1. The absolute value of either is 1, which is known as the *amplitude* of the curve. The 360° is called the *period* of the curve.

At this point, the reader may need no prompting to consider the graphs of cos x (Fig. III–40) and tan x (Fig. III–41). The tables are combined conveniently.

x	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \cos x$	1	0.9	0.5	0	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1
$y = \tan x$	0	0.6	1.7	$\pm\infty$	-1.7	-0.6	0	0.6	1.7	$\pm\infty$	-1.7	-0.6	0

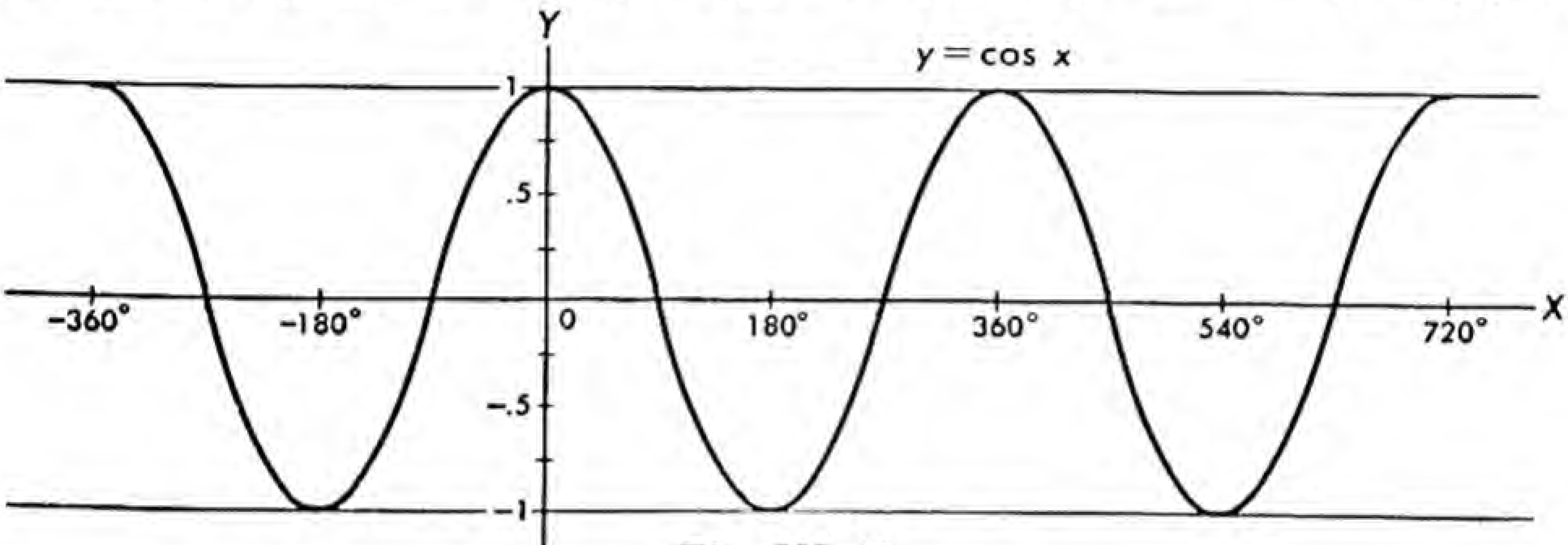


Fig. III–40

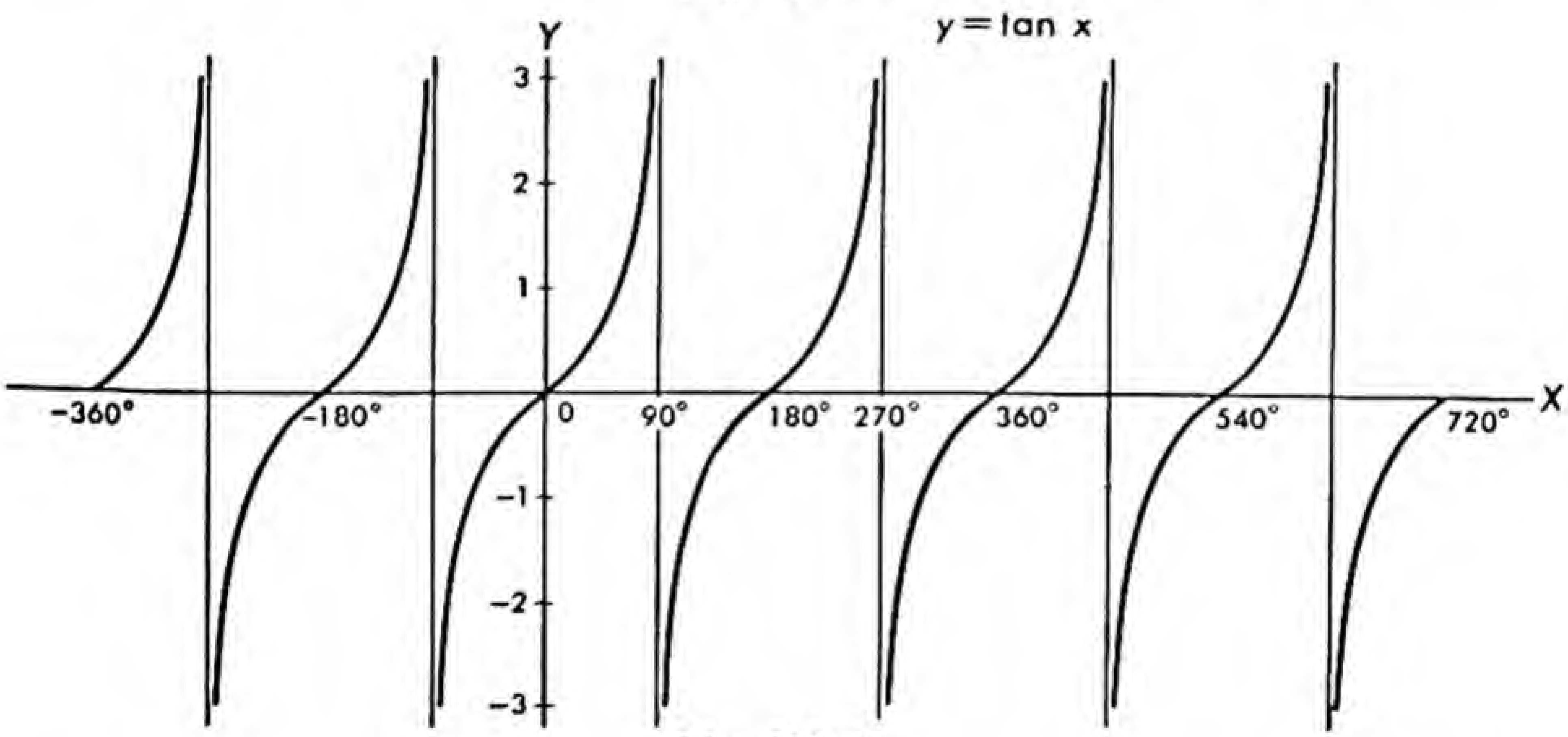


Fig. III–41

We have already seen that the tan x tends toward infinity as x approaches 90°. Of course only the tendency can be suggested in the graph. The curve will get forever nearer to the 90° line but never reach it. This is described by saying that the line is an *asymptote* to the curve. In the second quadrant, where the tangent is negative, tan x will appear to come from "$-\infty$". We can look at 90° as belonging to both the first two quadrants. If we approach it from quadrant I, the tan x (being positive therein) will

approach $+\infty$ at 90°. If we approach 90° from quadrant II (wherein the tangent is negative throughout), we are constrained to consider tan 90° = $-\infty$. Thus our definition for tan 90° should be $\pm\infty$. Of course the same situation prevails at 270°. The result of all this is that we have a *discontinuous curve.*

The amplitude of cos x as well as the sin x is 1. By attaching a coefficient other than 1 to each of these function values, the amplitude may be changed easily. Thus, if we make a table of values for 2 sin x or 2 cos x, each of the y values in the tables will be double those of the corresponding values above. The consequence is that the amplitudes are doubled. So, in general, the amplitudes of $\boldsymbol{a}$ **sin** $\boldsymbol{x}$ and $\boldsymbol{a}$ **cos** $\boldsymbol{x}$ is a.

Well, if the function value can have a coefficient, why not one for the angle itself? Consider, for example, sin $2x$. We know that the period for sin x is 360°, which means that a complete *cycle* or *wave length* is completed in that interval. This is true no matter what the angle is called or labeled. The actual symbol for the angle is, as we noted earlier, a place holder. So, if the symbol for the angle is $2x$, the cycle will still be completed in 360°. That is, $2x = 360°$, and so $x = 180°$. We say, therefore, that the period is 180°. Note that this means that when $x = 180°$, the angle value is 360°. Similarly, if we are dealing with cos $3x$, we let $3x = 360$, and so $x = 120°$, which is the period in this case. In general, to find the period of **sin** $\boldsymbol{bx}$, we let $bx = 360°$, and so $x = 360°/b$, which is the period.

We can illustrate both observations by considering $y = 2 \sin 3x$ (see Fig. III–42). The amplitude is 2 and the period is $360/3 = 120°$. This information, coupled with a mental picture of the sine curve, is adequate for a fairly accurate sketch. If one desires, a few values may serve as a guide in sketching.

x	0	30	60	90	120
$y = 2 \sin 3x$	0	2	0	-2	0

Note: When $x = 30$, $y = 2 \sin 3(30) = 2 \sin 90 = 2(1) = 2$, etc.

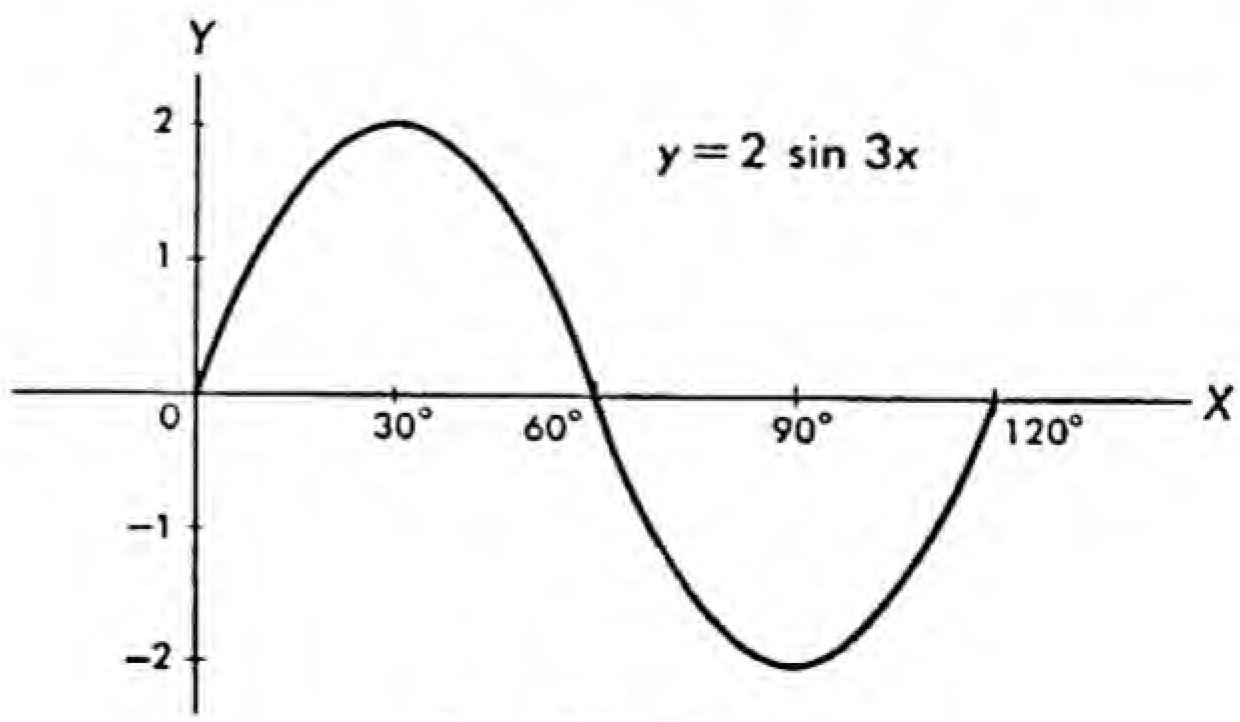


Fig. III–42

Expressions such as $y = 2 \sin x + \sin 3x$ are rather commonplace today as we deal mathematically with light, sound, radio waves, and other related phenomena. We could call y a composite function value. We shall graph this now with the intention of abstracting from the effort a technique that makes quick sketching feasible. For this purpose we show the values of $2 \sin x$ and $\sin 3x$ separately in the table as well as their sum which is to be graphed. In fact, we shall graph (Fig. III–43) all three: $2 \sin x$, $\sin 3x$, and $2 \sin x + \sin 3x$.

x	0	30	60	90	120	150	180	210	240	270	300	330	360
$2 \sin x$	0	1	1.7	2	1.7	1	0	-1	-1.7	-2	-1.7	-1	0
$\sin 3x$	0	1	0	-1	0	1	0	-1	0	1	0	-1	0
y	0	2	1.7	1	1.7	2	0	-2	-1.7	-1	-1.7	-2	0

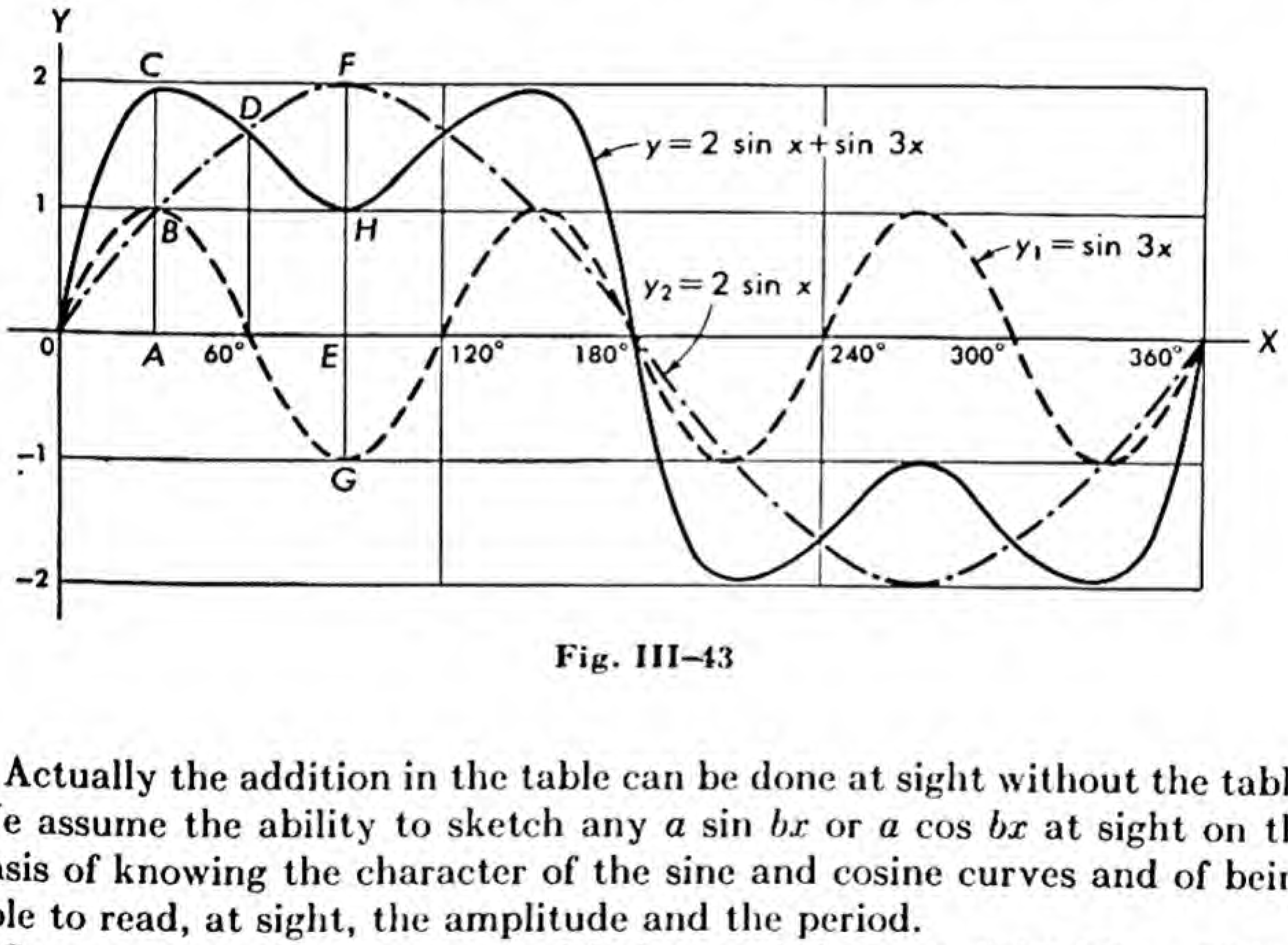


Fig. III–43

Actually the addition in the table can be done at sight without the table. We assume the ability to sketch any $a \sin bx$ or $a \cos bx$ at sight on the basis of knowing the character of the sine and cosine curves and of being able to read, at sight, the amplitude and the period.

Suppose then that we sketched $2 \sin x$ and $\sin 3x$ at sight. The addition on the graph consists merely of adding corresponding ordinates at sight. In fact, one need not even be cognizant of the numerical values. We need only add lines with due attention to their signs of course. We also select the places of addition where we see that the character of the resulting curve requires it.

At 0° both individual graphs are 0, so their sum or composite is 0. At A, one curve has the value AB, and the other has the same value because of

the intersection. So, we sketch AC double that of AB. At 60°, one of the readings is 0; therefore the composite will go through D. At E we have the two values EF and EG. The latter is negative, so we subtract that distance (at sight or with dividers) from FE, starting at F, and getting as a result the line EH. In brief, to the ordinate distances of one curve we add algebraically the ordinate distances of the other to get the graph representing the sum of the two function values. With a little practice this is easily done at sight.

EXERCISES (III-6)

1. Graph each of the following:

a. $y = \sin x$
b. $y = \cos x$
c. $y = \tan x$
d. $y = \csc x$
e. $y = \sec x$
f. $y = \cot x$

2. To which curves in exercise 1 may the concept amplitude not be properly applied?

3. State the amplitude and period of each of the following:

a. $y = \sin 4x$
b. $y = \frac{1}{2} \sin 3x$
c. $y = 3 \sin \frac{1}{2}x$
d. $y = 4 \cos 2x$
e. $y = \frac{1}{2} \cos \frac{1}{2}x$
f. $y = 0.2 \sin 200x$

4. Write the equations for each of the following:

a. Sine curve with amplitude 2 and period 60°.
b. Cosine curve with amplitude $\frac{1}{4}$ and period 45°.

5. Sketch curves for exercise 3.
6. Sketch (a) $y = \sin 4x + 4 \cos 2x$; (b) $y = 3 \sin \frac{1}{2}x + 2 \cos x$.

7. A NEW ANGLE UNIT

The 360° system of angle measurement, the *sexagesimal system*, is due to be supplemented with another. Basically the present system lacks an intimate relation with a linear measure and therefore with the real number system, although for practical purposes it is quite adequate.

An angle measure, adopted from another viewpoint, turns out to be a matter of worthwhile simplicity in analytical studies, as we shall see in the calculus. Consider a right triangle (Fig. III–44), with the hypotenuse equal to 1. The value of $\sin x$ is $BC : AB$, which is just BC. We shall see later that a very important ratio, $\sin x/x$, approaches 1 as x approaches 0, provided x is measured in the new unit of *radians*. The simplicity of this conclusion is indicative of certain advantages that will accrue from the use of radians.

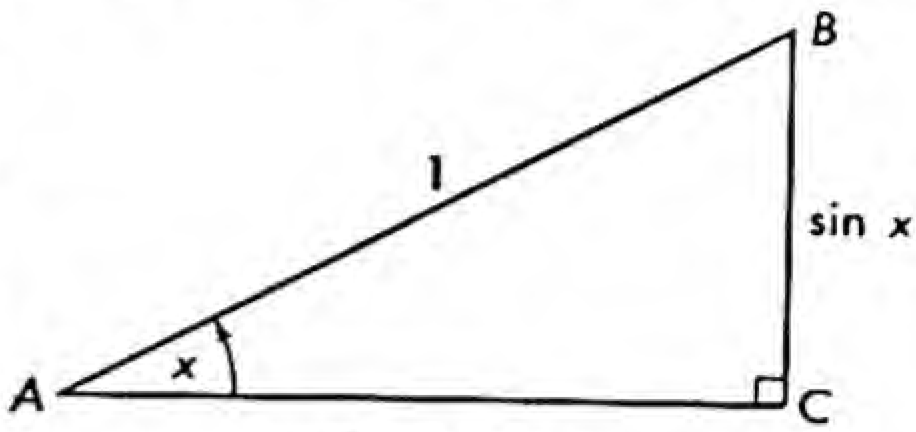


Fig. III–44

To present a definition of radians, we need to anticipate some information about a circle (Fig. III–45), specifically that its circumference is $2\pi r$, where r is the radius and π (pi) is an irrational constant equal to $3.14159265\cdots$. Also, by a *central angle*, we shall mean an angle whose vertex is at the center of the circle and whose sides are radii. If an arc AB (written $\widehat{AB}$) is equal in length to the radius, we say that the central angle has the measure of *1 radian.*

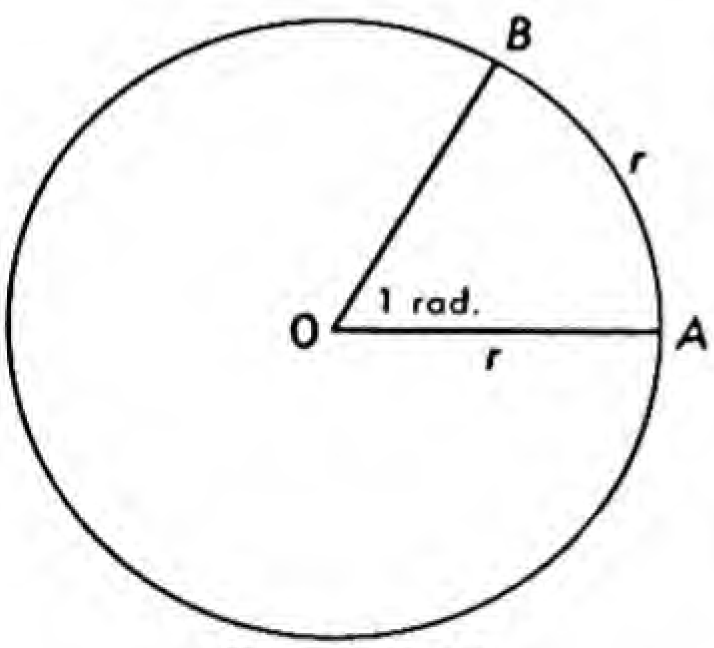


Fig. III–45

Of course it is desirable to know what relationship the radian bears to the degree. Since the circumference is equal to 2π times the radius, and the arc AB is one radius in length, there are 2π radians in the full 360° around $\angle O$. That is,

$$2\pi \text{ radians} = 360°$$

and so

$$\pi \text{ radians} = 180°$$

By dividing both sides of this equation either by π or by 180, we get the following relations, respectively:

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

Using 3.14 as an approximation for π, we find from the foregoing relations that $1° \approx 0.017$ radian and that 1 radian $\approx 57°$. However, in many situations where the answer can be left in radians, it may be expressed in terms of π. As a result we shall soon become familiar with such frequent equivalents as:

$$30° = \frac{\pi}{6} \text{ rad.,} \quad 60° = \frac{\pi}{3} \text{ rad.,} \quad 45° = \frac{\pi}{4} \text{ rad.,} \quad 90° = \frac{\pi}{2} \text{ rad.} \quad \text{etc.}$$

We sketch another composite curve (Fig. III–46) with the abscissas in units of radians.

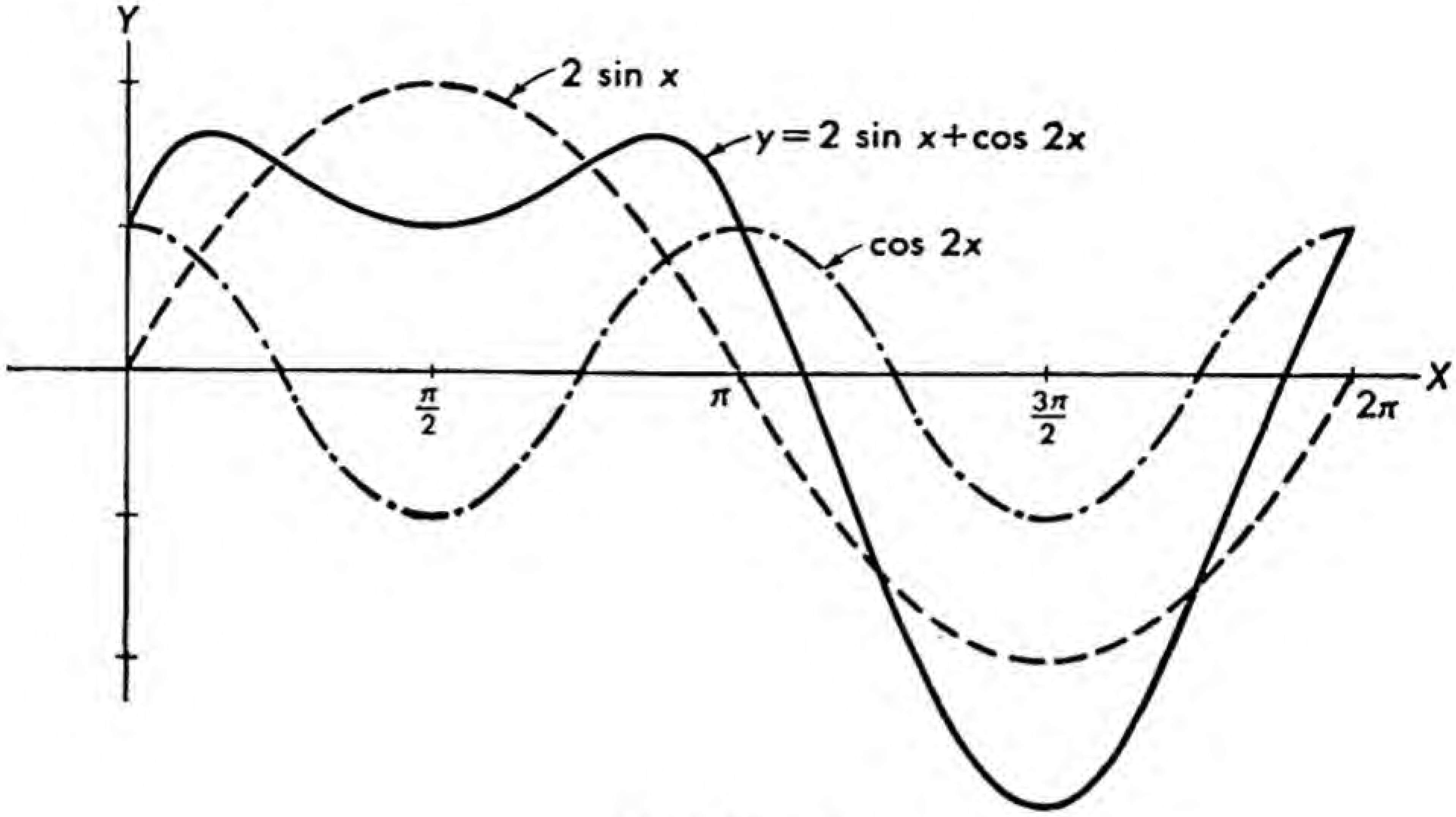


Fig. III–46

EXERCISES (III–7)

1. Change the following degree measures to radians:

   a. 60, 120, 45, 30, 150.
   b. 225, 90, 270, 20, 105.

2. Change the following radian measures to degrees:

   a. $\frac{3\pi}{4}$ b. $\frac{5\pi}{6}$ c. 3π d. $\frac{4\pi}{3}$ e. 1.2π

3. a. Find the length of an arc on a circle of 3 inch radius that is intercepted by a central angle of 2 radians; $1\frac{1}{4}$ radians.
   b. If s is the length of an arc, r the radius of the circle, and θ the number of radians in the central angle of the arc, write a formula relating s, r, and θ.

4. In a unit circle, one which has a radius equal to 1, we have $s = \theta$, according to exercise 3(b). This suggests the intimate relation between a length on an arc and an angle. The angle in radian measure has the same numerical value as an arc of this unit circle.
   a. How many radians has a central angle of the unit circle (radius, 1 inch), if it intercepts an arc of $\pi/4$ inch; 1.7 inches?
   b. How many inches in an arc of the same unit circle that is intercepted by a central angle of $\pi/8$ radian; 2.7 radians?

5. Find the numerical values of the following:

a. $\sin \frac{2}{3}\pi$ b. $\cos \frac{3\pi}{2}$ c. $\tan \frac{5\pi}{4}$ d. $\tan \frac{\pi}{5}$

6. Sketch the following; Alternate between degree and radian measure for the abscissas in the examples:

a. $y = 2 \sin x$
b. $y = 3 \cos \frac{1}{2}x$
c. $y = \tan 2x$
d. $y = \sec x$
e. $y = \csc x$
f. $y = \cot x$
g. $y = 2 \sin x + 3 \cos \frac{1}{2}x$
h. $y = 2 \sin x + \frac{1}{2} \cos 3x$
i. $y = \sin x + \cos x$
j. $y = \sin 2x + 2 \cos 3x$
k. $y = \sin x - \cos x$
l. $y = \cos x - \sin x$
m. $y = \sin x (\sin x) = \sin^2 x$

8. SOME NEW IDENTITIES

A comparison of the sin x and cos x curves indicates that if one or the other were shifted *translated* 90° or $\pi/2$ radian, the curves would coincide with each other. This means that a point on one curve corresponds to a point 90° ahead of it or behind it on the other. Thus,

$$\sin x = \cos\left(x - \frac{\pi}{2}\right) \quad \text{and} \quad \cos x = \sin\left(\frac{\pi}{2} + x\right).$$

We have also seen that the sine and cosine have equal values for complementary angles. That is,

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

The accumulation of instances such as these gives rise to the need to examine function values with angles that are binomial expressions. We should consider, in general, angles that are expressed as $x + y$ and $x - y$. Simple numerical substitutions show immediately that $\sin (x + y) \neq \sin x + \sin y$. The symbol *sine* is not to be confused with a numerical value such as "a", where $a(b + c) = ab + ac$ in accordance with the distributive postulate. *Sine* x is a numerical value, and if this were the multiplier of a binomial, it would be distributed as in

$$\sin x (\sin x + \cos x) = \sin^2 x + \sin x \cos x$$

In brief, sin x is a function value, while sine represents the function which refers to the pairs of values $\{x, \sin x\}$. It remains to determine whether

$\sin(x + y)$ can be expressed somehow as function values in x and y. To reach a decision, we shall go back to our basic definitions. For this we need three right triangles to contain separately each of the angles x, y, and $x + y$. For convenience we take x and y adjacent to each other, so that their sum is automatically included, Fig. III–47.

We pause momentarily to establish a couple of relations that will aid us here as well as at other times. Since any trigonometric function represents a

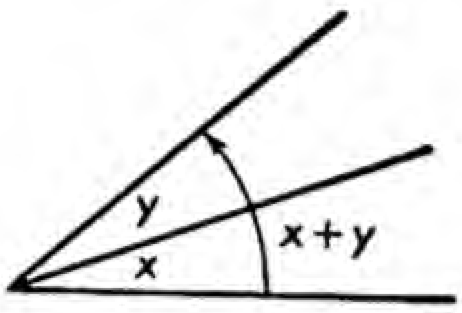


Fig. III–47

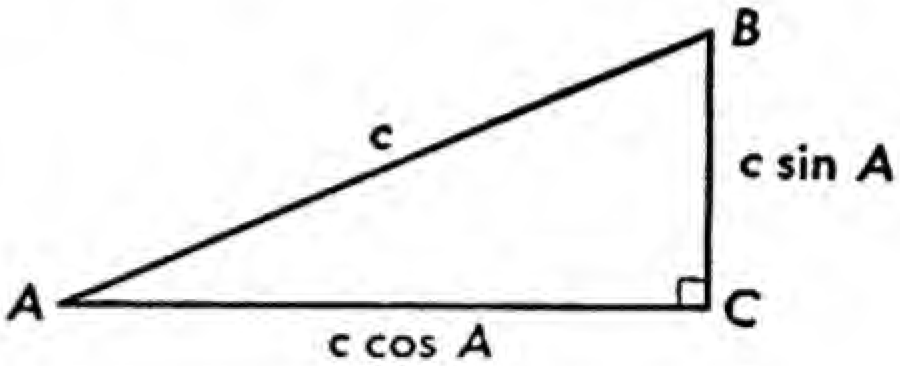


Fig. III–48

relation between two sides of a right triangle, any side of the triangle (Fig. III–48) can be expressed in terms of the other two. The following two cases are particularly valuable; Since

$$\frac{BC}{c} = \sin A$$

$$BC = c \sin A$$

Also

$$\frac{AC}{c} = \cos A$$

$$AC = c \cos A$$

Again, Fig. III–49 (a) suggests Fig. III–49 (b) and also Fig. III–49 (c).

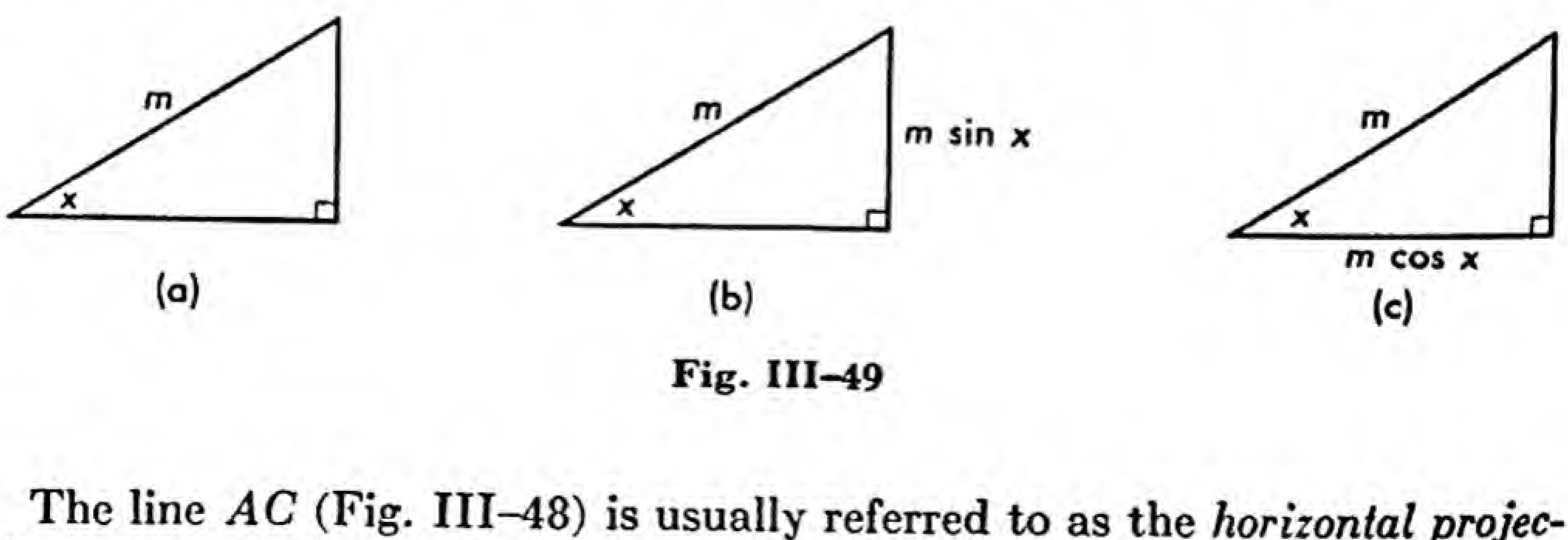


Fig. III–49

The line AC (Fig. III–48) is usually referred to as the *horizontal projection*; BC, as the *vertical projection* of the hypotenuse AB. The words horizontal and vertical need not be taken in the physical sense.

Since the angles x and y are taken adjacent to each other (Fig. III–50), $\angle A = x + y$. The perpendicular DC provides a right triangle setting for $\angle x$. Similarly the right $\triangle AED$ contains $\angle y$ and has a convenient common side with the previous right triangle. The perpendicular EB introduces a right triangle for $\angle(x + y)$. In addition, $FD \perp EB$ will prove quite convenient.

It should be noted that $\angle GED = x$, since $\triangle EGD \sim \triangle AGB$ by *aa* because of right angles and vertical angles. Finally, as on other occasions, there is no loss in generality by assigning the value of 1 to AE. Now

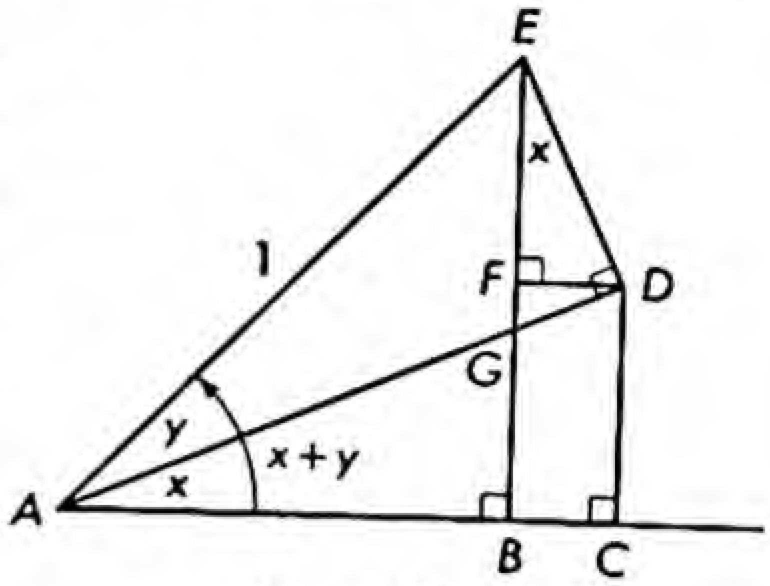


Fig. III–50

$$\sin(x + y) = EB = FB + EF = DC + EF$$

and since	$DC = AD \sin x$	and	$AD = \cos y$
so	$DC = \sin x \cos y$		
Also	$EF = ED \cos x$	and	$ED = \sin y$
so	$EF = \cos x \sin y$		
Thus	$\mathbf{\sin(x + y) = \sin x \cos y + \cos x \sin y}$		

This is the first of our *addition* identities which is true for all real values of x and y, although it has been developed with angles that are acute.

We computed earlier the values of the functions for 30° and 45°. The new identity puts us in the position of being able to determine the value for sin 75°.

$$\begin{aligned}\sin 75 = \sin(30 + 45) &= \sin 30 \cos 45 + \cos 30 \sin 45\\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

Other formulas may be derived from the one just developed. If we take (as a special case) $x = y$, we obtain by substitution the first *double angle* identity:

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$
$$\mathbf{\sin 2x = 2 \sin x \cos x}$$

If y is in quadrants I or II, $-y$ is in quadrants IV or III, respectively, and conversely. Since the sine of an angle is opposite in sign in the top quadrants as compared with the bottom quadrants, it follows that

$$\mathbf{\sin y = -\sin(-y)}$$

for any value of y. For example, $\sin(-23°) = -\sin 23°$, and $\sin(-140) = -\sin 140$. Either term of the last equality could have been reduced to $-\sin 40°$, at sight, if our intention had been to obtain an acute angle. In that case, all we would need do would be to refer to the reference angle and signs as we had done earlier.

The cosine function values, on the other hand, agree in sign in quadrants I and IV and in quadrants II and III. If y is in I, then $(-y)$ is in IV; and if y is in IV, then $(-y)$ is in I. A similar situation prevails for II and III. Consequently

$$\mathbf{\cos(-y) = \cos y}$$

and, illustratively, $\cos(-145) = \cos 145$.

Now, if we substitute $(-y)$ for y in the addition formula, we have

$$\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y)$$

which, in the light of the last two identities, becomes

$$\mathbf{\sin(x - y) = \sin x \cos y - \cos x \sin y}$$

The net effect has been a change in a sign in the right-hand member.

It is fairly certain, because of the desire for completeness, that we ought to find the corresponding formulas for the other functions. Actually the cosine and tangent cases will be adequate for our needs.

The $\cos(x + y)$ could be obtained from the same diagram used for finding $\sin(x + y)$, by starting with $\cos(x + y) = AB = AC - BC = AC - FD$. Instead, and for the purpose of illustrating available flexibility, we shall start with the formula for $\sin(x - y)$.

One observation will be helpful to us. We have seen that the sine of an angle equals the cosine of its complement.

$$\sin\left(\frac{\pi}{2} - M\right) = \cos M \qquad \text{and} \qquad \cos\left(\frac{\pi}{2} - M\right) = \sin M$$

The quantity M may even be a binomial such as that to be developed shortly. So, as we mentioned, we start with

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Since this is an identity, we can replace x and y by any values we desire. Indeed it was pointed out that the x's and the y's are only place holders. We substitute $\pi/2 - x$ for x, getting

$$\sin\left(\frac{\pi}{2} - x - y\right) = \sin\left(\frac{\pi}{2} - x\right)\cos y - \cos\left(\frac{\pi}{2} - x\right)\sin y$$

Because of the identities involving complementary angles, we get, by using the complementary functions,

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

[Note: $(\pi/2) - x - y = (\pi/2) - (x + y)$.]

Now this formula can produce two others by means of specialization. We set $x = y$ for one case and $y = -y$ for the other.

$$\mathbf{\cos 2x = \cos^2 x - \sin^2 x}$$

$$\mathbf{\cos (x - y) = \cos x \cos y + \sin x \sin y}$$

The tangent functions of $x + y$ and $x - y$ can be obtained by dividing the corresponding functions of sine and cosine in the light of an earlier identity. Thus

$$\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

The result can be expressed in terms of tangents if one notices that this is the outcome when numerator and denominator are divided by the product $\cos x \cos y$. To illustrate this, let us divide just the first term of the numerator by $\cos x \cos y$.

$$\frac{\sin x \cos y}{\cos x \cos y} = \tan x$$

Once this is seen, the rest of the division is possible at sight, and we get

$$\mathbf{\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

For $\tan (x - y)$, we can either substitute $-y$ for y in the last formula or go through a similar division for $\sin (x - y)$ and $\cos (x - y)$. Either way, only a change in the middle signs results.

$$\mathbf{\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}$$

Finally, if we substitute $x = y$ in the $\tan (x + y)$ formula, we get

$$\mathbf{\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}}$$

Here, as everywhere else, the values of the variables in the denominators must exclude those possibilities that would result in a zero denominator. In the last identity, for example, x must not be a $(\pi/4)$ reference angle in

any of the quadrants, for then $\tan^2 x = 1$, and the denominator would be zero. However, in this particular case, $\tan 2x = \tan(\pi/2)$ has been defined separately as infinity.

EXERCISES (III–8)

1. Find the following in radical form:

a. $\sin 105°$ b. $\cos 15°$

2. If x is the smallest acute angle in a 3, 4, 5 triangle, and y is the largest acute angle in a 5, 12, and 13 triangle, find

a $\sin(x + y)$
b. $\cos(x - y)$
c. $\sin 2x$
d. $\cos 2x$
e. $\tan(x + y)$
f. $\sin(x - y)$
g. $\tan 2x$

3. Using the double-angle formulas and the addition formulas, develop the formulas for:

a. $\sin 3x$ b. $\cos 3x$

4. Develop the $\cos(x + y)$ formula from the diagram used for $\sin(x + y)$.

5. A 16 inch line segment makes an angle of 27° with another segment. Find the horizontal projection of the first line on the second. Find the projection of the second line on the first.

6. If $\sin x = 2/3$ and x is acute, find the values of

a. $\sin 2x$ b. $\cos 2x$ c. $\tan 2x$

7. Prove that the following are identities:

a. $\tan y + \cot y = \sec y \csc y$

b. $\tan\theta + \cot\theta = 2\csc 2\theta$

c. $\tan x = \dfrac{1 - \cos 2x}{\sin 2x}$

d. $\sec^2 x - 1 = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

e. $-\sqrt{2}\sin\left(\dfrac{\pi}{4} - x\right) = \cos x(\tan x - 1)$

f. $\dfrac{1}{1 + \tan^2 m} - \sin^2 m = \cos 2m$

8. Prove that an angle bisector of a triangle divides the opposite sides into two segments which are proportional to the adjacent sides. (Use the Law of Sines on the two triangles that are formed.)

9. Solve each of the following equations for all positive values between 0 and 2π radian inclusive:

a. $2 \sin\left(x + \frac{\pi}{3}\right) = 1$

b. $\sin(\pi - x) = \cos(\pi - x)$

c. $\dfrac{2}{\cos y} - 2 = \dfrac{1}{\cos y}$

d. $2 \cot y - \dfrac{1}{\sin y} = 0$

e. $\sec^2 \theta = 2$

10. a. If we let $x + y = A$ and $x - y = B$, it is possible by adding the two equations to find a value of x in terms of A and B. By subtracting the two equations, it is likewise possible to solve for y.

b. Use the values in (a) to rewrite the identities for $\sin(x + y)$ and $\sin(x - y)$ in terms of functions of A and B.

c. Show that

(1) $\sin A + \sin B = 2 \sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(2) $\sin A - \sin B = 2 \cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

11. Using the results of exercise 10(a), and in a manner similar to the rest of exercise 10, develop the formulas for

a. $\cos A + \cos B$ b. $\cos A - \cos B$

9. COMPLEX NUMBERS AND TRIGONOMETRIC FUNCTIONS

The trigonometric functions are now intimately related or integrated into the real number system. Is there a place for the complex number?

We recall that the complex number $x + iy$ is representable in the complex plane as a point or as a vector. The right triangle (Fig. III-51) is easily introduced into the picture, which permits the entry of trigonometric relations. Let θ be the angle, made by the vector with the positive part of the X-axis, and let the length be $OP = r$, where $r = \sqrt{x^2 + y^2} = |x + iy|$. We now read

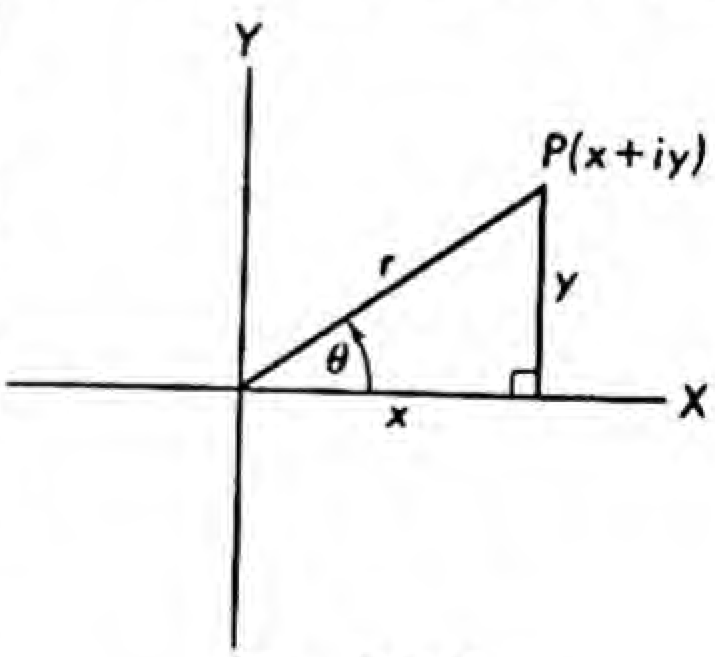


Fig. III-51

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta$$

which, by substitution in $x + iy$ and by factoring the r, yields

$$x + iy = r(\cos \theta + i \sin \theta)$$

This relation provides the means of expressing a complex value either algebraically or trigonometrically. The latter form is sometimes called the *polar form*, but we shall see more of that later.

Let us become familiar with this all important relation through concrete representations (Fig. III–52). Suppose that we start with $1 + i$ and repre-

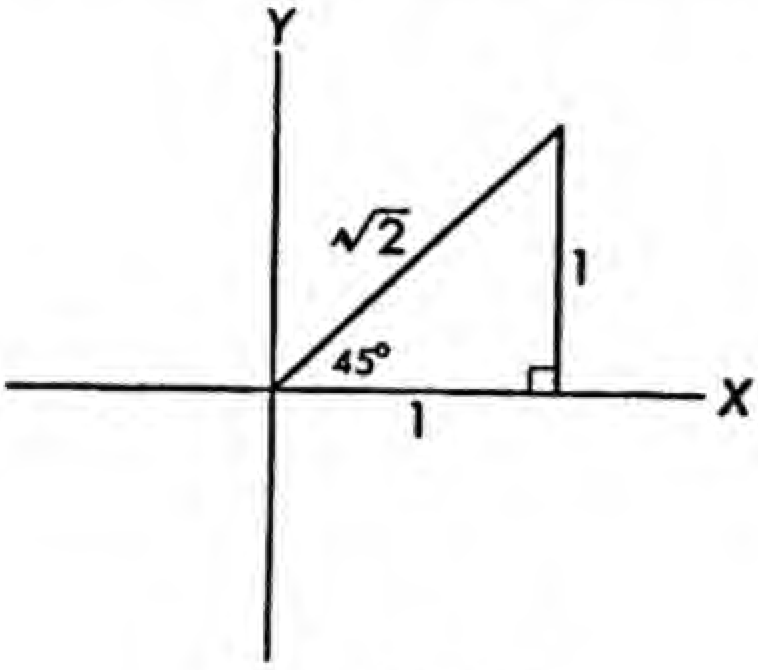


Fig. III–52

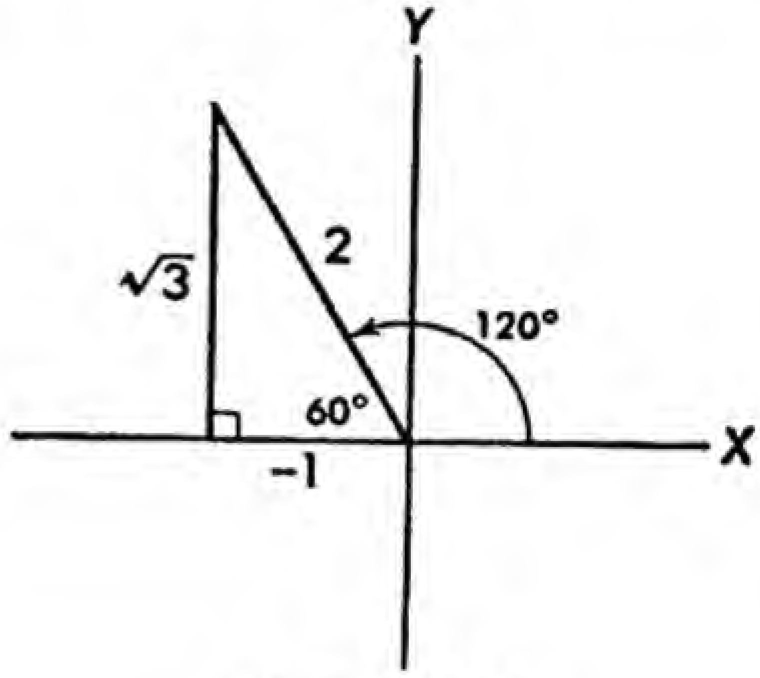


Fig. III–53

sent it in trigonometric form. We see, in this case, that $x = 1$, $y = 1$. Because the triangle is isosceles, $\theta = 45° = \pi/4$ radian. By the Pythagorean formula, we get $r = \sqrt{2}$. Consequently, by substitution, we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Here are other illustrations (see Fig. III–53):

a. For $-1 + i\sqrt{3}$,

$$x = -1, \quad y = \sqrt{3}, \quad r = 2, \quad \theta = 120° = \frac{2\pi}{3}$$

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

b. To find the algebraic equivalent of $5[\cos(5\pi/4) + i\sin(5\pi/4)]$, we note that $\cos(5\pi/4) = \cos 225° = -(\sqrt{2}/2)$ and that the $\sin(5\pi/4)$ has the same value. So,

$$5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 5\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

c. For $-4i$,

$$x = 0, \quad y = -4, \quad r = 4, \quad \theta = 270° = \frac{3\pi}{2}$$

$$-4i = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

The value r is frequently called the *modulus;* and θ, the *argument*.

The product of complex numbers, via the trigonometric forms, leads to some interesting and valuable results. We approach this on a general level.

$$\begin{aligned}(x + iy)\,(x' + iy') &= r(\cos\theta + i\sin\theta)\cdot r'(\cos\theta' + i\sin\theta')\\ &= rr'[\cos\theta\cos\theta' - \sin\theta\sin\theta' +\\ &\qquad i\,(\sin\theta\cos\theta' + \cos\theta\sin\theta')]\end{aligned}$$

In the light of the addition identities, this becomes (see Fig. III–54)

$$\boldsymbol{(x + iy)\,(x' + y') = rr'\,[\cos\,(\theta + \theta') + i\sin\,(\theta + \theta')]}$$

Thus the polar form of the product of two complex numbers has a modulus which is equal to the product of the moduli of the two numbers and an argument which is equal to the sum of the arguments of the two numbers. This result may be pictured as the product of two vectors.

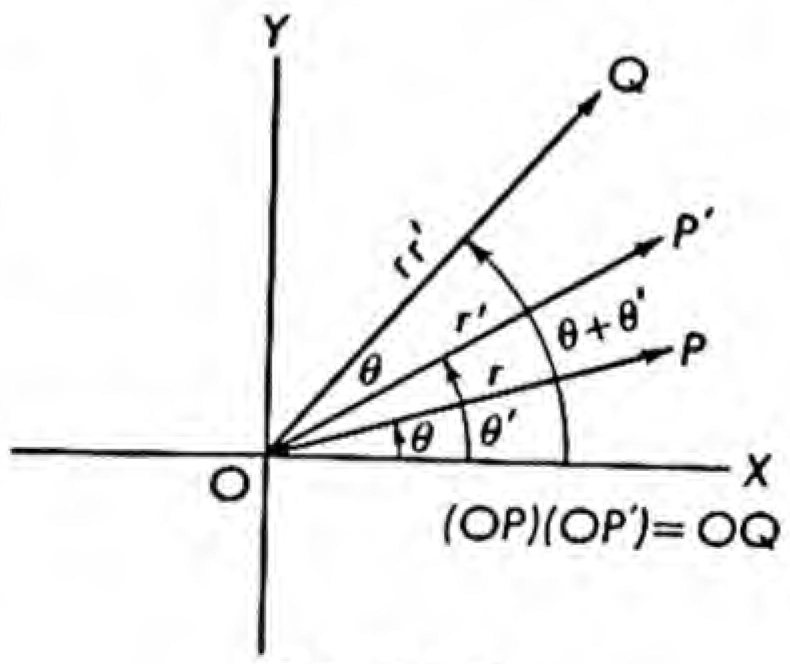


Fig. III–54

Should the last result be multiplied by another complex number, the character of the result would be unchanged. However, there would be another factor for the moduli and another addend for the argument. That is,

$$\begin{aligned}(x + iy)\,(x' + iy')\,(x'' + iy'') &= rr'r''[\cos\,(\theta + \theta' + \theta'')\\ &\qquad + i\sin\,(\theta + \theta' + \theta'')]\end{aligned}$$

We have seen that special cases frequently provide us with valuable results, and such is the case here. Suppose that the complex numbers which we are multiplying are equal to each other. The results would then be (keeping in mind that $r = r' = r'' = \cdots$ and $\theta = \theta' = \theta'' = \cdots$)

$$(x + iy)^2 = r^2(\cos 2\theta + i\sin 2\theta)$$
$$(x + iy)^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

And, in general, for any positive integral value of n,

$$(x + iy)^n = r^n(\cos n\theta + i\sin n\theta)$$

or

$$\boldsymbol{[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)}$$

This remarkable and useful result is called *De Moivre's formula*. Lately the foregoing basic expression has been telescoped for convenient usage. We let

$$\cos\theta + i\sin\theta = \mathbf{cis}\ \theta$$

where first letters are permitted to form a mnemonic. The last formula may be written as

$$(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$$

It is possible to prove the formula true for any real value of n. The following are additional illustrations with particular attention to the new symbolism.

a. $$\text{cis } \pi = \cos \pi + i \sin \pi = -1$$

b. $$\text{cis } \frac{\pi}{3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

c. $$\frac{1}{\text{cis } \theta} = (\text{cis } \theta)^{-1} = \text{cis } (-\theta)$$

Illustration (c) leads to an effective view of division of complex numbers:

$$\frac{r \text{ cis } \theta}{r' \text{ cis } \theta'} = \frac{r}{r'} \text{ cis } \theta \, (\text{cis } \theta')^{-1} = \frac{r}{r'} \text{ cis } \theta \text{ cis } (-\theta')$$

$$\frac{r \text{ cis } \theta}{r' \text{ cis } \theta'} = \frac{r}{r'} \text{ cis } (\theta - \theta')$$

To obtain the quotient of two complex numbers, we need only divide the moduli and subtract the arguments. This could have been anticipated in the light of the results with multiplication. By way of an illustration,

$$\frac{1 + i}{-1 + i\sqrt{3}} = \frac{\sqrt{2} \text{ cis } (\pi/4)}{2 \text{ cis } (2\pi/3)} = \frac{\sqrt{2}}{2} \text{ cis } \left(-\frac{5\pi}{12}\right)$$

Because of the periodicity of the trigonometric functions, we know that

$$\text{cis } \theta = \text{cis } (\theta + 2k\pi)$$

for any integral values of k. Concretely this indicates clearly that the addition of any multiple of 360° to the argument yields the same vector. This observation is of considerable moment when n in De Moivre's formula is a fractional value. We can demonstrate this by computing the cube roots of 1. Now,

$$1 = \text{cis } 0 = \text{cis } (0 + 2k\pi)$$

and so

$$\sqrt[3]{1} = \text{cis } \frac{2k\pi}{3}$$

Then

$$\text{cis } \frac{0}{3} = 1 \qquad \text{for } k = 0$$

Also

$$\text{cis } \frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad \text{for } k = 1$$

and

$$\text{cis } \frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \qquad \text{for } k = 2$$

If we continue beyond this point, using $k = 3$ or larger values, we see that no new results arise for the cube root of 1. For $k = 3$, for example, the argument is 2π and cis $2\pi = 1$. So, there are precisely three cube roots of 1, one real and two complex. The modulus of the roots is 1, and the arguments are 0, $2\pi/3$ and $4\pi/3$. If we graph these (Fig. III–55), we note that they are the vertices of an equilateral triangle inscribed in a unit circle at the origin.

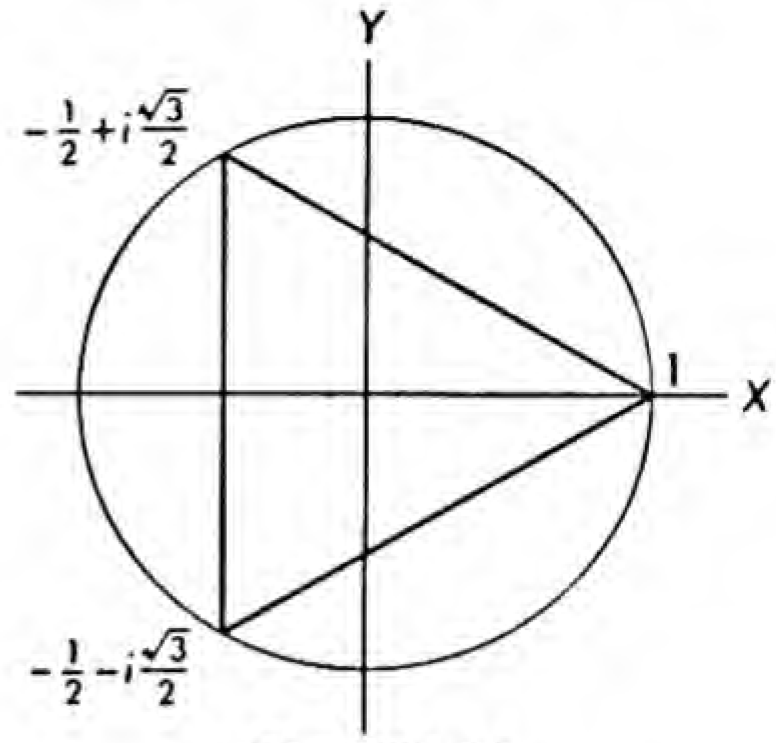


Fig. III–55

In similar fashion, it can be shown that there are four fourth roots of one which lie at the vertices of an inscribed square. Indeed the fourth roots of any number, real or complex, lie at the vertices of a square inscribed in a circle with center at the origin and radius $= r$. The fifth roots of a number lie at the vertices of an inscribed equilateral pentagon. From

$$1 + i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$

we get

$$\sqrt[6]{1 + i} = \sqrt[12]{2}\,\text{cis}\left(\frac{\pi}{24} + \frac{2k\pi}{6}\right)$$

From this we see that one root will have the argument $\pi/24 = 7\frac{1}{2}°$, and the other five will be at intervals of $2\pi/6 = 60°$ from each other. All lie on a circle of radius $\sqrt[12]{2}$. The six points determine an inscribed regular hexagon, Fig. III–56.

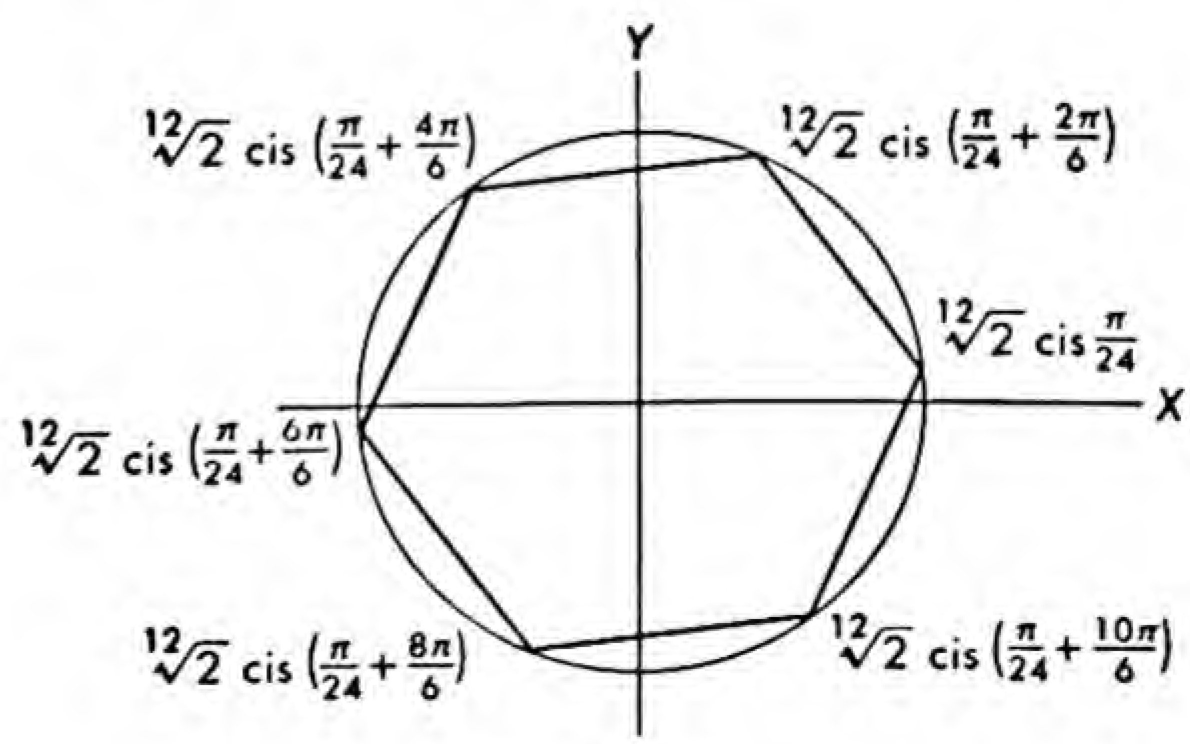


Fig. III–56

EXERCISES (III–9)

1. Sketch each of the following, showing the factors as well as the result:

a. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ b. $(1 - i)(1 + i)$

c. $2i\left(\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)$ d. $\frac{1 + i}{1 - i}$

e. $\frac{\sqrt{3} - i}{\sqrt{2} - \sqrt{2}i}$ f. $i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

g. $-i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

2. The exercises 1(f) and 1(g) suggest the effects on vectors when multiplied by i and $-i$, respectively, state the effects and establish this in general by using the multipliers i and $-i$ on $a + bi$.

3. Express each of the following as algebraic complex numbers:

a. $\text{cis}\ \frac{\pi}{4}$ e. $5\ \text{cis}\ \frac{\pi}{2}\ \text{cis}\ \frac{3\pi}{4}$

b. $\text{cis}\ \pi$ f. $2\ \text{cis}\ \frac{3\pi}{2}\left(3\ \text{cis}\ \frac{\pi}{2}\right)$

c. $2\ \text{cis}\ \frac{4\pi}{3}$ g. $\frac{\text{cis}\ \pi/2}{\text{cis}\ \pi/4}$

d. $3\ \text{cis}\left(-\frac{5\pi}{3}\right)$ h. $\frac{6\ \text{cis}\ (4\pi/9)}{2\ \text{cis}\ (5\pi/18)}$

4. a. Find the three cube roots of 8.
   b. Compare the results with the three cube roots of 1 in the text. Does comparison suggest a hypothesis for study?
5. Show graphically the three cube roots of $1 + i$; i.
6. a. Find the four fourth roots of 1.
   b. Find the cube roots of -1.
   c. Find the fourth roots of $1 - i$.
   d. Find the cube roots of $1 + i\sqrt{3}$.
7. Complete the general formula for integral values of m in $(r\ \text{cis}\ \theta)^{1/m}$.

III–9 REVIEW

1. Write two trigonometric equations whose graphs have 2 for an amplitude and $\pi/4$ for the period. Sketch both.

2. Solve each of the following equations for values of the angle between 0 and 2π:

a. $\sin x = \cos x$ b. $\sin 2x \sec 2x = 2$ c. $2 \cos 2x + 1 = 0$

3. Verify the following identities:

a. $\cot \theta - \tan \theta = 2 \cot 2\theta$ b. $\dfrac{1}{2} \tan 2x = \dfrac{\sin x}{\sec x - 2 \tan x \sin x}$

4. Express the fourth power of 2 cis $(\pi/4)$ in trigonometric as well as algebraic form.

5. Find the four fourth roots of $-i$, and graph all the quantities involved.

6. Show that $\sin \theta = \cos [(\pi/2) - \theta]$ for any value of θ.

7. If $(\pi/2) < M < \pi$, $\sin M$ equals (a) $-\sin M$; (b) $\sin (180° - M)$; (c) $-\sin (M - 180°)$; or (d) $\sin (M - 180°)$. Which?

8. If $\sin \theta > \frac{1}{2}$ and θ is an angle of a triangle, indicate in radians the possible range of values for θ.

9. Find the values of (a) $|2 - 3i|$; (b) $|-2 - 3i|$

10. a. Use the Pythagorean theorem to express the value of AC in Fig. III-44 in terms of $\sin x$.

b. Use the same diagram to express the value of $\tan x$ in terms of $\sin x$.

11. Use the same techniques as in the preceding exercise to express the sides of a triangle in terms of $\cos x$, and then express the $\cot x$ in terms of the $\cos x$.

12. Express the following trigonometrically:

a. $1 - i$ b. $2 - 2\sqrt{3i}$ c. $-3 - 3i$

13. Change each of the following to algebraic form:

a. 2 cis 150° b. 3 cis 315° c. 5 cis 270

d. cis $\dfrac{2\pi}{3}$

14. Find the results of the following:

a. cis $\dfrac{\pi}{6} \cdot 2$ cis $\dfrac{5\pi}{4}$ c. $\dfrac{8 \text{ cis } 160°}{4 \text{ cis } 20°}$

b. 4 cis 142° · 3 cis 18° d. $\dfrac{3 \text{ cis } (\pi/4)}{5 \text{ cis } (3\pi/2)}$

15. Sketch the graphs of each of the following:

a. $\sin \left(x + \dfrac{\pi}{6}\right)$ c. $\sin x + \sin 2x$

b. $\sin^2 x$

16. If the ratio of two angles of a triangle is 2:1, show that the ratio of the opposite sides, taken in the same order, is $2 \cos x:1$.

17. From a point on the ground, the angle of elevation of the top of a building, whose height is h feet, is θ. The angle of elevation of the top of a vertical pole d feet high on the edge of the building is θ'. Express the height of the pole in terms of θ, θ', and h

18. a. Two sides of a triangle are 6 inches and 10 inches and include an angle of 64°. Find the area of the triangle.

b. Show that the area of $\triangle ABC$ is given by: Area $= \frac{1}{2} ab \sin C$.

c. Show that the area of an equilateral triangle is $(s^2/4)\sqrt{3}$, where s is the length of a side.

19. Find the area of an isosceles trapezoid with two adjacent sides of 4½ inches and 14 inches and an included angle of 63°.

IV — NON-EUCLIDEAN GEOMETRY

1. THE DAWN OF A NEW ERA

The topic of parallel lines occupied a unique position in the historical development of mathematics, and certain outcomes (some of which we shall examine now) had great influence on the science of thought.

One of our early postulates concerned the similarity of triangles with proportional sides. For us, the implications of this postulate were far reaching. Among other derivations we found that the sum of the angles of a triangle is 180°; the straight line is the shortest distance between two points; and through a point not on a line, there is one and only one parallel to the line.

The undefined terms that are embedded in our postulates find their mathematical meaning and existence through the postulates. Through the medium of the deductive method of reasoning, and abetted by additional definitions, we find many theorems of mathematics. The entire structure is called a **postulate system.**

It is conceivable that some alteration in postulates may lead to the same or to a different system. Indeed we have adopted a departure from the traditional in our selection of postulates in this text. The classical Euclidean geometry, as systematized by Euclid, started with various postulates and certain concepts of superposition, which presumed certain subtle notions about motion. On this basis, congruence emerges early in our study, leading to parallel lines, to the 180° sum of the angles of a triangle, to similar figures, and to the Pythagorean theorem.

The development we pursued is wholly consistent with Euclid's, and so our development is Euclidean. However, there is one point of significant difference. Whereas Euclid's facts about parallel lines are directly deducible from similarity, as in our development, they are not deducible from

congruence alone without an additional postulate. It is this postulate of parallels that was the center of attention for over a thousand years.

Euclid postulated that through a point not on a given line, there is one and only one parallel to the line (see Fig. IV–1). In the years following Euclid's development, mathematicians did not find in the parallel postulate the same kind of simple or obvious abstraction from experience upon which the other Euclidean postulates were based. The self-evidence of the postulate, even on an abstract and philosophical plane, was questioned. Although the quality of self-evidence was not subject to definition, postulates at that time had to be self-evident.

Many attempts were made to prove the postulate of parallels by means of the other postulates, but all were unsuccessful. The literature is extensive. Finally, in the early part of the nineteenth century, Gauss, Lobachevsky, and Bolyai separately and almost simultaneously conceived of another approach. Suppose they reasoned, that through some external point there is more than one parallel to a line. That would indeed represent a contrary hypothesis fraught with contradiction. But no, this contrary assumption did not lead to the expected contradiction but rather to a totally consistent set of theorems that differed from Euclid's in those places where the theorems were dependent on the parallel assumption. A few years later Riemann produced another geometry with no parallel at all.

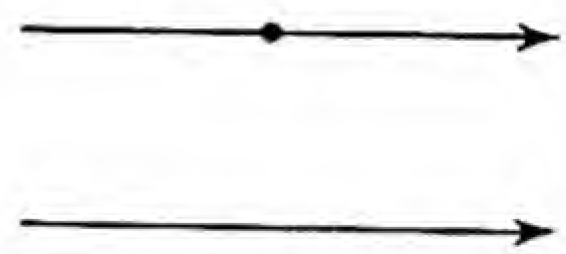


Fig. IV–1

This marks the birth of **non-Euclidean** geometry and the beginning of the downfall of the view of the absoluteness of the postulates. In its stead, slowly and painfully, came the notions of the relativity of postulational thought. The study of the nature of postulational (axiomatic) systems was subsequently begun with great seriousness. There is hardly a field of thought today that has not been affected by this dramatic development. In practical terms, and beginning with Einstein's Theory of Relativity, science found non-Euclidean geometry a tremendously valuable logical tool with which to probe the universe.

2. THE FAMILIAR IS QUESTIONED

Consider PA perpendicular (Fig. IV–2) to line m. Imagine that P is connected to the points B, C, $\cdots$ to the right of A. Since the line m is endless in extent, there will always be points that can be connected to P. The line has no end point, no last point. How, then, can we ever reach a condition of parallelism?

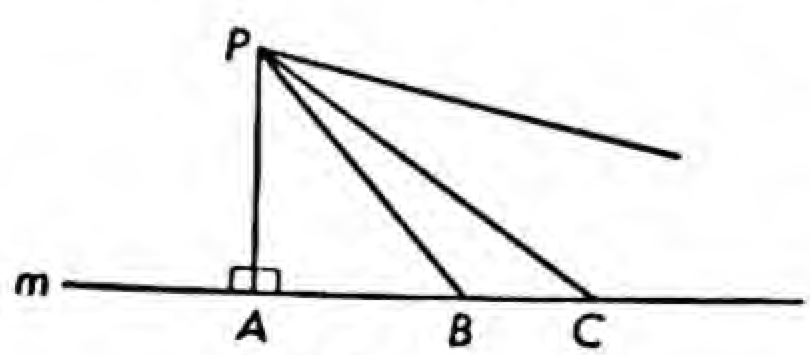


Fig. IV–2

3. CONSEQUENCES—RIEMANN

Suppose, then, that **we assume with Riemann that there are no parallel lines**; that all lines intersect. It follows at once that two perpendiculars to AB (Fig. IV–3) meet at some point P. This indicates immediately that *the sum of the angles of a triangle is greater than 180°.*

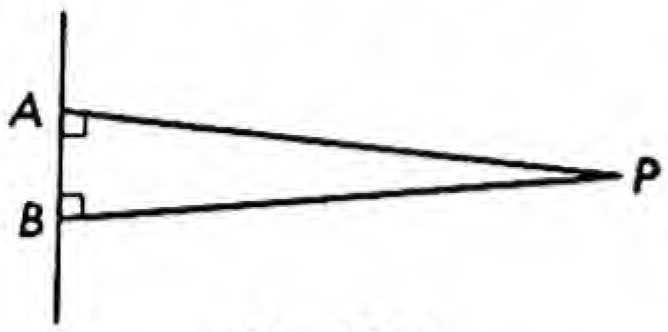


Fig. IV–3

Let us call the point P the **polar vertex** of the triangle and AB the **base line.** We may also name the sides PA and PB as **quadrants.** (The special names are selected in the light of subsequent applications.)

The $\triangle PAB$ is self-congruent by *asa;* and PA and PB, lying opposite equal angles, are therefore equal corresponding parts. It should be noted that the change in the parallel postulate does not affect most of the Euclidean conditions of congruence. (See exercise 1 at the end of this article.) This **birectangular triangle** is isosceles. It can be shown too that any isosceles triangle has equal base angles, and conversely.

Pursuing this observation, we note further that all perpendiculars to a base line meet at the same polar vertex (Fig. IV–4). The contrary position would not be tenable, for suppose that we draw another perpendicular at C, which is taken so that $AB = BC$, and that this perpendicular meets BP at P'. This creates two congruent triangles. Then, $BP' = BP$, and therefore P and P' are one and the same point.

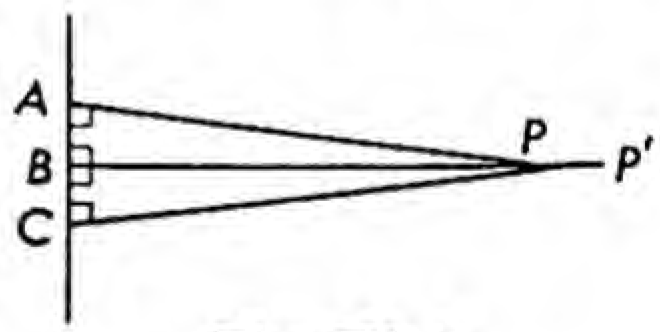


Fig. IV–4

These observations lead to the conclusions that any line has only one polar vertex on one side of itself; that all perpendiculars to it meet in the same point, and that all quadrant distances are equal. (If B is between P' and P on the line PP', then P' and P are on opposite sides of AB.)

The equality of all quadrant distances can be demonstrated by taking two different base lines. If on these (Fig. IV–5) we take $AB = A'B'$ and draw the perpendiculars as before, we have congruent bi-rectangular triangles, from which it follows that $A'P' = B'P' = AP = BP$.

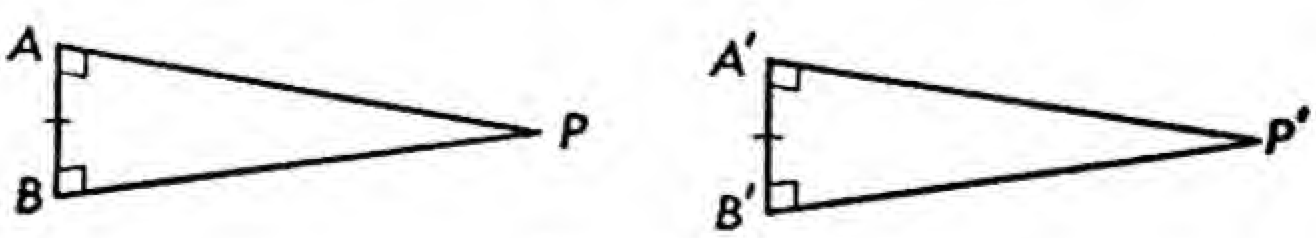


Fig. IV–5

We can go even further. *Any* line from the polar vertex to the base line is perpendicular to the base line for if PX is such a line (Fig. IV–6), the

triangles formed are congruent as marked. (XA can always be taken equal to XB, and the quadrant distances PA and PB are equal.) The equality of angles 1 and 2 makes them right angles. The diagram also suggests, incidentally, that the median and angle bisector from the polar vertex angle are one and the same line.

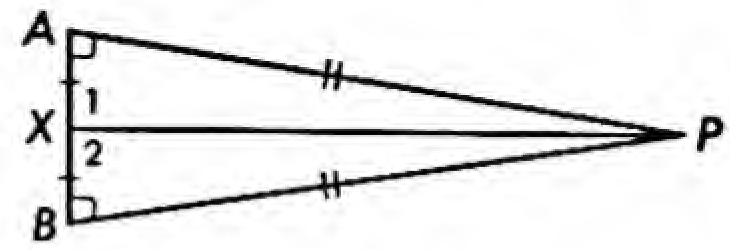


Fig. IV–6

If a point R is not a polar vertex (Fig. IV–7), then one and only one perpendicular is possible from R to the base line. Suppose that P is the pole of AB; then PR must meet the base line in, say, point B, since there are no parallel lines. P, being the polar vertex of AB, indicates that PB is perpendicular to line m. Consequently there is one perpendicular from R to m. There cannot be any other perpendicular from R to m. If there were another, then according to a foregoing conclusion, R would be a pole, which is contrary to the initial condition.

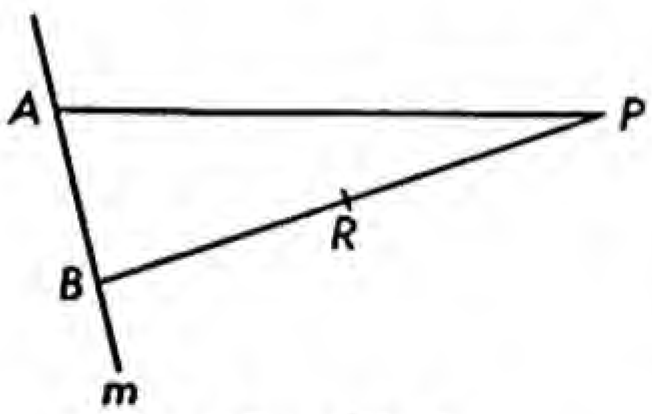


Fig. IV–7

Let us take the side PA of the birectangular triangle (Fig. IV–8) and extend it by its own length through A to P'. Join P' to B. The two triangles formed are congruent by *sas*, making $\angle 1 = \angle 2 =$ a right angle. As a consequence, PBP' is a straight angle and PBP' is a straight line. $PAP' = PBP' =$ two quadrants in length. Further, **two straight lines meet in two points.** We could consider P' to be the image of P on the other sides of the base line. Because of the congruence and the equalities, we can certainly consider P' as a polar vertex of AB too.

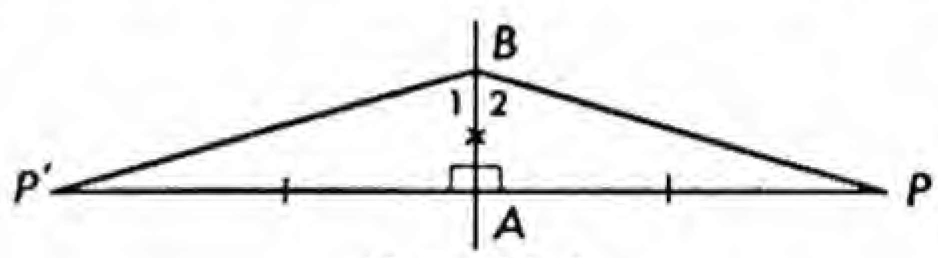


Fig. IV–8

We learned before that an infinite number of perpendiculars are possible from P to AB. The point P' could be joined to all the points on AB which were formed by the perpendiculars. Such connections all lead to straight lines, as they did with $P'BP$. In brief, between P and P' (Fig. IV–9), there are an infinite number of straight lines, all being perpendicular to AB and all having two quadrants of length.

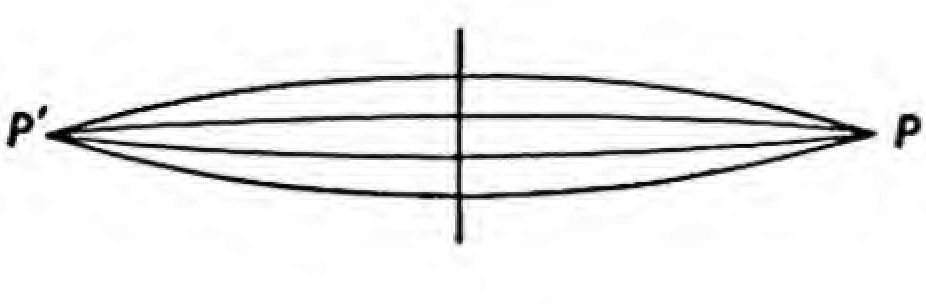


Fig. IV–9

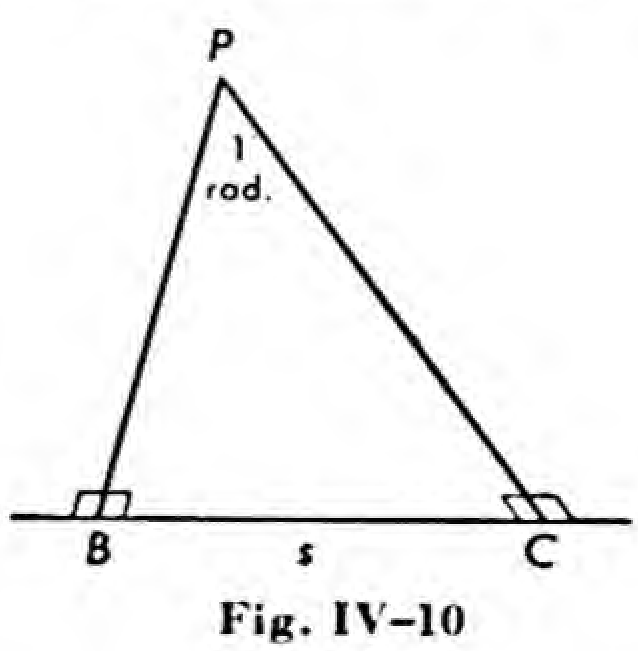


Fig. IV–10

Suppose now that we take a birectangular triangle with a 1-radian polar vertex angle, Fig. IV–10. This is a uniquely determined triangle, and so BC is a constant having, say, s length. At the vertex P we can continue to measure off adjacent 1-radian vertex angles which lead to adjacent congruent triangles (Fig. IV–11). At point P we have 360° or 2π radians. Thus we shall have 2π nonoverlapping triangles and consequently 2π adjacent, nonoverlapping segments on XY. The length of XY is then a constant having exactly $2\pi s$ units in length. Since XY is any line whatsoever, the length of the line in Riemann geometry is this finite amount, $2\pi s$.

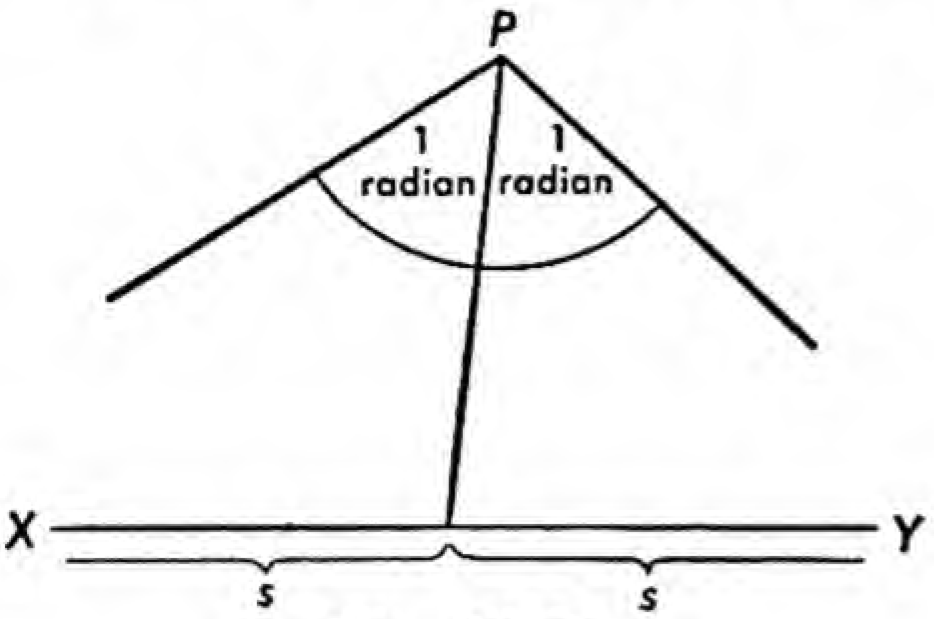


Fig. IV–11

EXERCISES (IV–3)

1. The Euclidean case of congruence by *saa* is not applicable in Riemannian geometry. Sketch an illustration of this.

It was pointed out before that Euclid's postulate of parallels is a theorem in this textbook as a result of our postulating the case of *sss* similarity. To deny this theorem, as is done in non-Euclidean geometry, is to deny the postulate on which it rests. As a result, not only *sss* but all cases of similarity are lost in non-Euclidean geometry. This does not necessarily mean that all cases of congruence are also lost.

We could start all over, in the more traditional sense, with congruence, *sss*, *sas*, and *asa*. Then, in time, we could introduce Euclid's postulate. To deny the postulate in this context would be to deny the pursuant theorems that depend on it. This would include such theorems as: the sum of the angles of a triangle is 180°, triangles are congruent by *saa*, and all the similarity theorems.

2. Prove that the base angles of any Riemannian isosceles triangle are equal.

3. Can there be an equilateral birectangular triangle? If so, what additional conclusion can you foresee?

4. Quadrilateral $ABCD$ (Fig. IV–12), in which $A = B = 90°$ and $DA = CB$, is called an *isosceles birectangular quadrilateral*. Line DC is called the *summit*; and angles D and C, the *summit angles*.

   a. Prove that $\angle D = \angle C$.

   b. State the conclusion as a theorem.

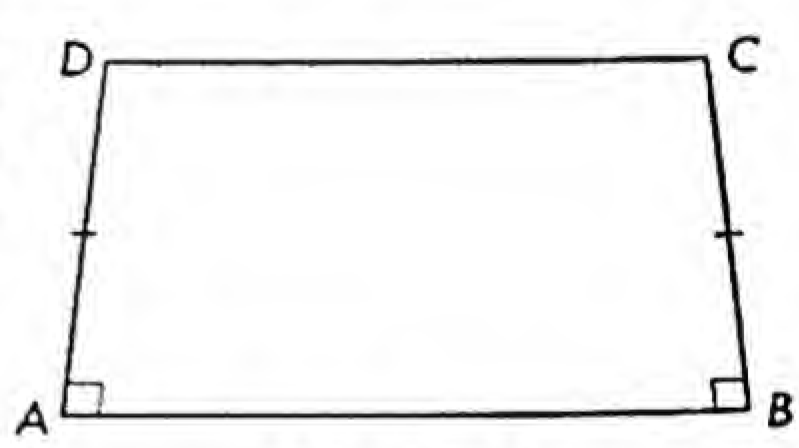


Fig. IV–12

5. Prove that the angle bisector of the vertex angle of an isosceles triangle is perpendicular to the base Compare your proof to the Euclidean counterpart.

6. $\triangle ABC$ is a right triangle with $C = 90°$. Support the following statements informally:

   a. $A < 90°$ if $BC <$ quadrant.

   b. $A > 90°$ if $BC >$ quadrant.

Express the conclusions in words.

7. Using the conclusions in exercise 6, prove that the base angles of an isosceles triangle are:

   a. Acute, if the altitude to the base is less than a quadrant.

   b. Obtuse, if the altitude is greater than a quadrant.

   c. Right, if the altitude is equal to a quadrant.

8. Prove that the line joining the midpoints of base and summit of an isosceles birectangular quadrilateral is perpendicular to both.

9. a. Prove that the summit angles of any isosceles birectangular quadrilateral are obtuse.

   b. What may be said of squares and rectangles in Riemannian geometry?

10. $ABCD$ is a trirectangular quadrilateral (three right angles) with right angles at A, B, and D. Show that:

   a. $DC < AB$ b. $CB < AD$

11. Prove that any two intersecting lines have a unique common perpendicular.

4. THE INTUITIVE SPATIAL SENSE

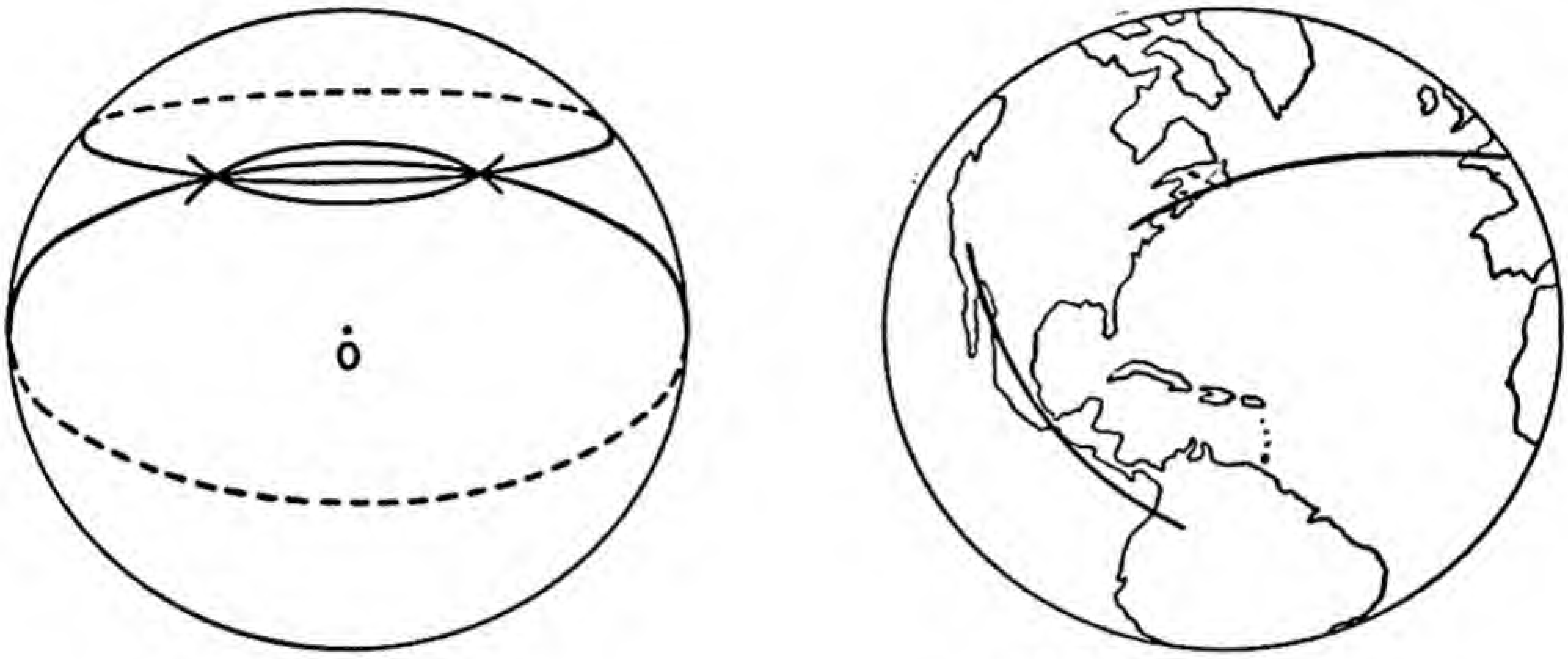
Enough has been indicated already for us to glimpse the nature of a non-Euclidean geometry, which in this case is a Riemannian geometry. The reader may have become increasingly dubious about the diagrams, for they do not *fit* into one's spatial sense. However, we have seen that our geometric idealization of points and lines is derived from material things and that our drawings continue to represent material things.

The light rays that helped to orient us to straight lines have been found, by prediction of the Theory of Relativity, to traverse curved paths in space. They move in **geodesics**, which are the shortest distances in a space-time framework. The geodesic path in a region of space is dependent on the curvature of the space in its region, which in turn is determined by the presence and distribution of matter in and around that region. A ray of light from a distant star will deviate from a Euclidean path as it passes near the sun. Can we conclude from this that our Euclidean idealizations may have exceeded the conditions of reality from which the Euclidean assumptions stemmed?

In Euclidean analysis, we had to idealize a flat, planar surface to provide the arena for the Euclidean story. While the same plane could have been used for the non-Euclidean account, taxing our visual sense, we could have found other surfaces that might have been more rewarding.

The surface of the sphere provides us with illustrative opportunities for Riemannian geometry. The straight lines, the geodesics, are the great circles of the sphere. The great circles are those circles whose centers coin-

cide with the center of the sphere. Through any two points on the surface, an infinite number of circles can be drawn on the sphere, giving an infinite number of arcs between the two points (see Fig. IV–13). The arc that lies along the great circle is the shortest between the two points.



(a) (b) Great circle course

Fig. IV–13

The $\triangle PAB$ (Fig. IV–14) has every characteristic of the Riemannian bi-rectangular triangle. The point P is the polar vertex; the sides are quadrants (of a great circle); and the sides, extended, meet in another point P' which provides another triangle. Thus the straight lines meet in two points; the length of a straight line is finite, being equal to a great circle whose circumference is $2\pi r$.

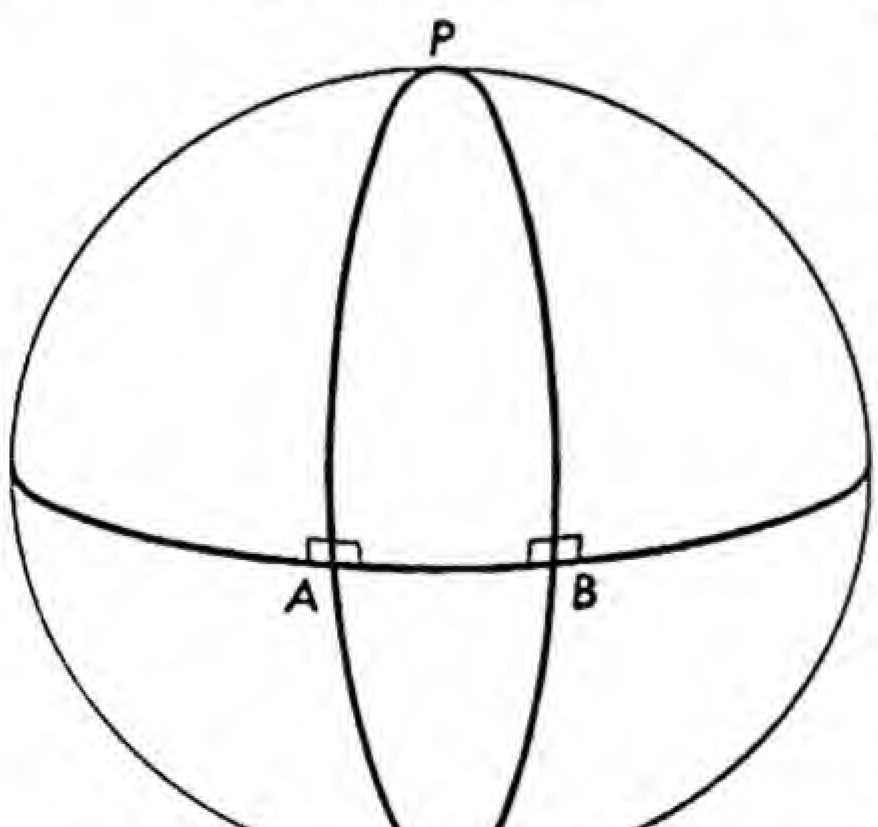


Fig. IV–14

The geometry of the surface of the sphere is strictly a Euclidean development. The surface is a two-dimensional surface embedded in a three-dimensional Euclidean framework. However, the surface of the sphere represents a complete model of the Riemannian geometry. Every theorem of two-dimensional Riemannian geometry is valid on the surface of the sphere. In this sense we recognize the logical validity of Riemannian geometry, for it concurs with a consistent Euclidean system. The two systems have to stand or fall together.

5. OFF AGAIN

We can look at the parallel line situation in still another way. As lines are drawn from P to B, C, $D, \cdots$ (Fig. IV–15), we assume that one and only one limiting position is reached, PX, which is parallel to line m. The same consideration could be put forth for the points on the other side of PA, where PA is perpendicular to m.

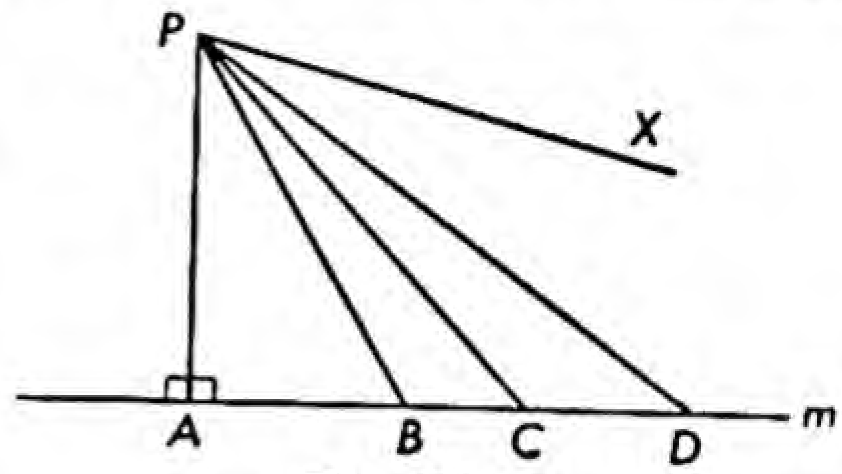


Fig. IV–15

Consistency suggests that we allow some limiting line, such as PY on the left, Fig. IV–16, to be parallel to m too. Now, if PX and PY were to lie on the same straight line, we would have the single Euclidean parallel. So let us assume instead that these are two distinct lines; which in effect means that each angle x and z is less than $\pi/2$ (Fig. IV–17).

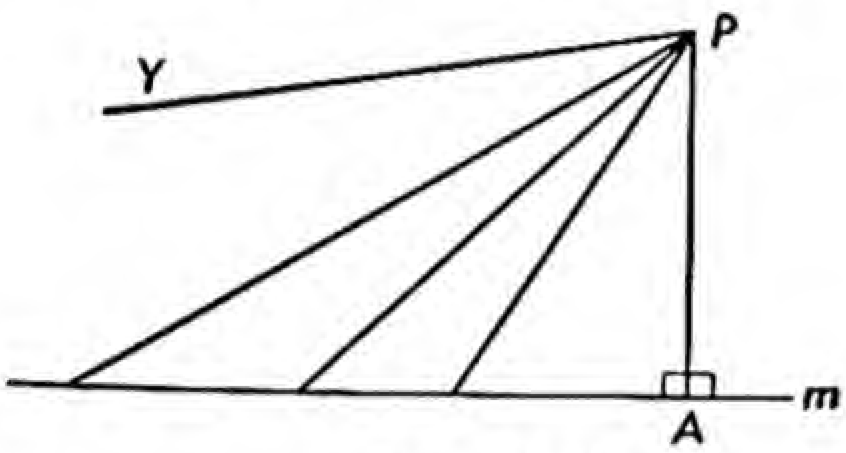


Fig. IV–16

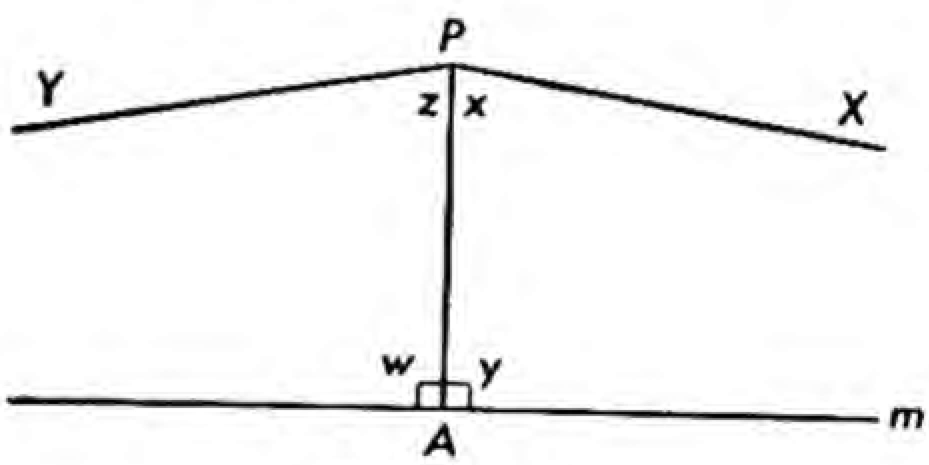


Fig. IV–17

From the fact that x is less than $\pi/2$ and $y = \pi/2$, we see that the sum of the interior angles on the same side of PA is less than π in this conception. Now this conclusion is not restricted to a perpendicular but holds for any transversal. If $RX \parallel AB$ in the *Lobatchevskian sense* (Fig. IV–18), then $\angle R + \angle A < \pi$. If the sum were larger than π, a Euclidean parallel to AB would be possible within $\angle R$. And if the sum were exactly π, then RX would be a Euclidean parallel. Both these possibilities are contrary to the Lobatchevskian concept of parallel lines. So we consider the Lobatchevskian postulate: that *parallel lines form with a transversal interior angles on one side, whose sum is less than 180°*. Let us refer descriptively to PX and PY as the **right** and **left** parallels.

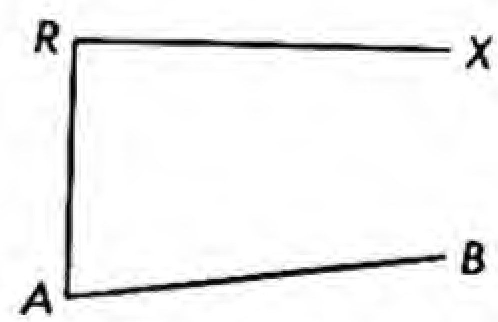


Fig. IV–18

Consider now the situation created by the extensions of PX and PY (Fig. IV–19). The two parallels separate the plane into distinct regions. Since lines within the angles XPY' and YPX' cannot meet line m, they will be known as nonintersectors. Outside of these regions all lines through

P intersect m. Now we have intersectors, parallels, and nonintersectors. If PA is drawn perpendicular to m, $\angle XPA$ is called the **angle of parallelism.**

6. MUTUAL PARALLELISM

It is bound to occur to the reader that the point P (Fig. IV–19) could have been farther down or up on the perpendicular, creating the possibilities of a number of right and left parallels to a line m. He might also have expected that the angle of parallelism would vary in some way under these circumstances. Should the angle remain the same, the situation would be in agreement with a Euclidean theorem on parallelism. This is not plausible, since, like **Lobatchevsky**, we started with a contrary postulate. However, before we can reach a decision on this, we have to take a closer look at a three-line parallel situation.

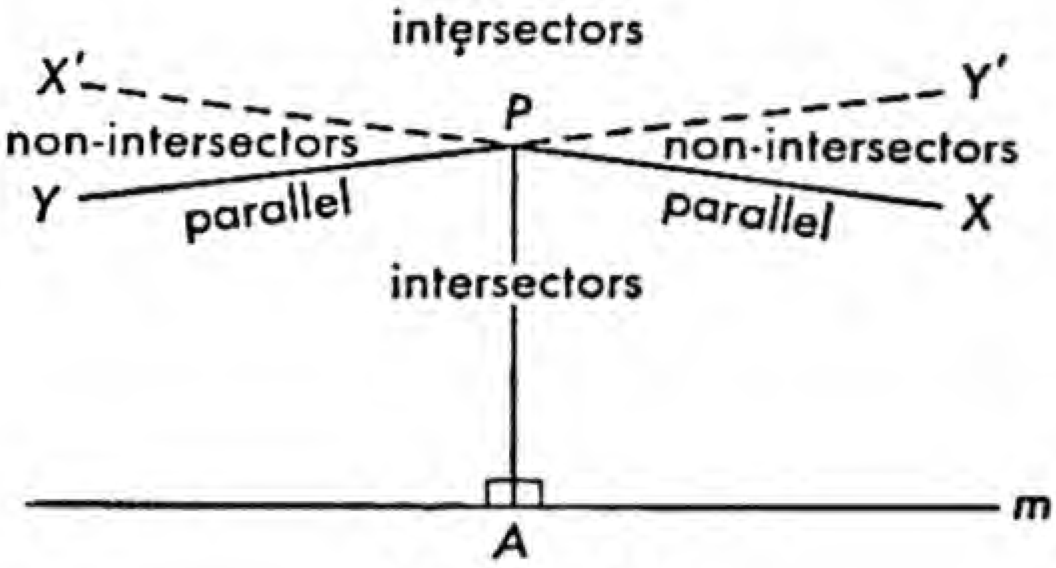


Fig. IV–19

Suppose that both a and c are right parallels to b in Fig. IV–20. Are a and c parallel to each other? The lines a and c may be (1) intersecting lines, (2) nonintersecting lines, or (3) parallel lines. Suppose that (1) were the case, i.e., that a and c intersect each other at some point K. This is impossible because it would mean that at K we would have two right parallels to b. Through a point outside a line there is only one right, or left, parallel to the line.

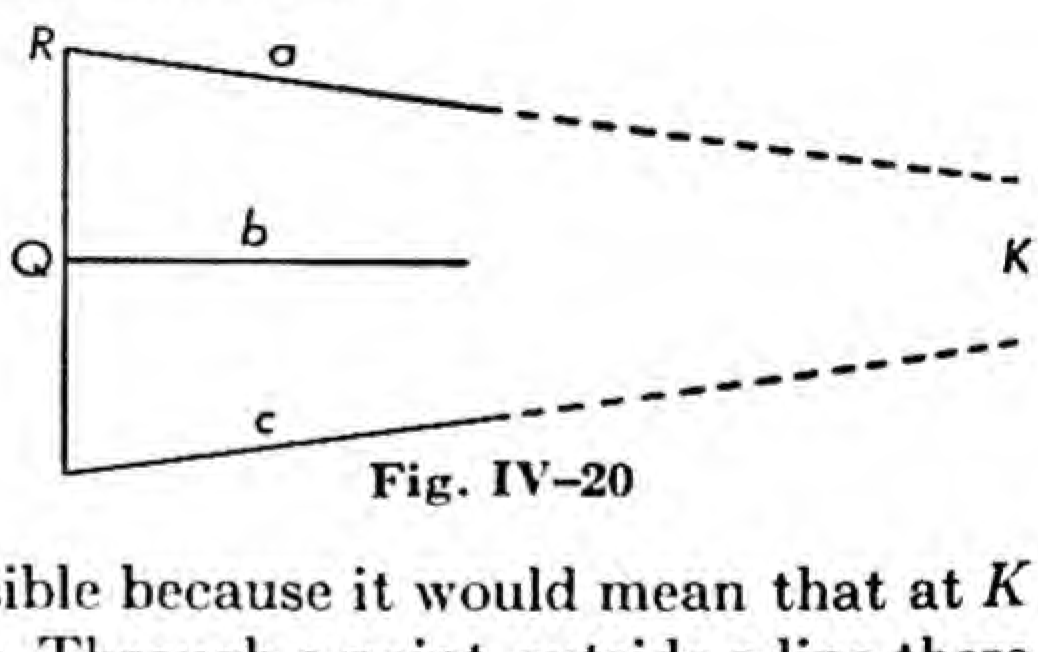


Fig. IV–20

Let us try again. Suppose that a and c are nonintersecting lines. It is possible, then, to draw a line a' within $\angle R$ which is parallel to c (Fig. IV–21). With respect to b, this line must be an intersecting line, meeting b somewhere in K'. This brings us to the same contradiction we found in the preceding paragraph. We cannot have two right or two left parallels through a point K' to the same line c. There is only one possibility remaining, and that is that a is parallel to c.

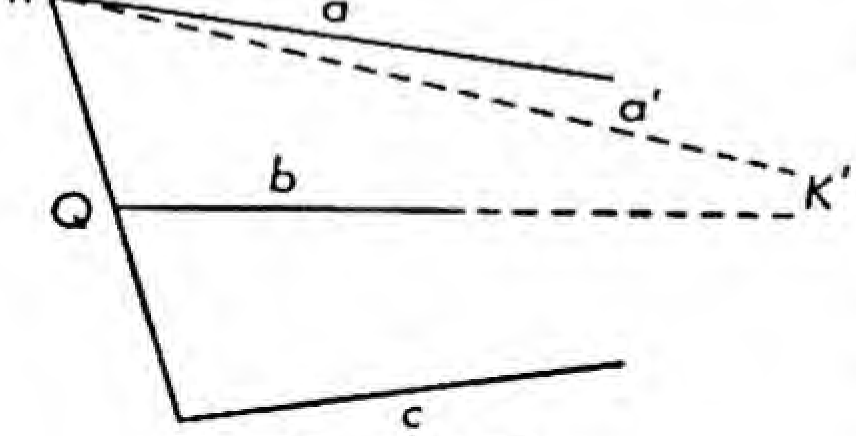


Fig. IV–21

We are led to the conclusion that two lines parallel to a third line are parallel to each other. Had we started with $a \parallel c$ and $b \parallel c$, the conclusion that $a \parallel b$ would be similarly obtainable.

EXERCISES (IV-6)

1. As with the Riemannian geometry, it is desirable to conceive of a model for the new geometry. In both cases the model is borrowed from Euclidean geometry, with altered interpretation of some of the undefined terms. Thus the great circles of Euclid become the lines of Riemann.

Felix Klein proposed an interesting model for Lobachevky's geometry. The Euclidean parallel requires the concept of the infinite line and plane because, in the postulate, parallels never meet. The Klein model arbitrarily draws the infinite into the visual fold. Let us imagine our plane enclosed by a circle, every point of which is at an infinite distance from any point within the circle. (The concept of distance in the last sentence indicates that this notion will have to be redefined for Lobachevskian geometry. However, we proceed with the model on an intuitive basis.)

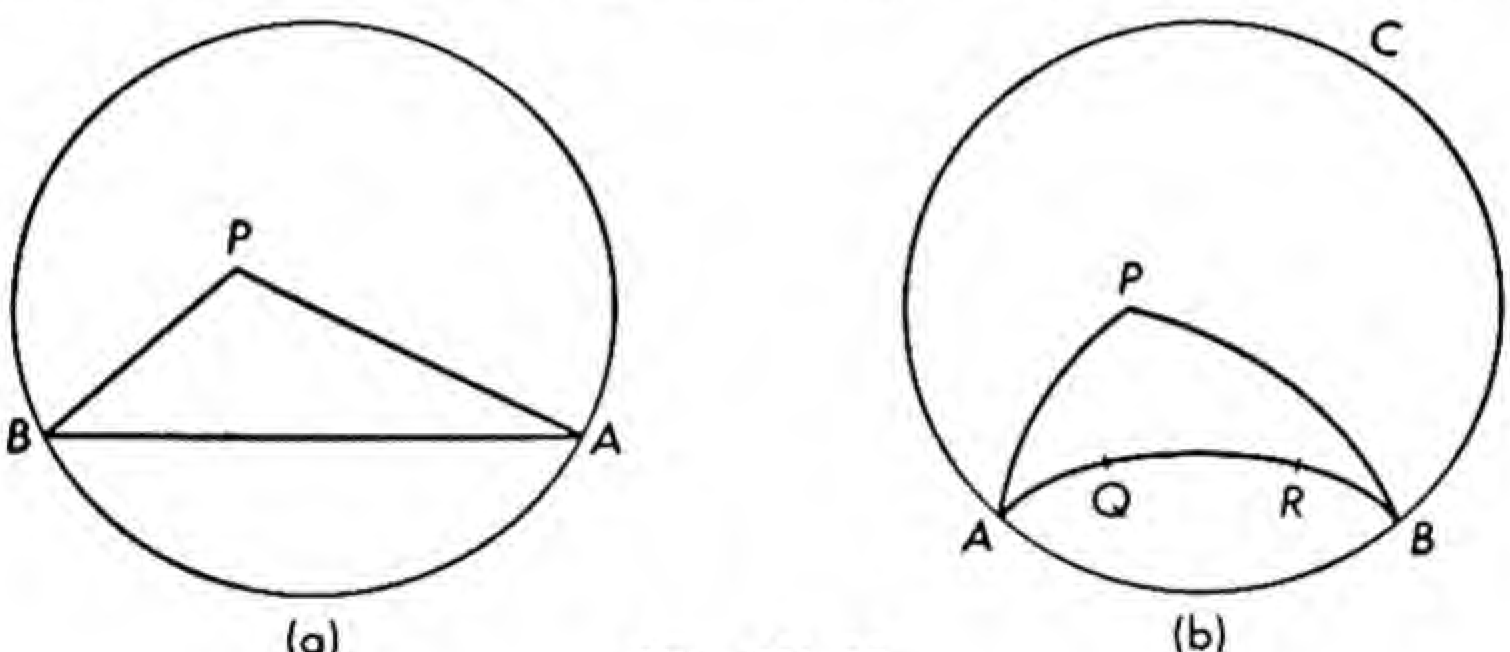


Fig. IV–22

a. We let Euclid's chord AB (Fig. IV–22a) become the Lobachevskian line. AB is infinite in length. If P is any point in the Lobachevskian plane, within the Euclidean circle, then PA and PB represent the right and left parallels, respectively.

b. Use the model to illustrate the intersectors and nonintersectors.

c. Draw any other right parallel to AB and explain, model-wise, why the new line is parallel to PA.

2. We define the angles A and B, the angles at infinity, to be 0° each. What is the sum of the angles of the $\triangle PAB$?

3. Another model was contributed by Poincaré. Here, too (Fig. IV–22b), the entire plane is contained within a circle which, as in Klein's model, represents the points at infinity. Distance between any two points Q and R is measured along the geodesic QR. If continued, this and all other geodesics in this model will lie on a portion of a circle which meets C in two points, A and B. Actually C and the extended geodesic QR meet at right angles. Although the concept of angles between two curves is something that still lies many pages ahead of us, the lack of its explanation here need not detain us now.

a. If we take any point P, not on QR or QR extended, the points P and A determine another geodesic, as do the points P and B. Indeed any two points determine a geodesic. The lines (meaning geodesics, of course) PA and PB represent the left and right parallels to QR.

b. Use the model to illustrate intersectors and nonintersectors.

c. Draw any other right parallel to QR and explain, model-wise, why the new line is parallel to PB.

d. As in Klein's model, the angles A and B may be taken as 0° each. What limitation shall we place on the size of $\angle P$? What then may be said about the sum of the angles of $\triangle PAB$?

7. PERPENDICULAR DISTANCE AND PARALLELS

We can go further now and consider the two sets of parallels, both right or both left, with equal perpendiculars, as in Fig. IV–23. How do the angles

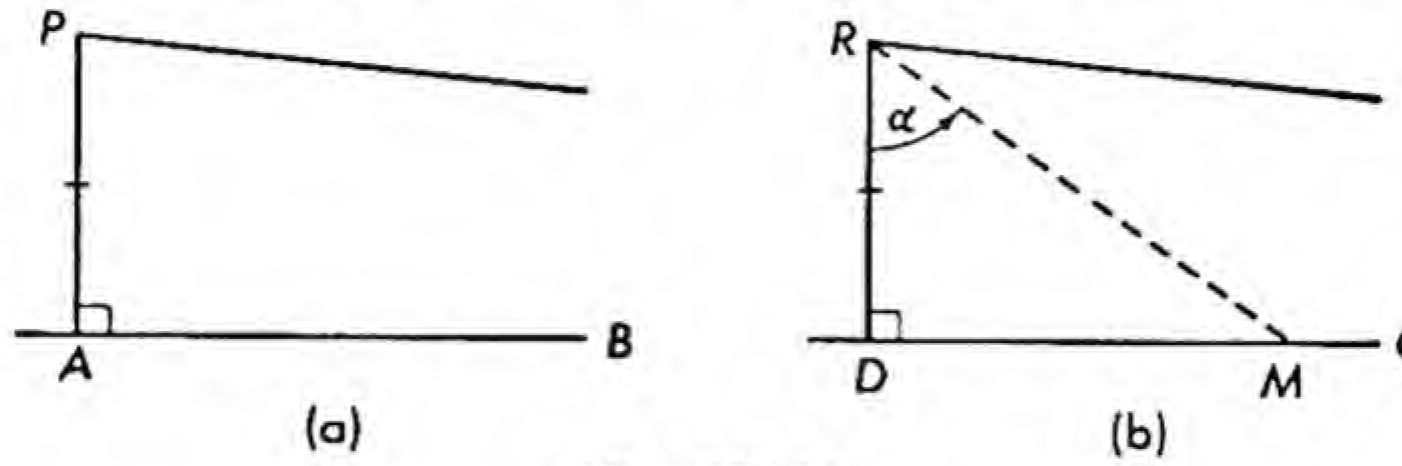


Fig. IV–23

of parallelism compare? Suppose that they are unequal and that $\angle R$ is the larger of the two. In this case some part of $\angle R$ is equal to $\angle P$. Then a line RM can be drawn to cut off a part of $\angle R$ (namely, α, alpha) which is equal to $\angle P$ (Fig. IV–23). Since RM falls within $\angle R$, it must be an intersecting line with respect to DC. This determines point M on DC.

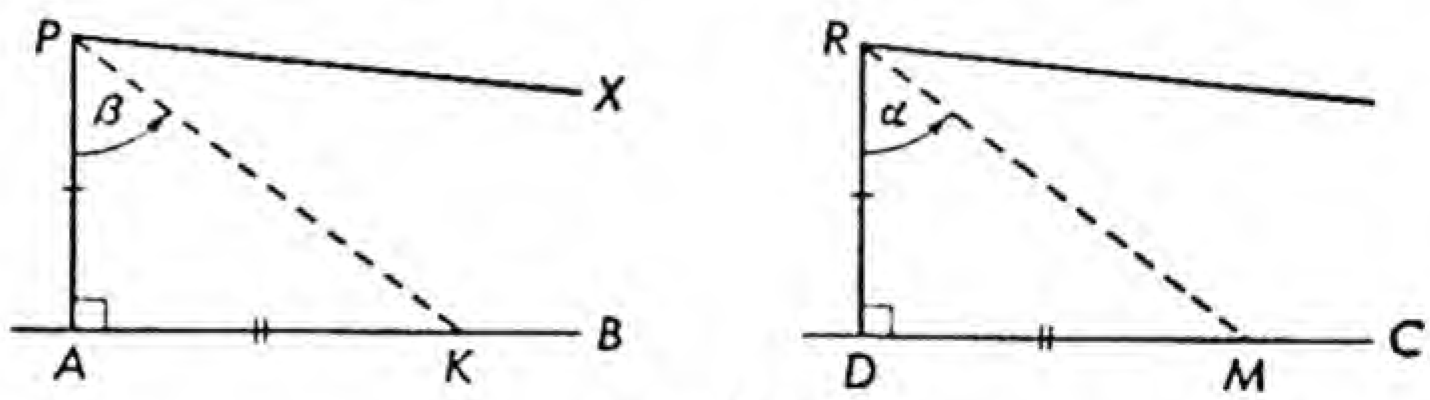


Fig. IV–24

We can now measure off $AK = DM$ and draw PK (Fig. IV–24). The triangles KAP and MDR are congruent by *sas*. Thus, $\angle \alpha = \angle \beta$ (beta); since α was taken equal to P, β and P are the same angle. Consequently PK coincides with PX, which is a parallel to AB. So, there is no point such as K. The assumption that the angles P and R are unequal leads to this contradiction. We take it, then, that $\angle P = \angle R$, and we conclude that, in Lobachevskian geometry, *two sets of parallels with equal perpendiculars have equal angles of parallelism*. The angle of parallelism is a constant for a given perpendicular. An equivalent view of this situation is that *if the angles of parallelism are unequal, the perpendiculars are unequal.*

Would the converse be true too? Would the perpendiculars be equal if the angles of parallelism were equal? This converse is true. The proof is left to the reader, who will find some suggestions in the exercise section that follows this article.

It now remains to determine whether the larger angles of parallelism are associated with the larger or the smaller perpendiculars.

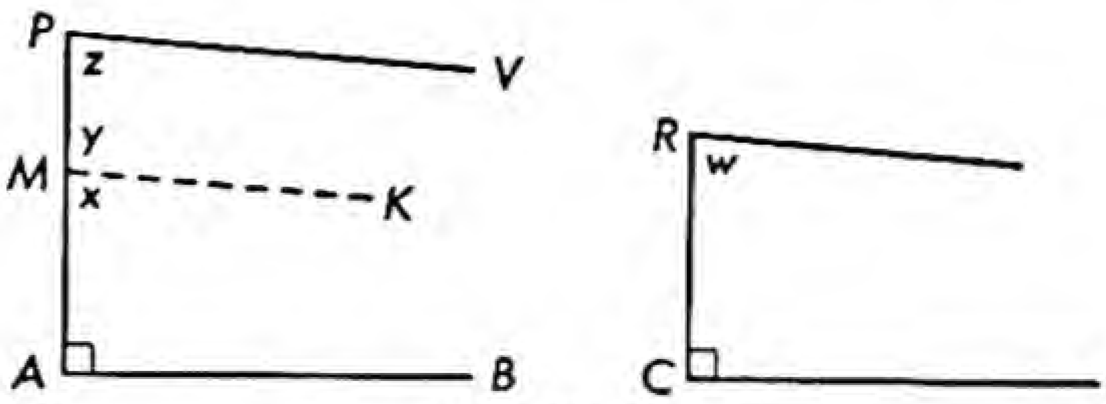


Fig. IV-25

Let us start with two sets of parallels and with the perpendiculars PA and RC, where $PA > RC$ (Fig. IV-25). We know that the angles of parallelism, z and w, are unequal. But, which is the greater?

We measure off $AM = RC$ and let MK be the right parallel to AB through M. This makes MK also parallel to PV and $x = w$. From our knowledge of parallelism we know that

$$z + y < \pi$$

but

$$x + y = \pi$$

By comparison, or by subtraction of equals from unequals, we obtain

$$z - x < 0$$

so that

$$z < x$$

and, by substitution,

$$z < w$$

We have shown that *the larger perpendicular has the smaller angle of parallelism.* The converse is also true (see exercises IV-7).

We can summarize much of what has preceded by saying that **the angles of parallelism and the perpendiculars vary in the opposite sense.** The larger of either is associated with the smaller of the other. The equality of either goes with the equality of the other.

EXERCISES (IV-7)

1. There has been some employment of a **contrapositive** of a proposition. This will be examined in greater detail later. Briefly, *a contrapositive of a proposition is a proposition with hypothesis and conclusion interchanged and each negated.*

PROPOSITION: If m, then p.

CONTRAPOSITIVE: If not p, then not m.

We shall see that these are always equivalent propositions. If one is true, so is the other. If one is false, so is the other.

List a proposition and its contrapositive that was utilized in this article.

2. Write the contrapositives of the two isosceles triangle theorems of Euclidean geometry.

3. Take any theorem and prove it through its contrapositive.

4. We have also employed a form of reasoning referred to variously as *indirect reasoning, reasoning by elimination,* and *reductio ad absurdum.* While variations are possible, essentially what takes place is that all the possibilities with respect to some relationship are studied. If it is possible to eliminate as false all but one, then that one is true. Thus, with respect to two angles of a triangle, A and B, we have as possibilities (1) $A > B$; (2) $A = B$; (3) $A < B$. If it is shown that suppositions (1) and (2) are false, then (3) is the correct result.

Try this approach to prove that two lines perpendicular to the same line are nonintersecting lines in Lobatchevskian geometry.

5. a. Prove that if two angles of parallelism are equal, then their perpendiculars are equal. The diagram (Fig. IV–26) suggests the use of the indirect proof. Assume that the perpendiculars are not equal, and develop the contradiction as implicitly indicated in the diagram.
   b. State the contrapositive of the theorem in (a).

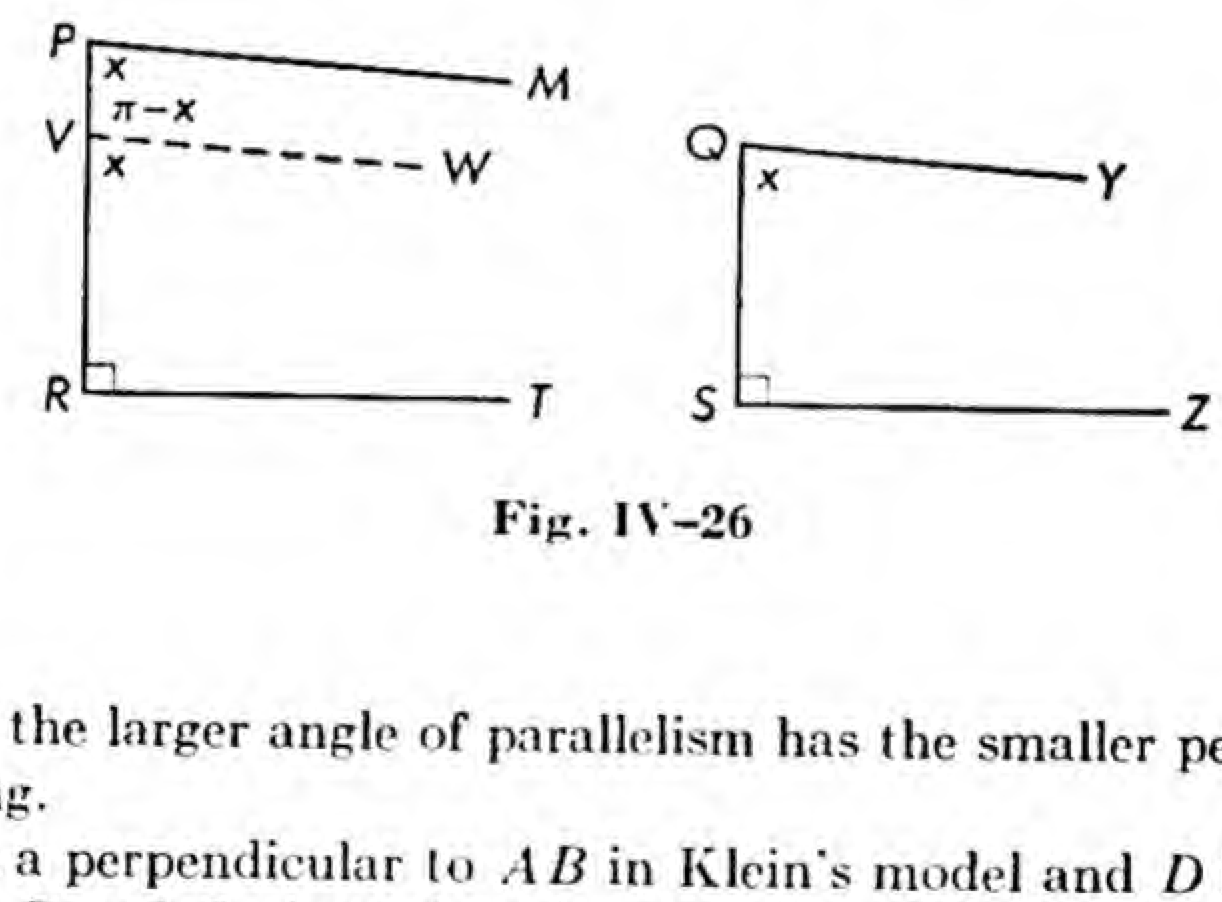


Fig. IV–26

6. Prove that the larger angle of parallelism has the smaller perpendicular. Use indirect reasoning.

7. a. If PC is a perpendicular to AB in Klein's model and D is a point on PC between P and C, show that the angle of parallelism at D is greater than that at P.
   b. Discuss the corresponding situation with D on PC extended past P.

8. Drop a perpendicular geodesic from P to AB in Poincaré's model (Fig. IV–22b). Take any other point on this perpendicular and connect it to B. Use this figure to illustrate at least one Lobachevskian theorem. (The Euclidean theorem, that the exterior angle of a triangle is greater than either remote interior angle, is applicable here.)

8. PARALLELS AND OTHER PERPENDICULARS

We know that in Euclidean geometry, parallels are everywhere equidistant; that perpendiculars dropped from one parallel to the other are

always the same distance between the parallels. The variations that have been taking place here in perpendiculars and angles of parallelism should put us on guard against hasty conclusions regarding other perpendiculars.

We consider two sets of parallels which may be both right, both left, or each one of two kinds (Fig. IV–27). We start with the perpendiculars AB and CD equal, and so the angles of parallelism are equal. By taking $BE = DF$, we can get two perpendiculars, GE and HF, at equal distances

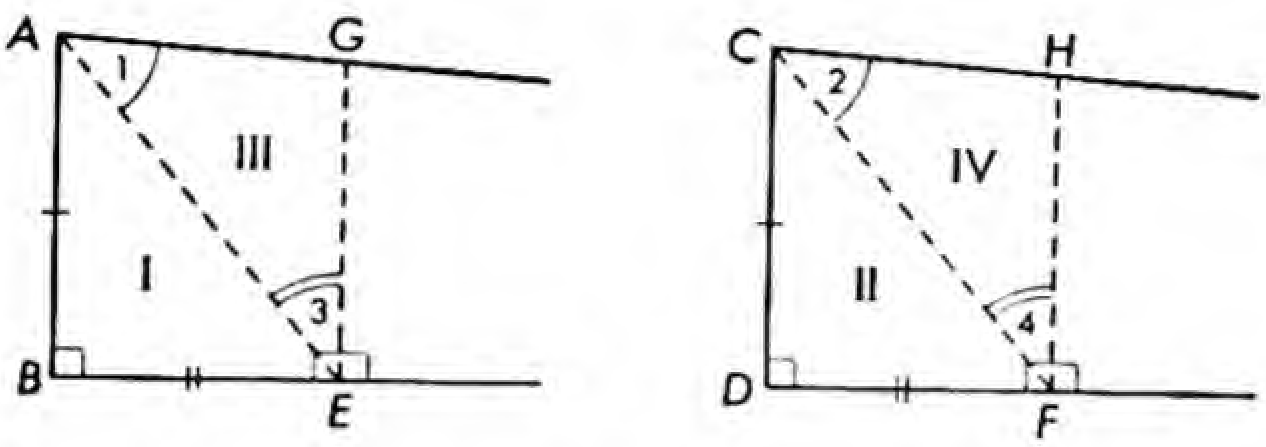


Fig. IV–27

from the first two perpendiculars. How do the new perpendiculars compare?

Triangles I and II are congruent by *sas*. This leads to the congruence of III and IV through $AE = CF$, $\angle 1 = \angle 2$ by subtraction of equals, and $\angle 3 = \angle 4$ by complements of equals. So, $GE = HF$. *If perpendiculars are drawn between parallels at equal distances from the feet of equal perpendiculars, then the new perpendiculars are equal.* With this information we go on to show that *perpendiculars between parallels grow shorter in the direction of parallelism* (Fig. IV–28). Suggestions for proof are given in the exercises for this article.

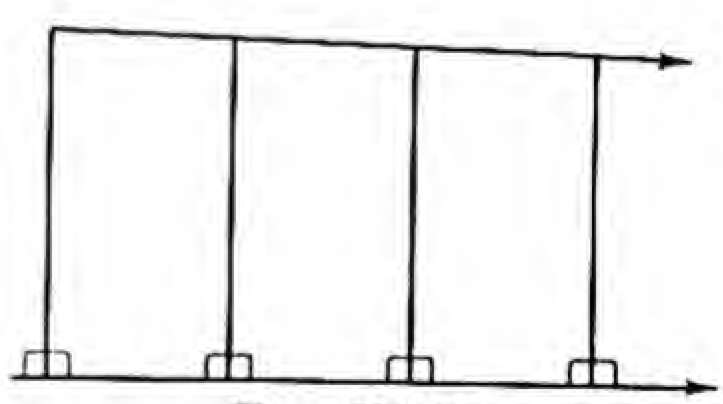


Fig. IV–28

The last conclusion yields almost immediately another startling consequence. If ED (Fig. IV–29) is the shorter of two perpendiculars, then by an earlier theorem, $y > x$ or equivalently

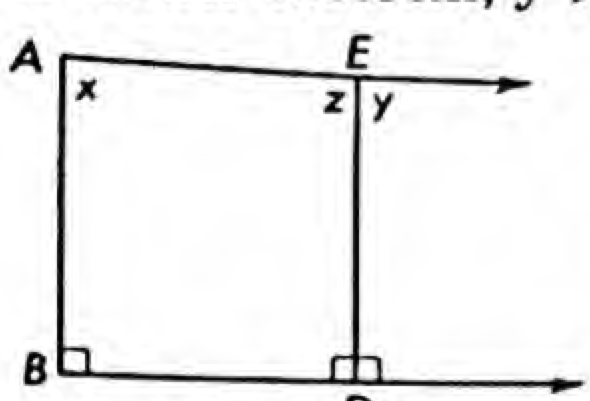


Fig. IV–29

	$x < y$
Since	$z = z$
it follows that	$x + z < y + z = \pi$

The sum of the nonright angles of a birectangular quadrilateral, including a pair of parallels, is less than π, or 180°. This is a hint of coming events, that is, that *the sum of the angles of a quadrilateral is less than* 360°.

EXERCISES (IV-8)

1. The lines AE and BD (Figs. IV-30a, b) are right parallels, and FC is the perpendicular bisector of BD.
   a. Show that $\angle 2$ is obtuse.
   b. If a left parallel is drawn to BD through F, why does it lie within $\angle AFC$?
   c. Why does $GB = ED$?
   d. Why is $AB > ED$?
   e. State the theorem just proved.

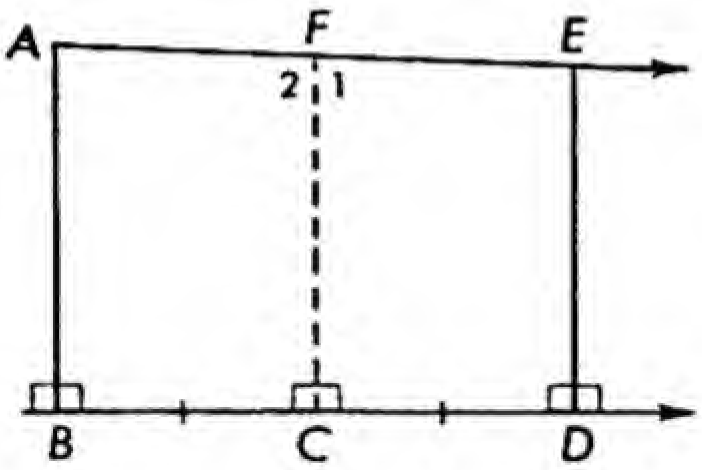


Fig. IV-30a

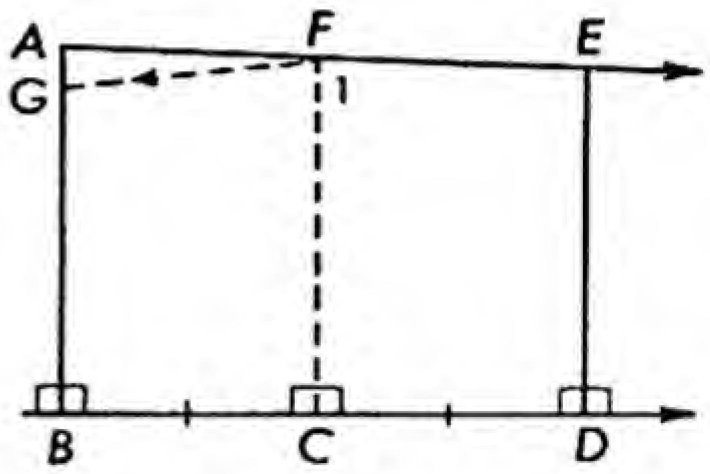


Fig. IV-30b

2. a. Use Klein's model to illustrate the theorem in the preceding exercise.
   b. Use Poincaré's model to illustrate the same theorem.
3. Compare the sum of the angles of the quadrilateral $ABDE$ with $FCDE$ in Fig. IV-30(b).
4. If BD in Fig. IV-30(b) were made continually larger, what effect would this have on the sums of the angles of the quadrilaterals $ABDE$ and $FCDE$?

9. THE ISOSCELES BIRECTANGULAR QUADRILATERAL

We return again to the isosceles birectangular quadrilateral, containing two right angles and two equal sides (Fig. IV-31). This time it is in a Lobatchevskian environment.

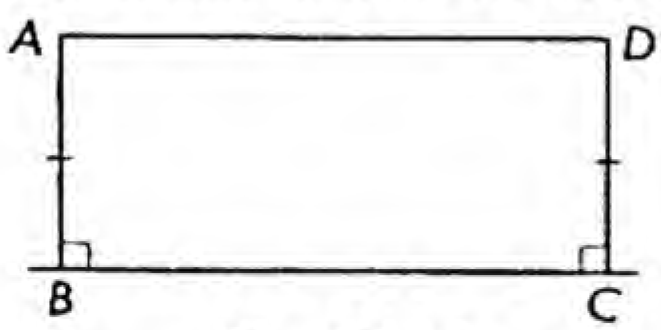


Fig. IV-31

If a parallel is drawn to BC through A, it cuts off a segment on DC less than AB by a previous theorem. The parallel, then, must fall within $\angle A$, making $\angle A$ larger than the angle of parallelism at A. By the same token, if we considered drawing a left parallel from D, $\angle D$ would turn out to be larger than the left angle of parallelism at D. The combination of these two conclusions means that AD is a nonintersecting line with respect to BC. As before, we call angles A and D the *summit angles* and AD the *summit*.

The symmetrical situation that takes place at both angles A and D suggests that *the two summit angles are equal.* The proof is suggested by the markings in Fig. IV–32. This leaves us with the question whether the angles are acute, right or obtuse.

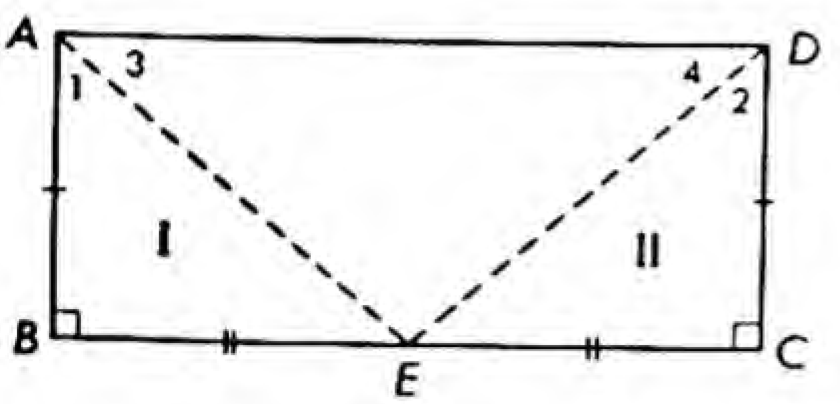


Fig. IV–32

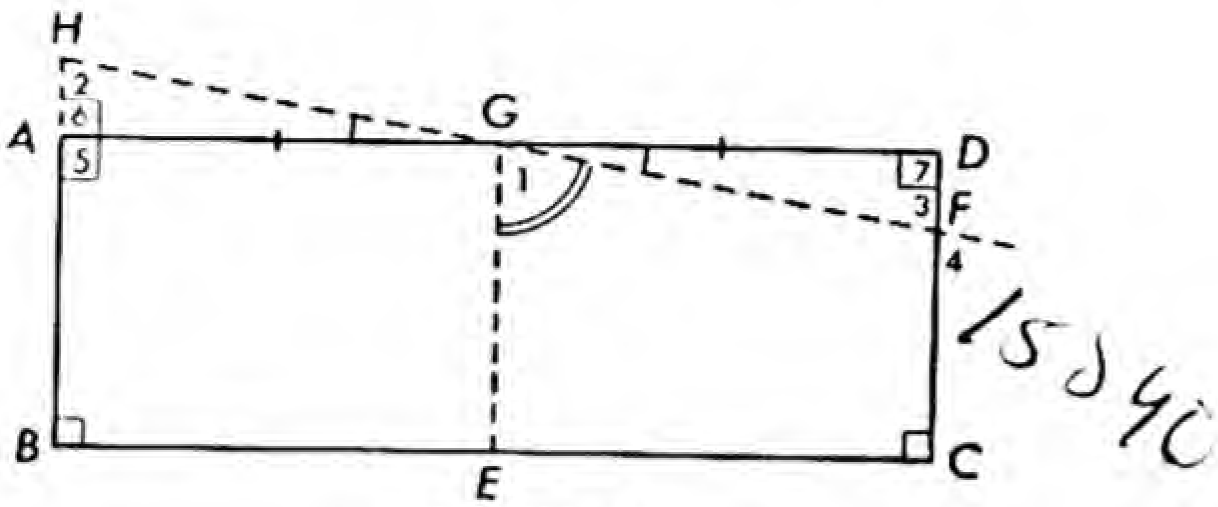


Fig. IV–33

We examine these possibilities now. Suppose that we assume, first, that the angles are right angles (Fig. IV–33). Through G, the midpoint of AD, we draw a right parallel to BC. Since AD is a nonintersecting line with regard to BC, the right parallel will cross DC at F and BA extended at H. The two triangles formed are congruent by *asa*, from which we get $\angle 2 = \angle 3$. Because of the vertical angles 3 and 4, this makes $\angle 2 = \angle 4$. Now, this is a definite contradiction. The lines HB and FC are unequal, and angles 2 and 4 are angles of parallelism for the same line CB. Therefore, the angles cannot be equal. In fact $\angle 4$ is supposed to be larger than $\angle 2$. No, the summit angles cannot be right angles.

Suppose that we consider the possibility that both summit angles are obtuse (Fig. IV–34). We draw the same right parallel through G, the midpoint of AD. Now, if $\angle 5$ is obtuse, $\angle 6$ is acute. Since D is also assumed to be obtuse, a line AK can be drawn, making $\angle GAK = \angle 7$, an obtuse angle. That is, we make $\angle GAK = \angle 7$. This leads to the congruence of the triangles AKG and GFD. Consequently $\angle 9 = \angle 3 = \angle 4$.

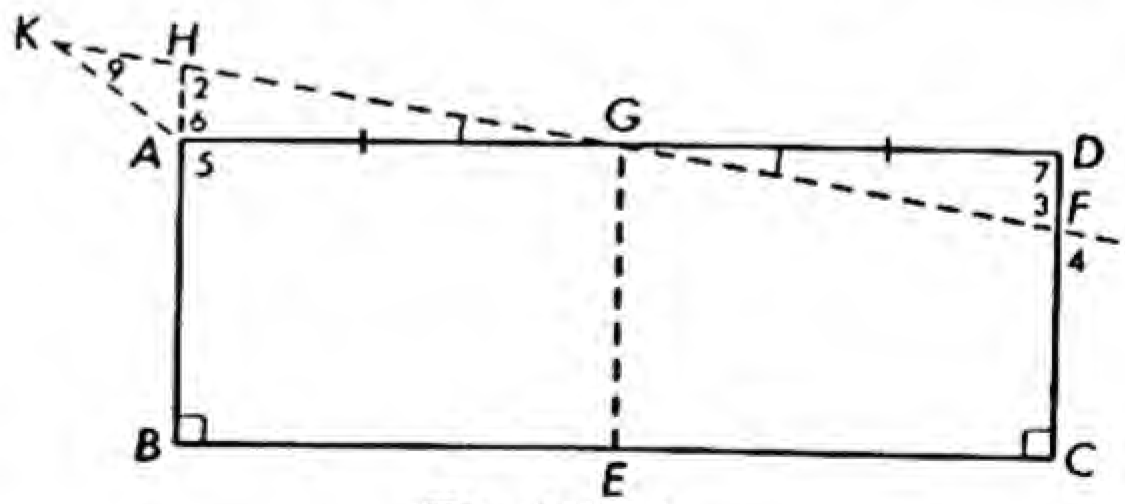


Fig. IV–34

If a perpendicular were dropped from K to BC extended, $\angle 9$ would be part of an angle of parallelism, all of which would be less than $\angle 4$. Consequently $\angle 9$ is less than $\angle 4$. But in the preceding paragraph we found the two angles to be equal. So, again our assumption is wrong. The summit angles cannot be obtuse. *The summit angles are equal acute angles.*

The chain reaction, deductively speaking, continues. The last conclusion informs us that the sum of the angles of an isosceles birectangular quadrilateral is less than 2π radians, or 360°. This, in turn, will permit us to reach a conclusion about the sum of the angles of a triangle.

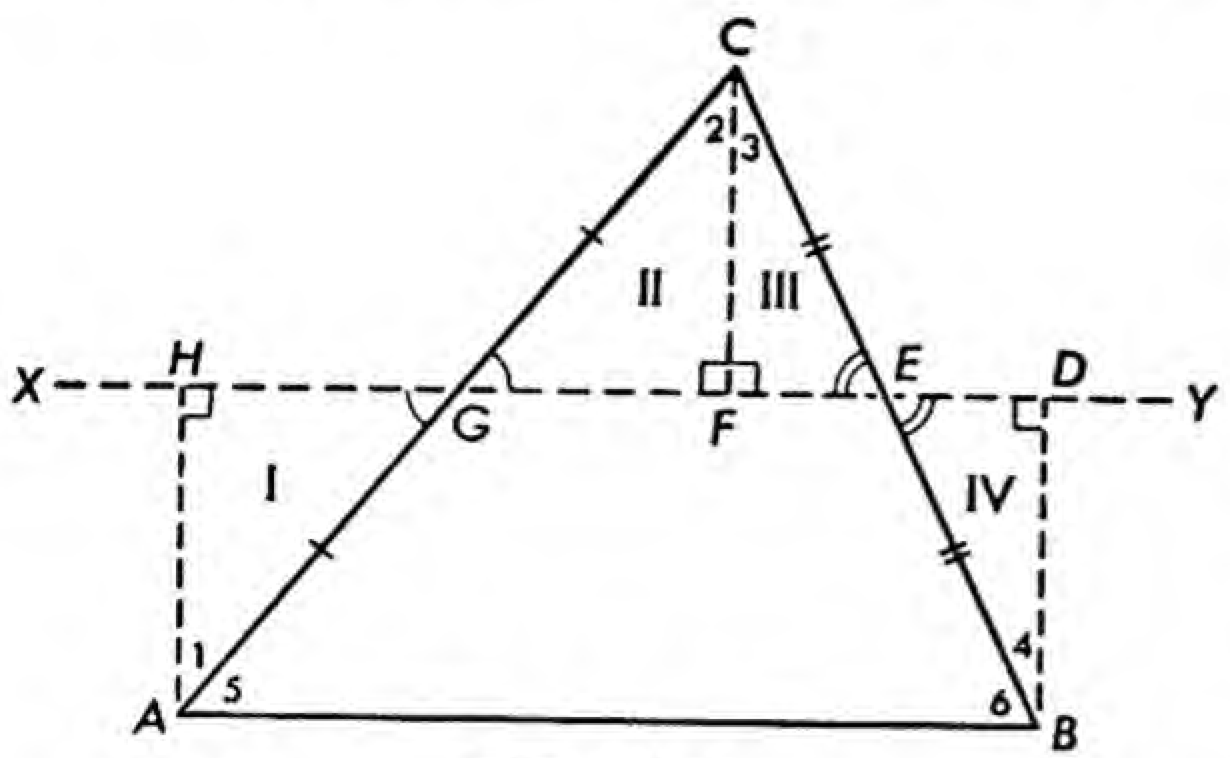


Fig. IV–35

Take $\triangle ABC$ (Fig. IV–35) and make AB the summit of an isosceles birectangular quadrilateral as follows: Through G and E, the midpoints of AC and BC, respectively, draw XY. Then we draw AH, DB, and CF, all perpendicular to XY. $AHDB$ is an isosceles birectangular quadrilateral, and the two pairs of right triangles (I and II, III and IV) are congruent by hypotenuse acute angles, which is a permissible case for congruence. From the congruent triangles, we get

$$\angle 1 = \angle 2$$
$$\angle 4 = \angle 3$$

We may add

$$\angle 5 = \angle 5$$
$$\angle 6 = \angle 6$$

So, by addition, we have

$$(\angle 1 + \angle 5) + (\angle 4 + \angle 6) = \angle 2 + \angle 3 + \angle 5 + \angle 6$$

or

$$\angle HAB + \angle DBA = \angle A + \angle B + \angle C$$

The sum of the angles of the triangle is equal to the sum of the summit angles of the isosceles birectangular quadrilateral. But the latter sum is less than π. Thus

the sum of the angles of a triangle in Lobachevsky geometry is less than 180°.

EXERCISES (IV–9)

Unless otherwise stated, problems refer to Lobatchevskian geometry.

1. Compare the known facts concerning the isosceles birectangular quadrilateral in Riemannian and Lobachevskian geometries.

2. In a number of places we have left to intuition the matter of where two lines intersect or whether they do in fact intersect. A series of postulates and theorems concerning *betweenness* is necessary to develop a thoroughly rigorous system. This,

unfortunately, would take us too far afield. Interestingly enough, Euclid's geometry was deficient for over 2000 years in just such respects until the works of David Hilbert and Oswald Veblen at the turn of this century.

Explicitly, suppose that CB and AD (Fig. IV–36) are perpendiculars and that CD is a nonintersecting line. Then $\angle C$ is larger than an angle of parallelism, and a line CF can be drawn parallel to AB. Will CF intersect DA? If so, where? If not, why not? Can you suggest some postulates that will be necessary to close the gap and remove intuition to at least some extent?

3. Compare angles 1 and 2 (Fig. IV–37) for the three geometries.

4. Prove that the line joining the midpoints of the base and summit of an isosceles birectangular quadrilateral is perpendicular to each of them.

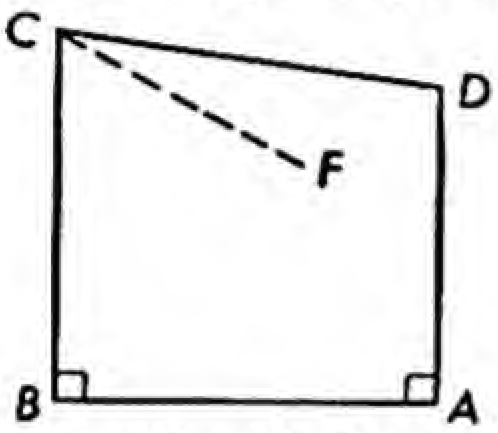


Fig. IV–36

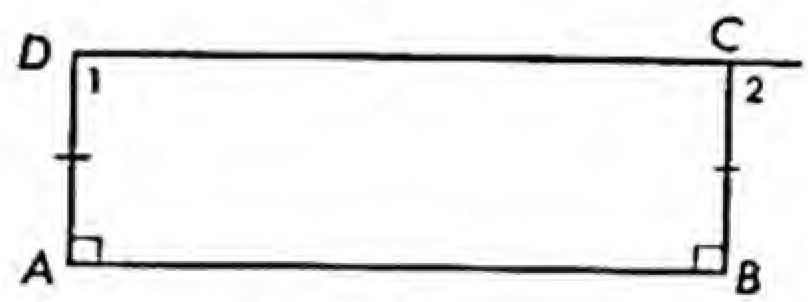


Fig. IV–37

5. $ABCD$ is a trirectangular quadrilateral with right angles at D, A, and B. Prove that (1) $CB > AD$ and (2) $DC > AB$.

6. Prove that the summit is longer than the base in an isosceles birectangular quadrilateral.

7. Prove that the line joining the midpoints of two sides of a triangle is less than one-half the third side in Lobatchevskian geometry and greater than one-half in Riemannian geometry.

8. Show that two nonintersecting lines have, at most, one common perpendicular.

9. Use the diagram in the text where the sum of the angles of a Lobatchevskian triangle was shown to be less than π, and show that the sum of the angles of a Riemannian triangle is more than π.

10. If $AD \parallel BC$ and $AB \perp BC$, then $\angle A$ is an angle of parallelism (Fig. IV–38). E is on AB, and EX is drawn perpendicular to AB. Explain why or why not

a. $\angle A$ is an angle of parallelism relative to EX.

b. $AD \parallel EX$

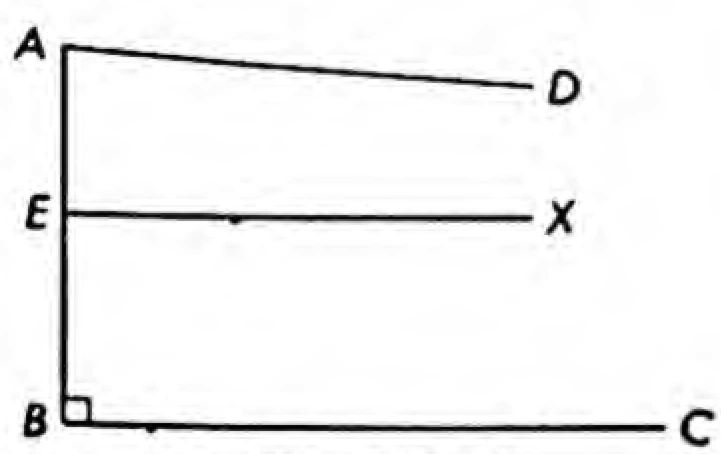


Fig. IV–38

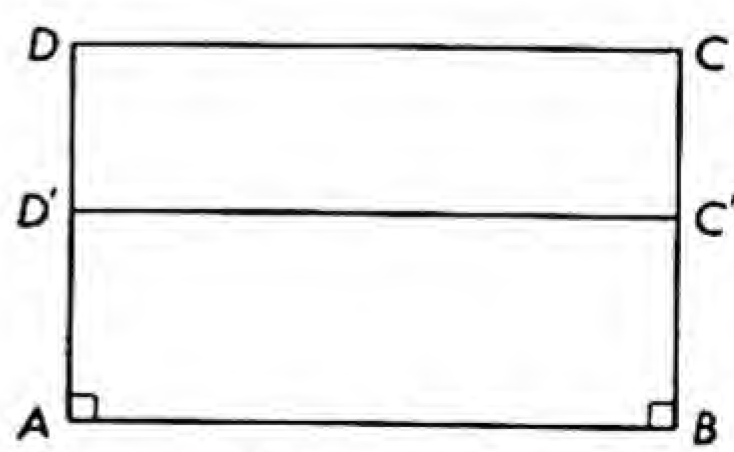


Fig. IV–39

11. Prove that the sum of the angles of a triangle is less than 180°, using an obtuse triangle. Take the figure in the text, but make either $\angle A$ or $\angle B$ obtuse.

12. $ABCD$ is an isosceles birectangular quadrilateral (it is called by some a *Saccheri quadrilateral*). $AD' = BC'$ (Fig. IV–39). Prove that $\angle AD'C' > \angle D$. This suggests that the deficit from 2π, or 360°, of a larger quadrilateral is more than that for a smaller quadrilateral. (Areas may be related to the excesses or deficits of a triangle above or below 180°.)

10. A LOBACHEVSKIAN WORLD

The reader who senses too intimately the flat plane view of Euclidean geometry may still want a three-dimensional model of a surface on which Lobachevskian geometry can be visualized. In part, this is possible with the kind of surface called a *pseudosphere*. Of course the straight line on the surface would be the geodesic—the generalization of the straight line concept on any surface.

If a curve such as ABC is rotated around its asymptote m, the surface formed would be a pseudosphere (Fig. IV–40). If three geodesics intersect to form a triangle (Fig. IV–41), the sum of the angles would be less than π.

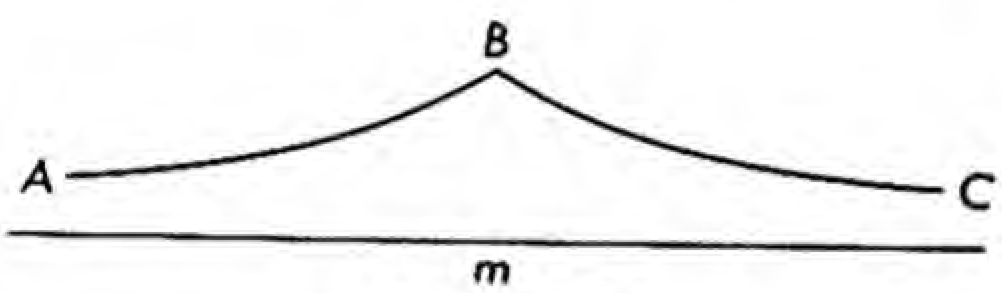


Fig. IV–40

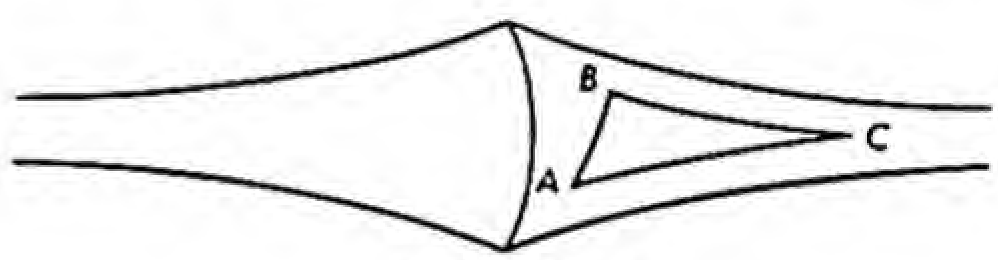


Fig. IV–41

This fact and the corresponding facts in the other geometries are not sufficient to decide experimentally the geometric nature of our space, for measurements, even on astronomic scales, show such slight deviations from 180° in a triangle as to be well within the experimental error.

For local purposes the major criterion in the selection of a geometry is that of simplicity. In that respect, Euclidean geometry is by far the best to use, and this we do. However, for astronomic purposes and for scientific microscopic purposes, a non-Euclidean geometry may be better for analytic reasons. We have mentioned earlier that Einstein employed a slight variation of Riemannian geometry with well-known success.

IV–10 REVIEW

1. If AB and BC are right and left parallels to line m, respectively, prove that the bisector of $\angle ABC$ is perpendicular to m.

2. If AX and BY are right parallels, as are $A'X'$ and $B'Y'$, with $AB = A'B'$ and $\angle A = \angle A'$, then $\angle B = \angle B'$.

3. A and B are two distinct points through which left parallels are drawn to each other with $\angle A = \angle B$. Prove that the parallels are parallel to the perpendicular bisector of AB.

4. If AX, BY, and CZ are parallel to each other in the same direction, and BY is the perpendicular bisector of AC, then $\angle A = \angle C$.

5. Prove that if the summit angles in a birectangular quadrilateral are equal, then the figure is isosceles.

6. Prove that if two triangles agree in *aaa*, they are congruent.

7. Keeping in mind the fact that the perpendiculars between a pair of parallels vary continuously in length, explain the significant, and perhaps surprising, statement that "two pairs of Lobachevskian parallels are always congruent."

8. Imagine $\triangle ABC$, with the "altitude" BD, becoming infinitely large.
   a. To what size is $\angle B$ approaching?
   b If vertices A and C recede infinitely from D at the same time, what do these angles approach in degrees?
   c. What is the limit of the sum of the angles of the triangle described in the parts (a) and (b)?
   d. How would you describe the area of triangle ABC?

V

SYMBOLIC LOGIC

1. INTRODUCTION

We have been engaged for considerable time now in deductive reasoning with its constituent elements of undefined terms, postulates, definitions, and theorems. We have met specialized logical propositions such as the converse and the contrapositive, and a variation in approach such as in indirect reasoning. It is time to pause and examine the reasoning process itself, to focus our attention on the nature of our reasoning. In this way we can better understand the motive power behind the scenes, that which links proposition to proposition and on to a conclusion. The techniques of **symbolic logic** are ideally suited for this undertaking.

The elements of geometry, at least in the early historical stages, were abstracted from observed phenomena. This has been discussed before. Eventually the entities, point and line, were divorced from the concrete, physical objects that prompted them. Point and line came to be two of the undefined elements of the three geometries, deductively developed. In somewhat similar fashion, we can think of many propositions of mathematics; for example:

Triangle ABC is isosceles
Quadrilateral $ABCD$ is isosceles and birectangular

and in abstraction we refer to them symbolically as propositions $p, q, \cdots$, where the letters are place holders for propositions. This evolution leads us to the undefined elements called *propositions*, which are symbolized by letters. Meaning is discarded and only form remains. Indeed some suggest that the word *proposition* ought not to be retained because it carries with it too strong a suggestion of meanings of the statements rather than only the forms of the statements. *Symbolic logic concerns itself with forms,*

logical forms. Of course we shall need other undefined elements and postulates.

2. PRIMITIVE TERMS AND TRUTH VALUES

We have had occasion to speak of the *negation* of a proposition. The negation of *the lines are parallel* is that *the lines are nonparallel*, or *the lines are not parallel*. If p represents the first proposition $\sim p$ will symbolize the negation of p and is read *non-p*. The *curl* symbol ($\sim$) becomes another undefined element.

If p is true, $\sim p$ is false, and if p is false, $\sim p$ is true. This may be summarized conveniently in the following table which is referred to as the **truth-value** table, where T and F become two more undefined elements. We postulate that

(a) **a proposition is either T or F but not both T and F.**

(b) **Negation Truth Values**

p	$\sim p$
T	F
F	T

A compound statement such as *the angles are vertical* ***and*** *the lines are perpendicular* illustrates the all-important logical word *and*, which connects the two simple propositions (1) the angles are vertical, and (2) the lines are perpendicular. Let this symbol be $\wedge$, which is read as *and*. The compound statement may be symbolized as $\mathbf{p \wedge q}$. This is called a *conjunction of two propositions*, or just a *conjunction*.

The conjunction can also be summarized in a truth-value table. If two propositions, p and q, are joined conjunctively, $\mathbf{p \wedge q}$ will depend somehow on the truth values of p and q separately. We postulate that

$p \wedge q$ is true only when p is true and q is true.

This is detailed in the following table:

Conjunction Truth-Values $p \wedge q$

	p	q	$p \wedge q$
Postulate	T	T	T
	T	F	F
	F	T	F
	F	F	F

The word **or** is another of the important logical terms. However, in ordinary usage, **or** may have one of two meanings. There is the **inclusive** meaning which refers to **either one or the other or both** and the **exclusive** meaning which refers to **either one or the other but not both.** For our purposes, the inclusive case is preferred. This is the **and/or** interpretation used in legal phraseology. We shall call this use of **or** a **disjunction** with its symbol as $\vee$.

In **either m is a root of the equation or m is a prime number,** we take it, unless specifically stated otherwise, that m may be a root or m may be a prime number or m may be both a root and a prime number, $p \vee q$. What is distinctly ruled out by this interpretation is that *it cannot be that m is not a root and not a prime number*. Thus a disjunction of p and q means that

It is false that		p is false	and	q is false	
$\sim$	(	$\sim p$	$\wedge$	$\sim q$	)

The last line is a symbolic translation of the line above it. Note the use of parentheses to show that the first negation symbol modifies the whole expression. We take the second line to be the definition of the disjunction $p \vee q$. Note that the definition involves only postulates and undefined terms.

DEFINITION: $p \vee q$ means $\sim(\sim p \wedge \sim q)$.

To obtain the full import of $p \vee q$, we set up a truth-value table in conformity with the denfition. The definition can be built up operationally by starting with the independent propositions p and q, which can have only four paired sets of truth values: $T\ T$, $T\ F$, $F\ T$, and $F\ F$. By negation of the first two columns of the table, we get the columns for $\sim p$ and $\sim q$. The postulate on conjunction and its truth table show us how to complete the column for $\sim p \wedge \sim q$. Finally, negation again allows us to complete the table to get $\sim(\sim p \wedge \sim q)$, which is synonymous with $p \vee q$.

Disjunction Truth Values $p \vee q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F
					$\sim(\sim p \wedge \sim q)$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The short table on the right represents a summary of the table on the left. The values of the disjunction are shown in reference to the original propositions p and q.

Thus we see clearly that the disjunction is false when both constituent propositions are false; otherwise, the disjunction is true.

It should be noted that there are two kinds of operations: *unary* and *binary*. The curl ($\sim$, negation) operation is unary and is applied to a single column. Thus column 3 is obtained by negating column 1; column 4 negates column 2, and column 6 negates column 5. In these cases, we automatically change from T to F and from F to T. The conjunction operation, and the others to follow, has reference to two columns. Column 5 in the preceding table is the conjunction of columns 3 and 4. By the conjunction postulate, column 5 will be true if preceded by T and T; otherwise, it is false.

The summary disjunction table provides a mechanical decision method for future disjunctions. The disjunction is F if it is a disjunction of an F and F; otherwise, T.

We have gone far enough to be able to introduce a few applications of the symbolism thus far. The propositions in the following are enclosed in parentheses.

a. (b is a root of the equation) and (it is a rational root): $p \wedge q$

b. It is not true that (the trinomial is factorable): $\sim p$

c. Neither (is the triangle a right triangle) nor (an obtuse triangle): $\sim p \wedge \sim q$

Neither· · ·**nor** means **not this**· · ·**and not that**· · ·

To permit certain forms to stand out clearly, it is sometimes necessary to take a little liberty with the form of the expression. We must also be aware of words that are implicitly understood. The last parentheses in (c) refers to the proposition (the triangle is an obtuse triangle) and not merely the three words shown.

d. (The quadrilateral is) equilateral but not (equiangular): $p \wedge \sim q$

But means **and** in this context.

e. Either (the set is denumerable) or (it has a larger cardinal number): $p \vee q$.

Should we know or intend the **exclusive** sense of **either**· · ·**or,** then we have to affix symbolically the conjunction that not both are true at the same time. This can be done as

$$(p \vee q) \wedge \{\sim(p \wedge q)\}$$

Since this is so cumbersome, it is desirable to modify the disjunction symbol to connote the exclusive sense. We may, for example, use $\underline{\vee}$; $p \,\underline{\vee}\, q$ is read as *either p or q but not both p and q*.

EXERCISES (V–2)

1. Construct the truth-value tables for:

a. $\sim p \wedge q$ (The curl refers only to p.)
b. $p \wedge \sim q$ c. $\sim(p \wedge q)$ d. $\sim p \wedge \sim q$

e. Do any of the above have the same truth values?

2. Construct truth tables for:

a. $\sim p \vee q$ b. $p \vee \sim q$ c. $\sim p \vee \sim q$ d. $\sim(p \vee q)$

e. Do any of these have the same truth values?

3. Each of the following contain two propositions which are to be substituted for p and q in exercises 1 and 2. Make the respective substitutions and state the new propositions in good form:

a. The roots of the equation are rational; the roots are equal.
b. The class of prime numbers is infinite; the class is denumerable.
c. The sum of two rational numbers is a rational number; their product is rational.
d. The sine is positive in quadrant II; the cosine is negative in II.
e. Education is essential for democracy; democracy is responsive to the needs of the individual.
f. Representatives are responsive to community needs; representatives are responsive to party policy.

4. By reference to your truth tables in exercises 1 and 2, construct summary tables for:

a. The negation of a conjunction of two propositions.
b. The negation of the disjunction of two propositions.

5. Use the conclusions of exercise 4 to write the negation of the following propositions, replacing a conjunction with a disjunction, and conversely (compare exercises 1c and 2c; also 1d and 2d):

a. The opposite sides of a figure are parallel and equal.
b. The angles are supplementary and equal.
c. The geometry is Lobachevskian or Riemannian.
d. The angle is acute and the lines are parallel.
e. The man is a dupe and a liar.

3. IMPLICATION

One of the most frequently used compound sentences in mathematics is of the *If* · · ·, *then* · · · form: If a triangle is isosceles, then the base angles are equal. In general, **If p, then q.** This is frequently referred to as p **implies** q. The *if* · · ·*then* form is an **implication.**

We should first attend to the need for a symbol for *implies*. This shall be taken as $\rightarrow$. For $p \rightarrow q$, we read p **implies** q.

Of course, in seeking a definition, we shall be mindful of usual intentions and of the logical symbols developed thus far. When we say, *if two angles are vertical, then they are equal,* we mean *that it cannot be true that the angles are vertical but that they are not equal;* the conjunction of (vertical angles and unequal angles) is false.

p* implies *q means that it is not true that (p is true and q is false).

$\boldsymbol{p \rightarrow q}$ means that $\boldsymbol{\sim(p \wedge \sim q)}$.

The isosceles triangle theorem means that we cannot have the conjunction of an isosceles triangle and a triangle that has not equal base angles. In Lobatchevskian geometry, to say that *If the angles of parallelism are equal, the perpendiculars are equal* means that in this geometry we cannot have equal angles of parallelism where the perpendiculars are not equal.

Once we grasp the definition of implication, we are in a position to construct its truth table too. By examining the proposition $\sim(p \wedge \sim q)$, we see that our table must have the following headings: p, q, $\sim q$, $p \wedge \sim q$, and finally $\sim(p \wedge \sim q)$.

Implication $\boldsymbol{p \rightarrow q}$

p	q	$\sim q$	$p \wedge \sim q$	$p \rightarrow q$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T
				$\sim(p \wedge \sim q)$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The third column is the negation of the second, and the fifth is the negation of the fourth. The fourth column is the conjunction of one and three, and by the conjunction table, a T is the conjunction only of two T's. The summary table will be used for reference.

The consequences of the definition of implication frequently leaves one with the feeling that we "got more than we bargained for." By ruling out the false implication represented by line two in the summary table, it turns out that all the other implications are true, as indicated in lines 1, 3, and 4. The first line is readily acceptable. The third line will be a little difficult to get used to; that is, that a true proposition q is implied by a false proposition. Likewise, the fourth line indicates that a false proposition implies a false proposition.

EXERCISES (V-3)

1. Construct the implication table for

   a. $q \rightarrow \sim r$ b. $\sim q \rightarrow r$

2 If q is the proposition that *x is an even number* and r is the proposition that *x is divisible by 3*, translate the following propositions:

a. $q \rightarrow \sim r$ b. $\sim q \rightarrow r$ c. $q \rightarrow r$

3. Let p be the proposition that *$\triangle ABC$ is isosceles* and q the proposition that *the vertex angle of $\triangle ABC$ is acute*. Express four implications using the propositions p, $\sim p$, q, and $\sim q$.

4. If p and q are translations of *m is an intersector* and *m is a perpendicular*, respectively, write the translations of

a. If m is an intersector, then it is a perpendicular.
b. M is a perpendicular if it is an intersector.
c. If m is an intersector or m is a perpendicular, then m is an intersector.
d. If m is not a perpendicular, then it is an intersector.

5. If p, q, and r are translations of $\angle A = \angle A'$, $\angle B = \angle B'$, and $\triangle ABC \sim \triangle A'B'C'$ respectively, write a translation of *two triangles are similar if two angles of one are equal to two angles of the other.*

6. If p is the proposition that *n is divisible by 2*, q is the proposition that *n is divisible by 3*, and r is the proposition that *n is divisible by 6*, express symbolically the implication that *if a number is divisible by 2 as well as by 3, then it is divisible by 6.*

4. EQUIVALENCES

We have had occasion to use expressions such as *means the same as*, *equivalent*, and similar connotations. Propositions are *equivalent* in the logical sense *if one proposition may be substituted for the other* (as x for y, if $x = y$ in the algebraic sense). We shall use the symbol $\leftrightarrow$ for equivalence.

In order that propositions p and q have the status of equivalence, we must define them so that p implies q and q implies p. Symbolically,

$$\boldsymbol{p \leftrightarrow q} \quad \text{means that} \quad \boldsymbol{(p \rightarrow q) \wedge (q \rightarrow p)}$$

A rectangle is an equiangular quadrilateral represents an equivalence between (*the figure is*) *a rectangle* and (*the figure is*) *an equiangular quadrilateral.* To be one is to be the other, $(p \rightarrow q) \wedge (q \rightarrow p)$. A figure cannot be one of these and at the same time not the other. This last sentence, in symbolic terms, is $\sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$. Each part of this conjunction is nothing more than the definition of an implication. So, this viewpoint leads to the same definition of equivalence.

Equivalence is frequently described by the expression *if and only if*. *If a triangle is a right triangle, then $a^2 + b^2 = c^2$.* We have also proved that *if $a^2 + b^2 = c^2$ in a triangle, then the triangle is a right triangle.* Thus the two propositions in these implications are equivalent. If one, then the other. This is summarized in words, as *$a^2 + b^2 = c^2$ if and only if the triangle is a right triangle.*

$$(\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2) \leftrightarrow (\textbf{right triangle})$$

It is time to develop the truth table for equivalences. We need five columns: p, q, $p \to q$, $q \to p$, $(p \to q) \wedge (q \to p)$.

Equivalence $\mathbf{p \leftrightarrow q}$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T
				$(p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Two propositions are equivalent when they are both true or when they are both false.

EXERCISES (V–4)

1. Construct a proposition of logical equivalence involving:
   a. Vertical angles.
   b. Binary number system.
   c. Area.
   d. a^n.
   e. School.
2. a. Show by means of a truth table that a negation of the implication $p \to q$ is $p \wedge \sim q$.
   b. Find another proposition that expresses the negation of the same implication.
3. Complete each of the following:
   a. $ax = b$ if and only if ______.
   b. $a + x = b$ if and only if______.
   c. $2^m = 1$ if and only if______.
   d. A line is a median of a triangle if and only if______.
   e. $m \parallel m'$ if and only if______.
   f. $\sqrt{a - b}$ is imaginary if and only if ______.
   g. If $\sim(\sim p)$, then______.
4. State three valid implications that are not equivalences.
5. Construct an equivalence involving each of the following:
   a. $x^2 > 0$.
   b. An equilateral triangle.
   c. A trirectangular triangle.
   d. A trirectangular quadrilateral.

6. Prove by means of truth tables that the following pairs of propositions have the same truth values:
   a. $(p \to q)$; $\sim p \vee q$
   b. $(p \to \sim q)$; $(q \to \sim p)$
   c. $\sim(p \wedge q)$; $\sim p \vee \sim q$
   d. $\sim(p \to q)$; $p \wedge \sim q$

5. AXIOMATIC BASES SUMMARIZED

I. Undefined terms

a. Proposition ($p, q, \cdots$)
b. Truth values (T, F)
c. Negation ($\sim$)
d. Conjunction ($\wedge$)

II. Postulates

a. Every proposition is either true or false (T or F) but not both.
b. If a proposition is true, its negation is false; and if a proposition is false, its negation is true. This is indicated in the negation truth table.
c. The conjunction truth-value table

III. Definitions

a. Negation of a proposition: $\sim p$
b. Disjunction: $p \vee q$ is defined as $\sim(\sim p \wedge \sim q)$.
c. Implication: $p \rightarrow q$ is defined as $\sim(p \wedge \sim q)$.
d. Equivalence: $p \leftrightarrow q$ is defined as $(p \rightarrow q) \wedge (q \rightarrow p)$.

It can hardly be expected that the foregoing list is a complete postulational basis for a symbolic logic or the only one for the same symbolic logic.

6. TAUTOLOGY

Let us consider a truth table for $p \wedge (\sim p \wedge q)$. In its construction we need five columns: p, q, $\sim p$, $\sim p \wedge q$, and the one desired.

p	q	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

$\sim[p \wedge (\sim p \wedge q)]$
T
T
T
T

Column 3 is the negation of column 1. Column 4 is the conjunction of columns 3 and 2, which yields a T only after T and T. Column 5 is the conjunction of columns 1 and 4.

It may be surprising to note that the proposition expressed in column 4 is always false. The respective truth values of p and q are irrelevant. The proposition $p \wedge (\sim p \wedge q)$ is always false. This means that the negation of this proposition is always true, as is shown in the separate table on the right.

We have now a proposition that is always true irrespective of the truth values of the component elements p and q. Only the schematic form of the proposition determines its truth. This is known as a **universally valid** proposition, or a **tautology.**

EXERCISES (V–6)

1. How do equivalences and tautologies differ?
2. Prove that the following are tautologies:

a. $\sim(\sim p) \leftrightarrow p$
b. $\sim(p \wedge \sim p)$
c. $(p \to q) \leftrightarrow (\sim p \vee q)$
d. $(p \to \sim q) \leftrightarrow (q \to \sim p)$
e. $[(p \wedge q) \wedge (\sim q)] \to p$
f. $[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$
g. $\sim[(\sim p) \wedge (\sim q)] \to (p \vee q)$
h. $[p \wedge (p \to q)] \to q$
i. $[\sim q \wedge (p \to q)] \to \sim p$

7. LOGICAL INFERENCE

In our mathematical efforts we have indulged frequently in the use of *hypothetical propositions*, propositions of the *if* · · · *then* variety. *If* (p) two lines are perpendicular to the same line, *then* (q) they are parallel. We have proved this proposition; this implication is true in plane Euclidean geometry. That is, $p \to q$ is a true implication. Now, this does not tell us anything about p and q separately but tells only about the two in this carefully defined relation. It is like knowing that the sum of two odd numbers is an even number, without having or knowing the individual odd numbers.

When we make an **inference**, that is, when we **infer** something in mathematics, and presumably elsewhere as well, we need more information than the implication. We may know, for example, in addition to the proved proposition in the preceding paragraph, that such and such two lines are in fact perpendicular to a line. Now we are in a position to infer that q is true, that the lines are parallel to each other.

Let us put the foregoing assumptions together in symbolic form. We have, as given, $p \to q$ and p. Thus our given is $(p \to q) \wedge p$. We reach the conclusion q; that is, we infer that $[(p \to q) \wedge p] \to q$ is true. Let us look to the truth table of the last composite proposition of whose truth we are confident.

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Our sense of logical correctness is symbolically underlined; we have a tautology. This tautology is known as the **Rule of Inference,** which is a basic ingredient of deductive reasoning. This warrants a summary: If

$$\left.\begin{array}{ll} & p \to q \\ \text{and if} & \underline{p\quad} \\ \text{then} & q \end{array}\right\} \quad \text{or} \quad [(p \to q) \wedge p] \to q$$

If a triangle is isosceles, its base angles are equal. $p \to q$
$\triangle ABC$ is isosceles ($AB = BC$). $\underline{p}$
Then, the base angles are equal; $\angle A = \angle C$. q

Historically, $p \to q$ was referred to as the *major premise;* p, the *minor premise;* and q, *the conclusion*. If the first two are asserted, then the conclusion follows. The tautologous character guarantees the form of reasoning. The Rule of Inference does not guarantee that the asserted premises are indeed true.

By way of contrast, the following form is not valid: If

$$\left.\begin{array}{ll} & p \to q \\ \text{and if} & \underline{\quad q} \\ \text{then} & p \end{array}\right\} \quad \text{or} \quad [(p \to q) \wedge q] \to p \qquad \text{(not a valid form)}$$

It is possible to obtain at sight the truth values of this composite proposition from the previous table. We get

$$\begin{array}{c} \underline{[(p \to q) \wedge q] \to p} \\ T \\ T \\ F \\ T \end{array}$$

Clearly this is not a tautology and is not a form of valid reasoning.

If two angles are vertical angles, then they are equal. Knowing that angles x and y are equal does not make them vertical angles.

The latter illustration reminds one of the converse of a proposition. We have seen that the truth of a converse does not necessarily follow from the truth of the proposition. A proposition and its converse are not equivalent. This shows up effectively in the truth table:

p	q	$p \to q$	$q \to p$	$(p \to q) \leftrightarrow (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The equivalence is not a tautology; it is not always true. On the basis of form alone, we cannot be certain of argumentation by converse.

EXERCISES (V–7)

Tell whether or not the following arguments are valid. (This does not entail a judgment of the truth or falsity of the premises or of the conclusion.) Where the argument is not valid change the form so that validity is effected.

1. If a and b are each negative numbers, then ab is a positive number. The numbers a and b are not negative. Hence ab is not a positive number.

2. The value $\sqrt[n]{x}$ is a real number if n is an odd number. The number n is odd. Therefore, $\sqrt[n]{x}$ is a real number.

3. If p, then q. $\sim(\sim p)$ is asserted. Then $\sim(\sim q)$.

4. Wages are increasing. If wages are increasing, then employees have more real income. Hence, employees have more real income.

5. Two lines intersect if they are perpendicular to the same line. Lines a and b intersect. Therefore a and b are perpendicular to the same line.

6. In time of war, steel workers work overtime. Steel workers are working overtime. Therefore this is a time of war.

7. The median in a triangle divides it into two equivalent figures. The line AB divides the figure into two equivalent figures. Consequently AB is a median.

8. The number $\sqrt[n]{-4}$ is imaginary if and only if n is an odd number. But the number $\sqrt[n]{-4}$ is imaginary. Therefore n is an odd number.

9. If there are numerous sun spots, radio transmission is poor. Radio transmission is poor, and therefore it must be that there are numerous sun spots.

8. CONTRAPOSITIVES

We have had occasion to employ contrapositives, particularly in non-Euclidean geometry, as equivalent to the propositions of which they are contrapositives. The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$. Let us see that these are equivalent.

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

So, the proposition and its contrapositive are equivalent because we have a tautology. If one is true, the other is true. If one is false, then the other is false. This makes no conclusions about the individual propositions p or q or their negations but only about the two implications, that is, that they are equivalent.

With an additional premise we can deduce a conclusion about a term in the proposition. Thus, if $p \rightarrow q$ and if $\sim q$, then $\sim p$. This follows from our knowledge of contrapositives and logical inference. By way of illustration: If a quadrilateral is a square, its diagonals are perpendicular to each other. The diagonals of $ABCD$ are not perpendicular to each other. Then $ABCD$ is not a square.

EXERCISES (V–8)

State the contrapositive of each of the exercises below.

1. If today is Wednesday, yesterday was Tuesday.
2. If $a > b$, then $A > B$.
3. If $|m| < |n|$, then $m^2 < n^2$.
4. If the investigation does not serve a legislative purpose, then it is improper.
5. If one is indifferent to success, then one will not be a good businessman.

Check the validity of the following arguments. Where possible, make changes to create validity if none exists.

6. If $(\frac{1}{3})m > 1$, then m is a negative number. The number m is non-negative. Therefore $(\frac{1}{3})^m < 1$.
7. If John drove at 60 miles per hour, he exceeded the speed limit. John did not exceed the speed limit. Therefore he did not drive at 60 miles per hour.
8. Art is not being encouraged. It must be encouraged if it is to fluorish. Consequently art will not flourish.

9. APPLICATIONS

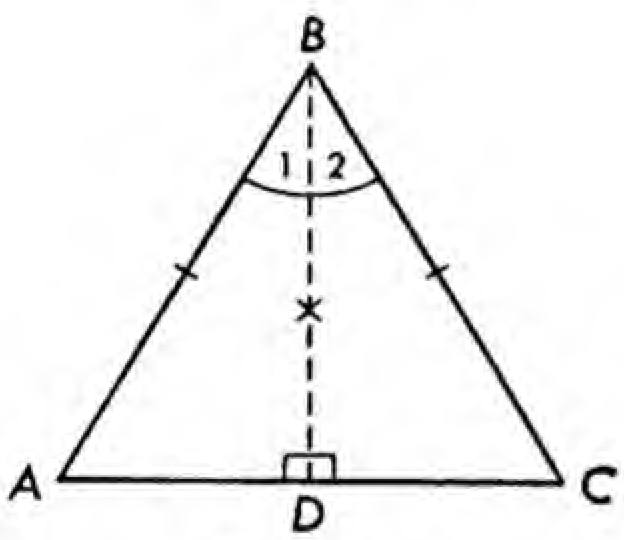


Fig. V–1

We have gone far enough to examine symbolically the kinds of proof we have used earlier. Consider the proof of the proposition that the *base angles of an isosceles triangle are equal* (see Fig. V–1).

GIVEN: p_1 $(AB = BC)$

PROVE: p_6 $(\angle A = \angle C)$

Asserted Proposition	*Step*	*Reason*
1. p_1	$AB = BC$	Given
2. p_2	BD bisects $\angle B$ (Draw)	Every $\angle$ has a bisector.
3. $p_2 \rightarrow p_3$		If a line is an $\angle$ bisector (it divides the angle into two equal parts, p_3).
4. p_3	$\angle 1 = \angle 2$	Rule of inference, steps 2 and 3.
5. p_4	$BD = BD$	Identity postulate.

6. $p_1 \wedge p_3 \wedge p_4$ (call this p')		Continued conjunction of 3 true props. is a true prop.
7. $p' \rightarrow p_5$		If two $\triangle$s agree in *sas* (p'), then they are $\cong$ (p_5).
8. p_5	$\triangle ABD \cong \triangle BCD$	Rule of inference, steps 6 and 7.
9. $p_5 \rightarrow p_6$		If $\triangle$s are $\cong$ (p_5), corres. parts are equal (p_6).
10. p_6	$\angle A = \angle C$	Rule of inference, steps 8 and 9.

We note that each step in the sequence is asserted as a given datum, a postulate, a definition, a true implication (theorem), a tautology, or a valid inference. The inference is the cement that binds the whole together.

The indirect method of reasoning eliminates all but one of the possibilities relevant to a situation. To see this symbolically, let us take the matter of two lines, a and b, in Lobachevskian geometry. The following propositions are the only ones possible:

p: a and b are parallel.
q: a and b are intersecting.
r: a and b are nonintersecting.

We have then	$p \veebar q \veebar r$	(noninclusive case of or)
if	$\sim p$	
and if	$\sim q$	
then	r	

What this amounts to is that the proposition $[(p \veebar q \veebar r) \wedge (\sim p \wedge \sim q)] \rightarrow r$ is a tautology that could be verified by a truth-value table.

To complete the preceding illustration, we consider as given the proposition that two lines, a and b, are perpendicular to the same line m. We also state the continued disjunction, $p \veebar q \veebar r$. We can demonstrate $\sim p$, since an angle of parallelism is less than a right angle. We can also demonstrate $\sim q$, since an intersecting line will have an acute angle with the perpendicular. In this way, briefly, we shall be left confidently with r.

EXERCISES (V–9)

1. If p and q are the only alternatives and if $\sim p$ then q. Set this up as a single proposition and prove it to be a tautology by a truth table.

2. Use the method of exercise 1 to prove that *if two lines are parallel then the corresponding angles are equal.*

Complete the following so that the reasoning is valid:

3. If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram. $ABCD$ is not a parallelogram.

4. If the opposite sides of a quadrilateral are not parallel or the diagonals are not perpendicular, then the figure is not a rhombus. The figure is a rhombus.

5. Sin x and tan x are both negative if x is in quadrant IV. Only one of sin x and tan x is negative.

6. Implications in mathematics are frequently stated in terms of **necessary** and **sufficient** conditions. In $p \rightarrow q$, q is said to be a necessary condition for p; p is true only if q is true. On the other hand, we say of p that it is a sufficient condition for q; q, if p.

In *If two angles are right angles then they are equal*, the equality of the angles is a necessary condition, and their being right angles is a sufficient condition. It suffices to know that the angles are right angles to conclude that they are equal. And it is necessary to have the angles equal so that they can be right angles.

Since the equivalence $p \leftrightarrow q$ is the same as $q \leftrightarrow p$, each component, p and q, is a sufficient and necessary condition for the other. This time we can say p, *if and only if* q, or q, *if and only if* p.

Consider the two propositions from the viewpoint of necessary and sufficient conditions: The quadrilateral is a rhombus. The quadrilateral has two adjacent sides equal.

Discuss each of the following from the viewpoint of sufficient, necessary, and sufficient and necessary conditions.

7. If ABC is a triangle, then the sum of the angles is 180° (in Euclidean geometry).

8. If $4^n = 2$, then $n = \frac{1}{2}$.

9. The sum of the angles of a quadrilateral is greater than 360° only if the geometry is non-Euclidean.

10. $i = \sqrt{-1}$ if $i^2 = -1$.

11. If $y^3 = 8$, then $y \neq 1 - i\sqrt{3}$.

12. If x and y are real numbers, then $|x + y| \leq |x| + |y|$.

Insert the word(s) necessary, sufficient, or necessary and sufficient:

13. A —— condition that a number be divisible by 9 is that the sum of its digits is 9.

14. The absence of parallel lines is a ____ condition for a Riemann geometry.

Examine the following for validity and correct as necessary:

15. A sufficient condition that a/b be an integer is that $b = 1$.

16. A necessary condition that $m = 3$ is that $2m - 5 = 1$.

17. A sufficient condition that $a^2 + b^2 = c^2$ in $\triangle ABC$ is that $\angle C$ be acute.

18. A sufficient condition that $\cos(\pi + x) = \cos x$ is that $x = n\pi/2$ for integral values of n.

19. A necessary and sufficient condition that $ABCD$ be a square is that it be a rectangle.

Prove each of the following propositions and show the logical steps symbolically as was done in the text:

20. The area of a triangle is equal to one-half the product of the base times the height.

21. If two lines are perpendicular to the same line in Riemannian geometry, then they meet in one point on the same side of the line.

22. In Lobachevskian geometry, if two sets of left parallels have equal perpendiculars, then they have equal angles of parallelism.

V-9 REVIEW

1. Express the implication $p \to q$ as (a) a conjunction and (b) as a disjunction.
2. If $p \wedge q$ is known to be true, what may be said of p and q?
3. Construct the truth table for the exclusive case of the disjunction.
4. If p, q, and r are the translations of the propositions "the set has an infinite number of terms," the elements of the set are in one-to-one correspondence with the non-negative integers," and "the set has a transfinite cardinal number," respectively, write a translation of the proposition: "If a set has an infinite number of terms or the elements of the set are in one-to-one correspondence with the non-negative integers, then the set has a transfinite cardinal number."
5. Determine which of the following pairs of propositions represent equivalent propositions:

   a. $(p \vee q) \vee r$ and $p \vee (q \vee r)$
   b. $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$
   c. $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$
   d. $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$

6. Prove each of the following:

   a. $(q \to p) \leftrightarrow (\sim p \to \sim q)$
   b. $\{(\sim q \to \sim p) \wedge (q \to p)\} \to (p \leftrightarrow q)$
   c. $\{(p \wedge q) \to r)\} \leftrightarrow \{(p \wedge \sim r) \to \sim q\}$
   d. $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$

7. Substitute propositions of your own choosing to illustrate the following tautologies. Make two translations of each.

   a. $\{(p \vee q) \wedge (\sim p)\} \to q$
   b. $\{[p \to (q \wedge r)] \wedge [\sim(q \wedge r)]\} \to \sim p$
   c. $\{\sim(p \wedge q) \wedge p\} \to \sim q$
   d. $\{(p \to q) \wedge (q \to r)\} \to (p \to r)$

8. The following are not tautologies. By substituting propositions of your own choosing, illustrate that the following are fallacious arguments:

   a. $\{(p \to q) \wedge q\} \to p$
   b. $\{(p \to q) \wedge (\sim p)\} \to \sim q$
   c. $\{(p \veebar q) \wedge p\} \to q$

VI — FUNCTIONS

1. COMMUNICATION NEEDS

In previous text we had occasion to examine various trigonometric functions. Let us return to $y = \sin x$ briefly for closer examination. This, in turn, will provide a basis for further analytic explorations.

The equation $y = \sin x$ contains two letters x and y, which we have called "variables," and the word *sine*, which by definition directed the performance of certain operations. We could indeed refer to *sine* as an **operator** in the same sense that $\sqrt{}$ is an operator.

When, in $y = \sin x$, we select some real number value for x, we find that a unique y value results. We are at liberty to select any real value for x. The letter x, then, represents *any value of the set of real number points on the X-axis*. This is the sense implied when we use the word **variable**; it refers to **any member of any given set** (*class* and *aggregate* are other names for set). If the set turns out to have no members (such as the set of even prime numbers that are greater than 2 or the set of round squares), we say that the set is a **null** or **empty** set. Should the set turn out to have only one member (such as the set of all multiples of 5 between 5 and 15), then we call the variable a **constant.**

The x-variable in $y = \sin x$ is further described as the **independent variable** to signify the freedom of choice of any x value from its set of available values. On the other hand, the value of y must be calculated. The y value is unique and dependent on the choice of an x value. So, y is referred to as the **dependent variable.**

We have seen in an earlier chapter that the set of y values for $y = \sin x$ consists only of the real values between $+1$ and -1, inclusive. This is pictured in Fig. VI–1.

The set from which values of x may be selected, over which it is defined,

is called the **domain,** X, and the set of values that result for y is called its **range,** Y.

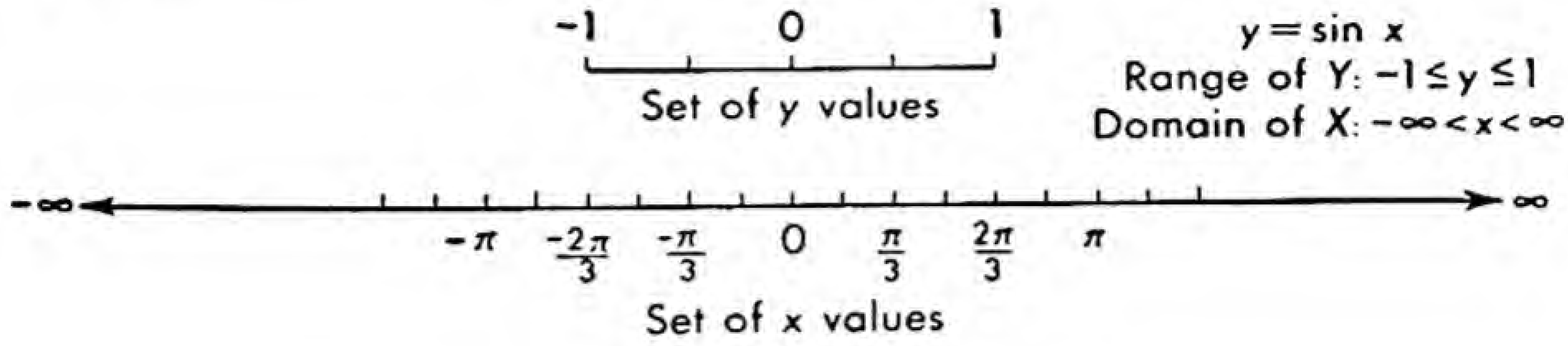


Fig. VI-1

We have seen earlier that when $x = \pi/6$, $y = \frac{1}{2}$. When referring to the graph (Fig. VI-1), we call the x value the **abscissa** and the y value the **ordinate.** The related pair of values is referred to as the **coordinates.** The pair above is represented simply as $[(\pi/6), (\frac{1}{2})]$, with the x value first. This is an **ordered pair** satisfying the equation $y = \sin x$. Other ordered pairs that we have found to satisfy this equation are: $[(\pi/3), (\sqrt{3}/2)]$, $[(\pi/2), 1]$, $(\pi, 0)$, and $[(3\pi/2), -1]$. Thus the equation yields a set of ordered pairs which may be symbolized, in general, by $\{x, y\}$.

The set of ordered pairs is called a **function.** The paired values are *functionally related* as defined by the equation. As with trigonometric functions, all our functions permit only a unique y value for each x value. This restriction is necessary to avoid ambiguity.

The symbol for function is a single letter as f, g, and H. In the present instance, the function f consists of the set $\{x, y\}$ that satisfies $y = \sin x$ for all real values of x.

It should be noted that y is only another name for $\sin x$, for according to the equation, y and $\sin x$ are equal. One could therefore describe the function more compactly as the set determined by $\{x, \sin x\}$.

$$f{:}\{x, \sin x\} \qquad \text{for all real values of } x.$$

The colon is a symbol for the phrase *whose set is*. The whole expression is read as *f is the function whose set is the ordered pairs* $\{x, \sin x\}$.

There are times when the equation is not known precisely or where it is too complicated for casual reference. In such cases, the functional representation may be given by

$$f{:}\{x, f(x)\}$$

The symbol $f(x)$ is a substitute for a dependent value. It stands for y, or in this case for $\sin x$. It is a *function value* whose independent variable is x. In $[(\pi/4), f(\pi/4)]$, $f(\pi/4)$ is the value of the dependent variable when the independent variable is $\pi/4$. $f(\pi/4) = \sqrt{2}/2$.

These many words and distinctions are necessary for accurate communication. So, in general, $f(x)$ is a function value of a function f whose independent variable is x and whose dependent variable is $f(x)$ or y, if $y = f(x)$. It must be remembered that $f(x)$ is a single entity and not a product of f and x.

We have seen that the symbol (x, y) refers to a particular but unspecified ordered pair. It is *any point in* $\{x, y\}$. The first represents any member of the set of ordered pairs indicated by the second symbol.

EXERCISES (VI–1)

1. List three functions of two variables, each taken from arithmetic or geometry. Indicate the independent and dependent variables, the domain and the range, and sample ordered pairs that satisfy the functions.

2. Do the same as in exercise 1 for two functions from business, science, or other fields.

3. Indicate the domain and range of each of the following for all real values of x:

a. $y = \tan x$
b. $y = \cos x$
c. $y = \sec x$
d. $y = \csc x$
e. $y = 2 \sin x$
f. $y = \sin 2x$
g. $y = 2 \cot 3x$
h. $y = 2$

4. a. Give any two illustrations of a null set.
b. Give any two illustrations of a set with only one member.

5. Given $f\colon \{x, \tan x\}$, find:

a. $f(0)$ b. $f\left(\frac{\pi}{4}\right)$ c. $f\left(\frac{\pi}{6}\right)$ d. $f\left(\frac{\pi}{2}\right)$

6. If $g\colon\{x, \tan x\}$ and $h\colon\{x, \cot x\}$, determine the function $k\colon\{x, y\}$ for which $g(x) = h(x)$.

7. In the light of our earlier experiences with graphing of composite curves, complete each of the following, where $f\colon\{x, \cos x\}$ and $g\colon\{x, \sin x\}$ for all real values of x:

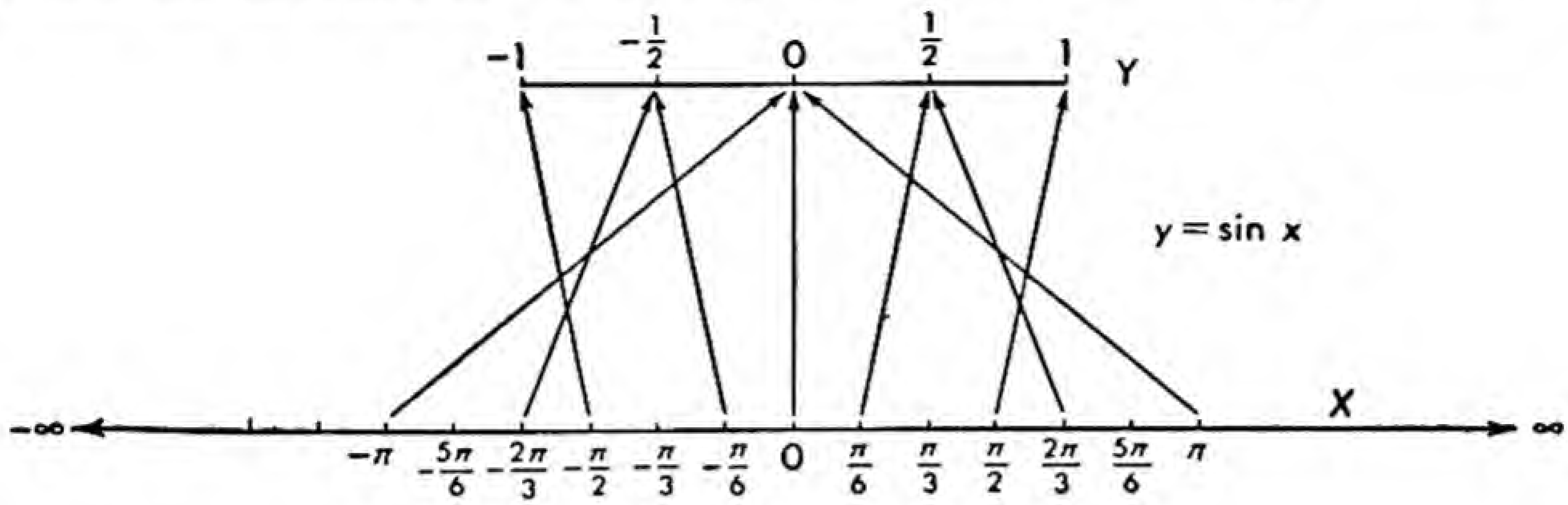
a. $(f + g)$:
b. $(f - g)$:
c. $\dfrac{1}{f}$:
d. Show that $fg\colon\{x, \frac{1}{2} \sin 2x\}$.
e. Show that $\dfrac{f}{g}\colon\{x, \cot x\}$; $\sin x \neq 0$.

2. MAPPING

In any of the trigonometric graphs we worked with, we were concerned with the 1-to-1 correspondence between the x and y values. We may represent (Fig. VI–2) the corresponding domain and range by two straight lines as before.

To each and every value of x there is one and only one value of y. This is indicated by the arrows in the diagram which *maps* the set of x on

the set of y. Though useful, this is not the best of graphical representations. If we invert the view, as we have done so often with other operations, and reverse the direction of the arrows, then to each value of y there are an



Mapping $\}x\{$ on $\}y\{$

Fig. VI-2

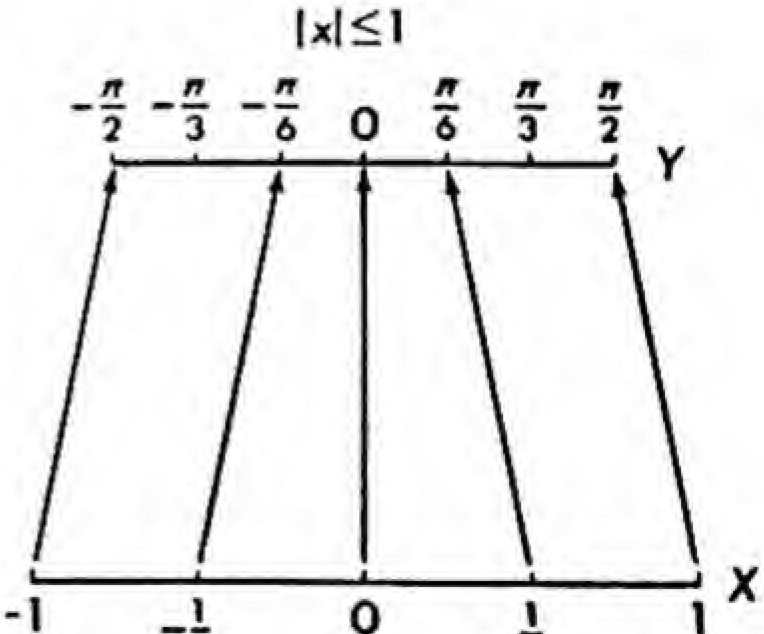
infinite number of values of x. The original function is a *single-valued* function. The function inverted is called a *many valued relation.* The word "function" is reserved exclusively for single-valued relations. The word **relation** will be used otherwise for $\}x, y\{$ where an x value has one or more y values.

We have long ago committed ourselves to the construction of inverses for each and every operation. To avoid many-valued relations as inverses of functions, we shall have to restrict the domains of some functions. In the case of the sine function, for example, we shall have only $\pi/6$ to go with $\frac{1}{2}$ and not $5\pi/6$ or others. This we begin to do presently.

If f represented the original function, we could use f^{-1} to represent the inverted, or *inverse,* function. It should be noted that we are inventing symbols for the inverse cases here just as we did earlier. As a rule the inverse function will also have an inverse equation to go along with it. In the case of $y = \sin x$, we need a symbol for the inverse of the operator *sin.* Some use **arc sin,** as we shall; others use sin^{-1}. So, the inverse equation of the sine equation is

$$y = \textbf{arc sin } x$$

which is read as y *is the angle whose sine is* x. The letters x and y have been retained in their original positions so that x remains the independent variable and y the dependent variable. The last equation could be interpreted as $\sin y = x$, which emphasizes the fact that the set of y-values is now the set of angle values.



Mapping $\}x\{$ on $\}y\{$

y = arc sin x

Fig. VI-3

Since we are interested in functions rather than relations, we arbitrarily eliminate some of the duplication. Thus, when $x = \frac{1}{2}$, we take for y only $\pi/6$. This eliminates such possibilities as $y = 5\pi/6, -7\pi/6$, and so forth. This is done (Fig. VI–3) most effectively by restricting the range to $|Y| \leq \pi/2$. In this way $y =$ arc sin x is described correctly as the "inverse sine function."

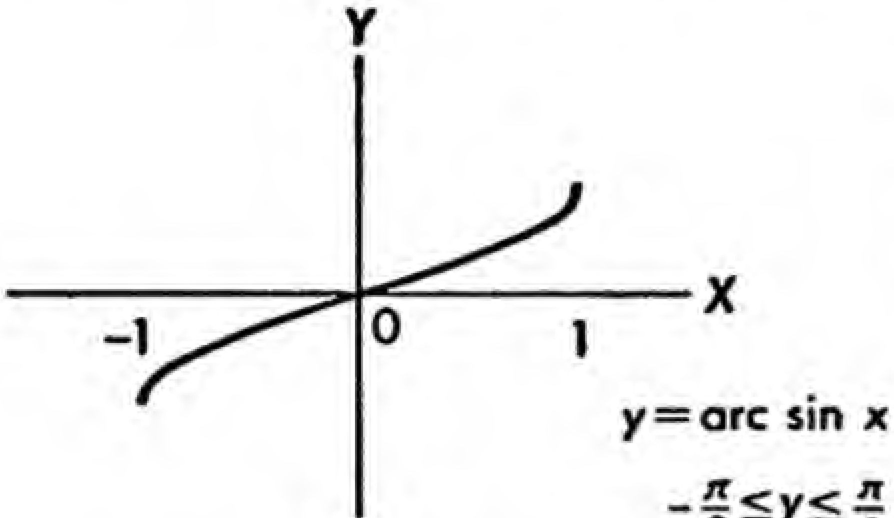


Fig. VI–4

With the foregoing restrictions in mind, we graph the arc sin x function either with the aid of a table of permitted values or by means of a sketch for which we have developed prior experience. The difference in orientation should not be an insurmountable obstacle. The graph (Fig. VI–4) shows further that we have a single-valued relation and so a function.

EXERCISES (VI–2)

1. Sketch a line to line mapping of

   a. $y = \cos x$ b. $y = 3 \sin \frac{1}{2}x$ c. $y = \tan x$

2. By distinctive shading or cross-hatching, graph the relations:

   a. $y > \sin x$ b. $y \leq \sin x$

3. Graph:

   a. $y =$ arc cos x b. $y =$ arc tan x

Indicate in each case the range of definition so that the equations define a function.

VI–2 REVIEW

1. a. Indicate two null sets.
   b. Describe two sets, each of which contains only one element.
2. List two members of each of the following sets where x is any real number:
   a. $\{x, 2x\}$ b. $\{x, x^2\}$ c. $\{x, \sqrt{x}\}$
3. Find one ordered pair that is a member of all three sets in the preceding example.
4. What is the range for each of the sets in exercise 2?
5. Which are the dependent variables in each of the sets in exercise 2?
6. A relation that includes in its set such elements as $(2,3)$ and $(2, -3)$ is not a function. Explain.
7. a. If the domain of $\{x, \sin x + \cos x\}$ is defined only for $x = k\pi/2$, where $k = 0, 1, 2, 3$, and 4, what is the range of the function?

   b. If the domain of the function is defined for $-\infty < x < \infty$, what is the range?

8. If $f:\{x, \cot x\}$ for all real values of x, what are the values of:

a. $f\left(\frac{\pi}{6}\right)$? b. $f\left(-\frac{\pi}{4}\right)$? c. $f(\pi)$?

9. $f:\{x, x+2\}$ is not a function without a further specification. Explain and illustrate. (The specification is called a **legend.**)

10. If $f:\{x, \sin x\}$ and $g:\{x, \cos^2 x\}$ for all real values of x, describe completely

a. $f+g$ b. $\frac{g}{f}$ c. $\frac{f}{g}$

11. If $f:\{x, 2x\}$ and $g:\{x, x^2\}$ for all real values of x, find:

a. $g(3)$ f. $f(x)+g(x)$
b. $f(\frac{1}{2})$ g. $f+g$
c. $g(\sqrt{3})$ g. $g(x)-f(x)$
d. $g(x^{1/4})$ i. $g-f$
e. $f\left(\frac{1}{a}\right)$

12. By means of parallel lines or line segments, map a few members of each of the following functions for real values of x:

a. $f:\{x, 3x\}$ b. $g:\{x, x^3\}$

13. Define an inverse function for each of the following, adding further restrictions as necessary:

a. If $f:\{x, 3x\}$ for all real values of x.
b. If $g:\{x, x^2\}$ for all real values of x.
c. If $h:\{x, x^3\}$ for all real values of x.
d. If $k:\{x, x\}$ for all real values of x.

3. ALGEBRAIC FUNCTIONS

Some time back we saw how the number system was extended from the natural numbers to rational numbers with the aid of the equation $ax = b$. Irrational numbers entered the scene accompanied by equations of the type $x^2 = a$. Should a be a negative number in this last equation, we have the setting for the imaginary number.

These thoughts eventually gave rise to the notion that numbers could be classified from the viewpoint of equations. For purposes of generalization, we begin with an equation as

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

where n is any positive integer, and the coefficients are integers. The roots of such equations are called **algebraic numbers** which are special cases of real and complex numbers. We have met some instances already.

The preceding equation is a **polynomial** equation with integral coefficients. Complex coefficients are also possible. A polynomial is an expression derived from a finite number of the operations of addition, subtraction, and multiplication on a letter x and the complex numbers. The following examples are illustrations of polynomials: $x^2 + 3x - \frac{1}{2}$; 8; $x\sqrt{3} - ix^2$. Any function whose dependent value is a polynomial is a **polynomial function.**

We have seen that the integers and the rational numbers are denumerable. Their cardinal number is aleph-null, $\aleph_0$. Cantor showed that algebraic numbers are also denumerable. It is certain from our development of the number system that the algebraic numbers include all integers, all rationals, and (as we have seen) at least some irrational and complex numbers.

We have also proved that the real numbers, which include all the rationals and irrationals, have a higher cardinal number, C, and therefore that the set is not denumerable. Hence, the real numbers must contain *more* numbers than the algebraic numbers which are denumerable. This is possible if and only if some irrational numbers are not algebraic. Thus there exists some irrational numbers which are not and cannot be roots of any algebraic equation. They *transcend*, so to speak, the field of our equations. Such numbers are called **transcendental** numbers.

Some time after this was surmised, it was proved that π is a transcendental number. Most trigonometric values are transcendental. Numbers like $2^{\sqrt{3}}, 3^{\sqrt{2}}, \cdots, a^b$, where a is algebraic and b is algebraic irrational, are all transcendental.

After the above interlude, we return to the subject of graphs and this time bring in *algebraic functions*. The first four operations we studied (addition, subtraction, multiplication, and division) are called the *rational operations*. If we add to these the taking of the nth root, where n is any positive integer, we have the *algebraic operations*.

If now we take some variable x together with the complex numbers and apply to them a finite number of algebraic operations, we get *algebraic expressions*. The following are indicative of the scope and variety of the algebraic expressions:

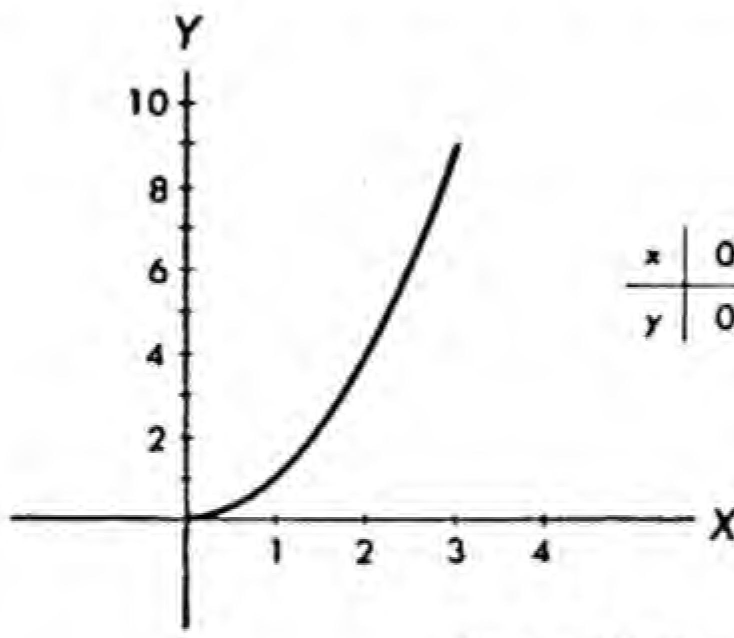
$$x^2 - 2x + 2, \quad \sqrt{x-2}, \quad \frac{8}{\sqrt[3]{x}} + x^2, \quad \frac{5}{x^2 - 4}, \quad \frac{2 - \sqrt{x}}{\sqrt{x-1}}$$

Finally, any function whose dependent value is an algebraic expression is an **algebraic function.** Thus, $f:\{x, y\}$ is an algebraic function if y is any value such as those just illustrated and where, unless otherwise restricted, the domain and range are the real numbers. We turn our attention in this direction now.

We begin by considering the class of number pairs determined by the

side of a square and its area. If we call x a side, then x^2 represents the area. The side x is the independent variable in this case, and the area x^2 is the dependent variable. For convenience, we assign the letter y to the area, and so we have the defining equation, $y = x^2$.

Assuming no physical barriers, the domain of X is any real positive number. This is indicated symbolically as $0 \leq x < \infty$, where the number 0 is included for the sake of completeness. The range is the same, since the square of a real number is a real number, and the squares of numbers from 0 to infinity will be numbers from 0 to infinity.

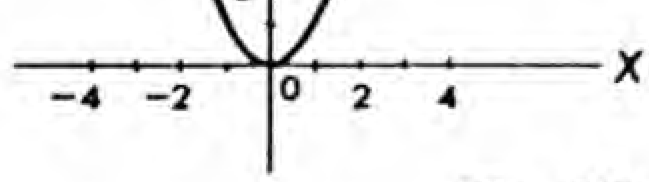


$y = x^2$

x	0	1	1.5	2	2.5	3
y	0	1	2.25	4	6.25	9

Fig. VI–5

The six points in the plane, Fig. VI–5, have been joined by a smooth *continuous* curve. Intuitively we can envision any value for the side of the square and a side for the square that has any real value for the area. Neither a side nor an area is impossible or undefined. For these reasons we feel justified in using the six points as guide points for a smooth, unbroken curve. However, we shall set down later a strict formulation for this subtle concept of *continuity*.



$y = x^2$

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

Fig. VI–6

We may separate the equation $y = x^2$ from its attachment to the square and view it as a function rule in the broadest sense. The x values have an unrestricted domain from $-\infty$ to $+\infty$.

We shall see later that this curve, called a *parabola*, has fascinating as well as useful properties, as near to us as our own car. The range of Y, we note, is the non-negative number. This is not an imposed restriction but rather a consequence of the fact that the square of any real number is a positive number or 0.

The curve is *symmetric* about the Y-axis. If the plane is folded on the Y-axis, the right and left parts of the curve coincide. This, too, is no accident but a consequence of the fact that the squares of any two numbers, which differ only in sign, are the same positive number. For example, $(-3)^2 = (+3)^2 = 9$. This is true for all values in X. Indeed, if we took $-x$ and substituted it for x in the original equation, $y = (-x)^2$, we get the very same equation: $y = x^2$. The substitution serves as a test for symmetry about the Y-axis.

On solving $y = x^2$ for x, we get two equations

$$x = \sqrt{y} \qquad \text{and} \qquad x = -\sqrt{y}$$

where $y \geq 0$.

This is the first step toward getting an inverse function equation. The next step, to keep x the independent variable, is to interchange x and y. We have met this procedure before in connection with the trigonometric functions. The inverse functions are $y = \sqrt{x}$ and $y = -\sqrt{x}$, whose graphs are shown in Fig. VI-7.

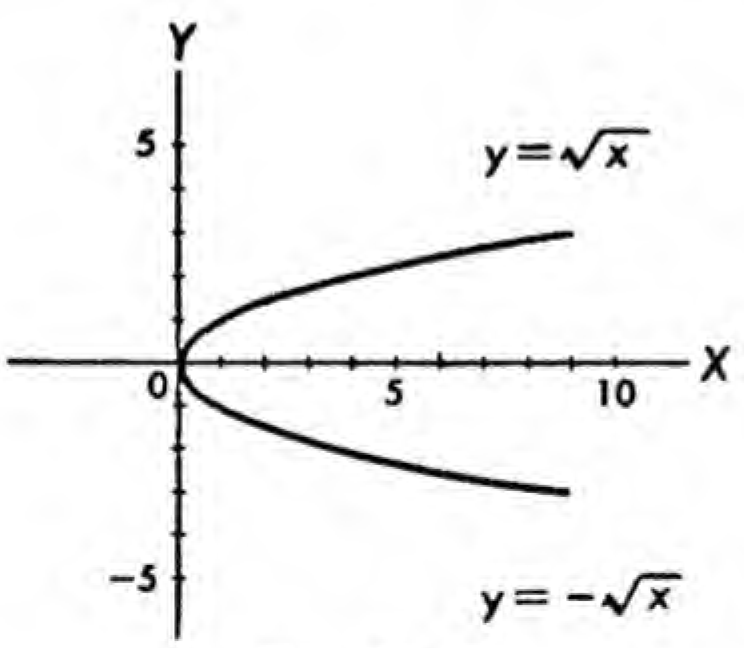


Fig. VI-7

A comparison of tables of values, graphs, and equations reveals the fact that what has been effected is an interchange of abscissas and ordinates for all members of the function. For example, the point member (2, 4) of the original equation has become the member (4, 2) of the inverse. The two equations define two functions, each of which is graphed in Fig. VI-7. If we consider the two equations simultaneously as $y = \pm\sqrt{x}$, then we have a two-valued relation whose graph is the parabola. We can take up a more general equation:

$$y = x^2 - 2x - 3$$

The intersection of the curve with the X-axis gives points that are referred to as the *x-intercepts*. In the graph (Fig. VI-8) the x-intercepts are given by the points $(-1, 0)$ and $(3, 0)$. They are also referred to as the **zeros of the function,** defined by the preceding equation. From the

viewpoint of the equation, the zeros of the function are those values of x where $y = 0$. Under this condition, the equation becomes

$$x^2 - 2x - 3 = 0$$

$y = x^2 - 2x - 3$

x	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5

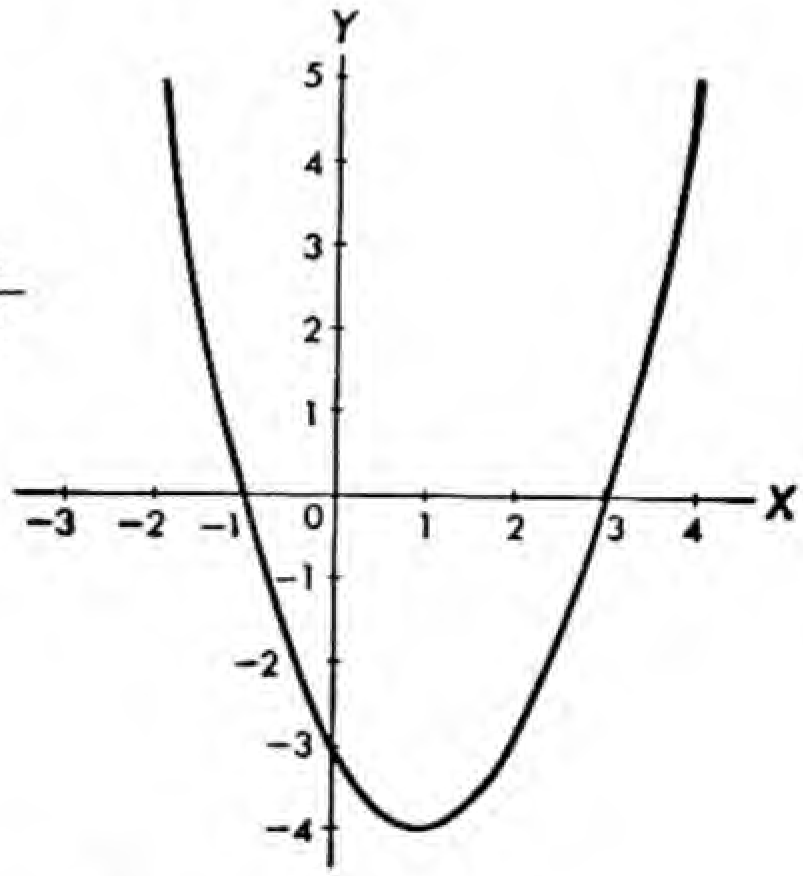


Fig. VI–8

and according to the graph, the "roots" of this equation, or the values that satisfy the equation, are $x = -1$ and $x = 3$.

EXERCISES (VI-3)

1. Express the inverses of each of the following:

a. $y = x - 2$ c. $y = \dfrac{1}{x}$

b. $y = 3x$ d. $y = 2x^2$

2. Construct a table of values for $y = x + 3$, and for its inverse, to bring out the inversion of the variables.

3. Graph each of the following and their inverses:

a. $y = 2x + 1$ b. $y = 4x^2$

4. Find the simplest (lowest powers) equations with integral coefficients that will have the following as roots:

a. $\frac{3}{5}$ b. ± 1 c. $\pm\sqrt{2}$ d. $-2\frac{1}{3}$
e. $3 \pm \sqrt{7}$

5. We saw earlier that the product of a complex number by its conjugate yields a real number. Use this fact to determine equations with integral coefficients that have the following as roots:

a. $2i$ c. $3 - 4i$

b. $-\dfrac{2i}{3}$ d. $1 + i$

6. Which of the numbers in exercises 4 and 5, if any, are algebraic numbers?

7. Gauss proved that any polynomial equation has at least one root. With this information it is simple to go on to show that every polynomial, $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, $a_0 \neq 0$, where $n \geq 1$, can be factored into the product of exactly n factors

$$f(x) = a_0(x - r_1)(x - r_2)\cdots(x - r_n)$$

The r's are the n roots of the polynomial equation $f(x) = 0$. The roots, of course, may be real or complex. When the coefficients are integers, the roots are the algebraic numbers we have been discussing.

Cantor defined the number h (called *height*) by $h = |a_0| + |a_1| + |a_2| + \cdots + |a_n| + n$. When the absolute values and n are positive integers, h is a positive integer. In this way, it is possible to assign a distinct positive number to each and every polynomial equation with integral coefficients.

To see the implications of these observations, let us take $h = 5$. There can be only a finite number of equations with this height. For example, $x^4 = 0$, $x^3 + 1 = 0$, and $x^2 - x - 1 = 0$ are a few samples. Each of these equations has a finite number of roots. Consequently the roots of equations of height 5 are finite. In this way we see that with any height, there is associated a finite number of algebraic numbers. By starting with height 2, then height 3, and so forth, all algebraic numbers can be arranged in a distinct and ordered sequence. Consequently, the algebraic numbers are denumerable.

a. Why is 2 the least value of h?
b. There is only one equation with height 2. What is it and what algebraic number does it define?
c. Write all the equations with $h = 3$ and $h = 4$. List the algebraic numbers they yield.
d. What is the highest degree of an equation that has $h = 7$?
e. If the coefficients of the polynomial equation $f(x) = 0$ are rational, by what procedure can they be made integral? What effect will this have on the roots?
f. If the coefficients of $f(x) = 0$ are complex, explain how this can be written as $k(x) + im(x) = 0$. What conditions with respect to $k(x)$ and $m(x)$ will make this last equation true?

8. Write an equation with integral coefficients, one of whose roots is $3^{1/2}$, and show thereby that this is not a transcendental number.

9. a. Show that $3^{\sqrt{2}}$ is a root of $x^{\sqrt{2}} - 9 = 0$.
b. Explain whether or not the equation is a polynomial equation.

10. Draw the graphs of:

a. $y = x$
b. $y = 2x^2$
c. $3y = 4x^2$
d. $y = x^3$
e. $y = x^4$
f. $y = 2x + 3$
g. $2y - 3x = 6$

In each case indicate the domain and range of the variables.

11. Write the inverses of each equation in exercise 10.

12. a. Which of the graphs in exercise 10 are symmetric about the Y-axis? What is the algebraic test of symmetry?
    b. Are any of the graphs symmetric about the X-axis? What is the algebraic test for this symmetry? Justify your answer. (Note the nature of the equations that possess this kind of symmetry.)
13. Draw the graphs of each of the following:
    a. $y = x^2 + 4x - 5$ b. $y = 2x^2 + 5x - 3$
    c. Determine the zeros in both cases.

4. THE QUADRATIC

We have just been concerned with a variety of polynomial function values of the second degree having *integral coefficients*. The two function values in the last exercise of the preceding section can also be written as $f(x) = x^2 + 4x - 5$ and $f(x) = 2x^2 + 5x - 3$. These are called *second-degree* or *quadratic* polynomials, since the highest power of the variable is 2. When there are only three terms in a polynomial expression, it is also called a *trinomial*. We saw earlier that trinomial expressions arise as a result of the product of two binomials. For example,

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

Because the need for the inverse process, factoring, is at hand, certain observations are desirable. First, the x^2 term comes from the product of x times x, which are the first terms of the two binomials. Similarly the term 6 in the final product comes from $2 \cdot 3$, the product of the end terms of the two binomials. The $5x$ comes from the sum of two products, $2x + 3x$.

Let us look at this now as though we were factoring the trinomial. Since the x^2 and 6 both come from products, each comes from a finite number of possibilities. For example, the 6 must be derived from "$1 \cdot 6$" or "$2 \cdot 3$". We are searching only for factors with integral coefficients. The middle term, $5x$, coming from a sum has innumerable possibilities; for example, $x + 4x$, $2x + 3x$, $6x - x, \cdots$.

As a result of these observations, we can plan a systematic approach at locating the factors. We use the first and third terms as sources for potential factors and the middle term as the means of discrimination among the possibilities. Consider, for example, $2x^2 - 5x - 3$.

Clues from $2x^2$: only $2x \cdot x$

Clues from -3: only $(+3)(-1)$ and $(-3)(+1)$

Therefore the possible products are:

$$(2x + 3)(x - 1)$$
$$(2x - 3)(x + 1)$$
$$(2x + 1)(x - 3)$$
$$(2x - 1)(x + 3)$$

Multiplication reveals that the third trial is the correct one; that is, $2x^2 - 5x - 3 = (2x + 1)(x - 3)$.

We return now to the equation $y = x^2 - 2x - 3$ of the preceding article. In finding the zeros of the function, we were finding in fact the roots of the equation $x^2 - 2x - 3 = 0$. The previous methods used for equation solving are of no help now because of the presence of an x^2 and x term in the equation. The first-degree methods were entirely dependent on the possibility of getting a single x term in the equation. The special second-degree equations of the type $x^2 = 5$ lacked the x term.

Well, the only thing that can be done with the trinomial in the equation is to factor it. Thus, for $y = 0$, we have

$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

Although the value of x is unknown at this point, we do understand that x does represent some number or numbers. In this case each of the binomials also represent numbers. The last equation indicates that the product of two numbers is zero. This is possible if and only if at least one of the numbers is 0. This means that either

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$

So, $$x = 3 \quad \text{and} \quad x = -1$$

Of course these values agree with what we found in the graph in preceding section 3.

Clearly, the zero in the right hand member of the quadratic equation is crucial. The predictability of the values of the factors depends on the unique property of the zero just employed. To check the answers involves the substitutions of the final results in the original equations:

If $x = 3$	If $x = -1$
$(3)^2 - 2(3) - 3 \stackrel{?}{=} 0$	$(-1)^2 - 2(-1) - 3 \stackrel{?}{=} 0$
$9 - 6 - 3$	$1 + 2 - 3$
$0 = 0$ *Check*	$0 = 0$ *Check*

In the solution given here, we saw that "if $x^2 - 2x - 3 = 0$, then x is an element of the set $\{3, -1\}$." Thus "x is an element of the set $\{3, -1\}$" is a necessary condition that $x^2 - 2x - 3 = 0$.

In the two preceding *checks* we saw that *If x is a member of the set* $\{3, -1\}$, *then* $x^2 - 2x - 3 = 0$. And so, *x is a member of the set* $\{3, -1\}$ is also a sufficient condition. Consequently *x is a member of the set* $\{3, -1\}$ is a *necessary and sufficient condition* for $x^2 - 2x - 3 = 0$. Or $x^2 - 2x - 3 = 0$ if and only if x is a member of the set $\{3, -1\}$.

The so-called check is therefore not a comforting addendum to the

solution but an integral part of the solution. It would be better to use the label "converse" for the word "check."

The solution of the quadratic by factoring depends on the fortuitous circumstance of a factorable expression. Suppose that the trinomial is not factorable. Fortunately a special case of factoring leads us to a general formula for the solution of all quadratic equations in one unknown.

It is known that some trinomials have identical factors, that the trinomials are *perfect squares*. For example,

$$x^2 + 2x + 1 = (x + 1)(x + 1) = (x + 1)^2$$
$$y^2 + 6y + 9 = (y + 3)^2$$
$$z^2 - 5z + \frac{25}{4} = \left(z - \frac{5}{2}\right)^2$$

We have restricted our illustrations to those cases where the coefficient of the leading term, the one with the highest power, is 1. We note in these cases that the constant term is the square of one-half the middle coefficient; that is,

$$1 = \left(\frac{2}{2}\right)^2, \quad 9 = \left(\frac{6}{2}\right)^2, \quad \text{and} \quad \frac{25}{4} = \left(\frac{5}{2}\right)^2$$

This is corroborated in the more general case where

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

In this observation we have a method of approach to a general trinomial equation with complex coefficients. Take

$$ax^2 + bx + c = 0$$

where $a \neq 0$.

After subtracting c from both members, and dividing both members by a, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

We add the term $(\frac{1}{2} \cdot b/a)^2 = b^2/4a^2$ to both members.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

The left member is now a perfect square, and so

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The right member has been combined into one fraction.

As in earlier cases (such as $y^2 = 7$, where $y = \sqrt{7}$ or $y = -\sqrt{7}$), we can take the square root of both members of the equation and get

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

so

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The substitution of these results in the original equation reveals that these results are indeed the necessary and sufficient conditions that $ax^2 + bx + c = 0$.

By way of application of the resulting formulas, consider the nonfactorable equation $2x^2 - 3x - 1 = 0$. Here $a = 2$, $b = -3$, and $c = -1$. Substituting these values for the respective letters in the general formulas, we have

$$x = \frac{3 + \sqrt{9 + 8}}{4} \quad \text{or} \quad x = \frac{3 - \sqrt{9 + 8}}{4}$$

$$x = \frac{3 + \sqrt{17}}{4} \qquad x = \frac{3 - \sqrt{17}}{4}$$

$$x \approx \frac{3 + 4.12}{4} \approx \frac{7.12}{4} \qquad x \approx \frac{3 - 4.12}{4} \approx \frac{-1.12}{4}$$

$$x \approx 1.8 \qquad x \approx -0.3$$

The symbol $\pm$ may be used to represent the disjunctive "either-or." This means that x could be 1.8 or -0.3, or both. The choice will depend not on the equation, in which both answers will always check, but rather on the meaning attached to the x.

As another illustration, we consider the following:

$$x^2 - 4x + 13 = 0 \qquad a = 1, b = -4, c = 13$$

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$x = 2 + 3i, \qquad x = 2 - 3i$$

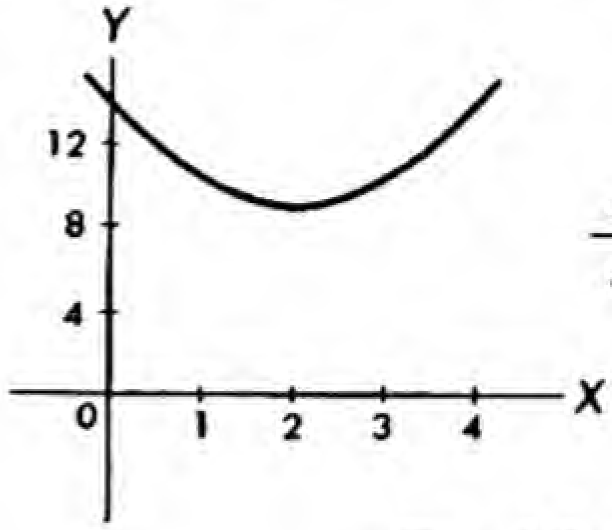
CONVERSE: If $x = 2 + 3i$, then

$$\begin{aligned} x^2 - 4x + 13 &= (2 + 3i)^2 - 4(2 + 3i) + 13 \\ &= 4 + 12i - 9 - 8 - 12i + 13 \\ &= 0 \end{aligned}$$

The substitution of $2 - 3i$, instead, will change both signs of the two $12i$ terms, causing their sum to be zero too. Otherwise, there is no change in the converse procedure.

A brief glance at the third step in the solution, or an analysis of the formulas, shows that when the roots of a quadratic equation are complex, they are conjugate complex.

The last illustration provides the opportunity for the extension of an earlier viewpoint. In solving $x^2 - 4x + 13 = 0$, we have also found the zeros of the function whose equation is $y = x^2 - 4x + 13$.



$y = x^2 - 4x + 13$

x	0	1	2	3	4
y	13	10	9	10	13

Fig. VI–9

Consider the graph of this equation (Fig. VI–9). The graph does not cross the X-axis. There are no x-intercepts. Consequently there are no real values of x for which $y = 0$. Nevertheless the function has zeros ($2 + 3i$ and $2 - 3i$) but in the field of complex numbers.

EXERCISES (VI–4)

1. Find the factors of each of the following:

a. $x^2 + 4x + 3$
b. $a^2 - 3a - 4$
c. $2m^2 + 3m - 5$
d. $h^2 - 11h - 12$
e. $12y^2 + 11y - 6$
f. $k^2 - 9$
g. $4t^2 - 25$
h. $x^2 + 2ax + a^2$
i. $3a^2 - 5ab - 2b^2$
j. $15 - t - 2t^2$
k. $4x^2 + 12xy + 9y^2$
l. $ka^2 - 3ak - 4k$

2. The expression $x^2 + 4$ is not factorable in the field of real numbers. However, it is factorable in the field of complex numbers. Find the factors.

3. Find the zeros of the following equations:

a. $y = 2x^2 - 3x - 5$
b. $y = x^2 - 6x + 9$
c. $y = 6x^2 + x - 12$
d. $y = 10 + 3x - x^2$

4. Solve the following equations and check the answers:

a. $x^2 - 8x - 20 = 0$
b. $4x^2 - 11x + 6 = 0$
c. $3h^2 + h = 1$
d. $5y^2 = 6 - 13y$
e. $\frac{2}{5}x^2 - \frac{3}{5}x = -1$
f. $10h - \frac{3}{h} = 1$
g. $\frac{35}{a} - \frac{12}{a^2} = 18$
h. $2x^2 - 0.5x = 0.25$
i. $16y^2 - 9 = 0$
j. $25x^2 - 16 = 0$
k. $x^4 - 1 = 0$

5. The dimensions of a rectangle differ by 7 inches and the area is $51\frac{3}{4}$ square inches. Find the dimensions.

6. The length of a rectangular sheet of tin is 2 inches more than the width. Two-inch squares are cut from each of the corners, and the sides are bent up to form a rectangular box whose capacity is 63 cubic inches. Find the dimensions of the tin.

7. Trigonometric equations may also take on the quadratic form. For example, $2 \sin^2 \theta - \cos \theta = 1$ is such an equation. However, before proceeding to the techniques appropriate to the quadratic, it is necessary to have only one function-value expression. By recalling that $1 - \cos^2 \theta$ is equivalent to $\sin^2 \theta$, we substitute and collect similar terms. We get $2 \cos^2 \theta + \cos \theta - 1 = 0$. This is factorable as $(2 \cos \theta - 1)(\cos \theta + 1) = 0$. The first factor yields $\cos \theta = \frac{1}{2}$, and so, $\theta = \pi/3$, $5\pi/3$. The other factor is handled similarly Solve each of the following for 0 to 2π, inclusive.

a. $2 \sin^2 x + \sin x - 1 = 0$
b. $1 - 2 \sin^2 x = \cos x$
c. $\tan^2 x = 3$
d. $\tan^2 y - 3 \tan y + 2 = 0$
e. $2 \sin^2 x + 3 \cos x - 1 = 0$
f. $\sec^2 \theta - 3 \sec \theta + 2 = 0$
g. $2 \cos^2 \theta = 3 + 5 \cos \theta$
h. $5 \cot t = 5 - 3 \csc^2 t$

8. Trace the steps used in developing the quadratic formula with the solution of the equation $4x^2 - 3x - 4 = 0$.

9. Solve each of the following:

a. $3x^2 + 2x = 5$
b. $x^2 + x - 3 = 0$
c. $3x^2 - 2x = 4$
d. $5y^2 + 3y - 1 = 0$
e. $x^2 + 1 = 0$
f. $t^2 - 6t + 25 = 0$
g. $1 - \dfrac{2}{h} + \dfrac{6}{h^2} = 0$
h. $\sin^2 x - 4 \sin x - 1 = 0$
i. $\sec^2 x = 6 \tan x - 6$
j. $9 \sin^2 x + 30 \cos x = 20$
k. $4 \sec^2 x - 16 \sec x + 13 = 0$

10. It is possible to reverse the process of solving quadratics to determine the equations which have given roots. Suppose, for example, that 3 is a root of a quadratic equation. This means that $x = 3$ or $x - 3 = 0$, and so $(x - 3)$ is a factor of the quadratic. If $x = \frac{1}{2}$ is the other root of the same equation, then $2x = 1$ or $2x - 1 = 0$, and so $(2x - 1)$ is the other factor. Then $(x - 3)(2x - 1) = 0$, or $2x^2 - 7x + 3 = 0$ is the second-degree equation whose roots are 3 and $\frac{1}{2}$. Note that x is, in effect, a place holder. Any other symbol would do as well.

Find the quadratic equations with the following roots:

a. $-3, -2$
b. $\frac{1}{2}, 4$
c. $\frac{1}{2}, \frac{1}{3}$
d. $-\frac{1}{2}, \frac{2}{3}$
e. $3i, -3i$
f. $1 + 2i, 1 - 2i$
g. $2 + \sqrt{5}, 2 - \sqrt{5}$
h. r_1, r_2

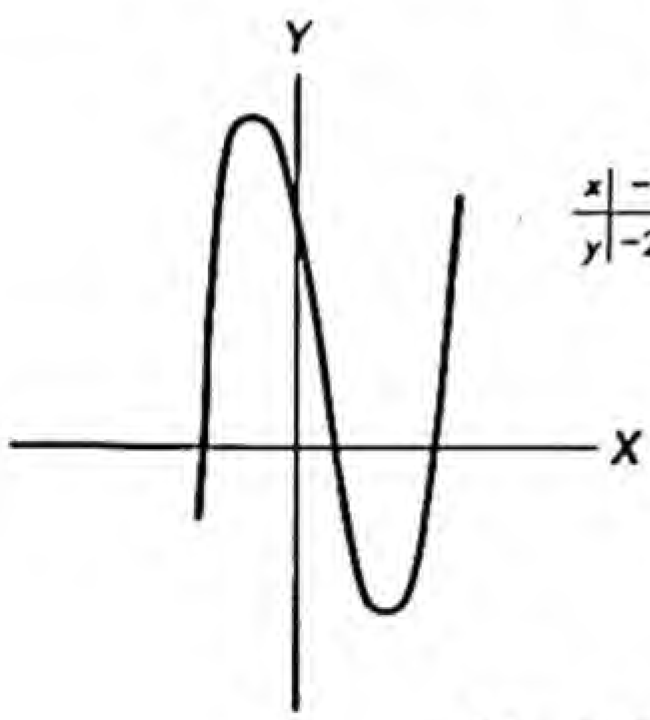
11. If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, prove that $r_1 + r_2 = -(b/a)$ and $r_1 r_2 = -c/a$.

12. Use the conclusion in exercise 11 or exercise 10(h) to find the quadratic equations whose roots are:

a. $5, -\frac{1}{2}$ b. $3 + 2i, 3 - 2i$ c. k, k^2

5. HIGHER DEGREE EQUATIONS

We turn our attention to a third-degree, cubic equation: $y = x^3 - 2x^2 - 5x + 6$. To obtain the graph, we need, of course, a sufficient number of tabular values. We could begin with $x = 0$ and then with some values to the right of this followed by some to the left, plotting the points as we go along. The distribution of the points on the graph will usually indicate the direction to pursue to get a fair portion of the curve. The process of finding a y-value is, as before, one of substitution. Thus, when $x = 4$, $y = (4)^3 - 2(4)^2 - 5(4) + 6 = 64 - 32 - 20 + 6 = 18$.



$y = x^3 - 2x^2 - 5x + 6$

x	-3	-2	-1	0	1	2	3	4
y	-24	0	8	6	0	-4	0	18

Fig. VI–10a

The determination of the tabular values provides an opportunity for a digression concerning a technique of considerable merit. Let us consider the general third-degree polynomial in one unknown: $ax^3 + bx^2 + cx + d$.

This expression can be evolved by continued multiplication by x in the following manner:

1. We take the first coefficient	a
2. Multiply by x and get	ax
3. Add to the next coefficient and get	$ax + b$
4. Multiply by x	$ax^2 + bx$
5. Add to the next coefficient	$ax^2 + bx + c$
6. Multiply by x	$ax^3 + bx^2 + cx$
7. And add to the next coefficient to get	$ax^3 + bx^2 + cx + d$

The whole process may be summarized in a horizontal form. There are just two operations: (1) the vertical addition with each result multiplied

by x, in the multiplier corner, and (2) the product entered in the next column, as shown by the arrows. This is continued until the last column.

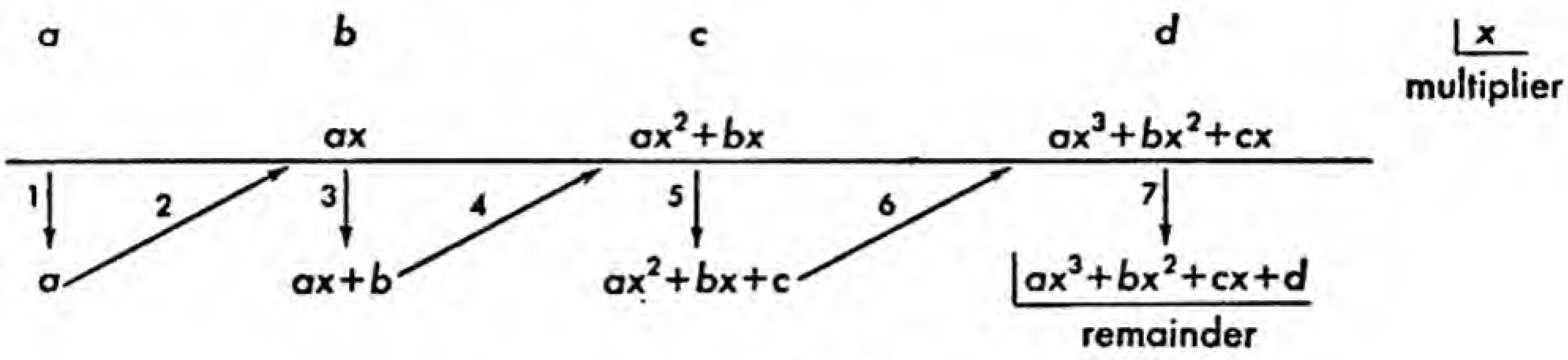


Fig. VI–10

The steps are numbered in Fig. VI–10 to correspond to the preceding vertical outline. The last sum, called the **remainder,** represents the value of the original expression and is obtained entirely from the coefficients and x.

Suppose that we wanted the value of the original expression, $ax^3 + bx^2 + cx + d$, for $x = 3$. We would follow the same procedure excepting that we would use 3 as the multiplier instead of x. We would get $27a + 9b + 3c + d$, which is precisely what we would get if we substituted 3 for x. Finally, if the coefficients of the trinomial were particular integers, the remainder would be an integer which (and this is important!) represented the value of the expression for the particular value of x that was to be used.

We return now to the equation $y = x^3 - 2x^2 - 5x + 6$, which we have just graphed. We had to find the value of y for various values of x by direct substitution. Let us illustrate how this can be done for three cases by means of **synthetic division.**

Given: $y = x^3 - 2x^2 - 5x + 6$.

If $x = 2$:

$$\begin{array}{rrrrl} 1 & -2 & -5 & 6 & \underline{|2} \\ & 2 & 0 & -10 & \\ \hline 1 & 0 & -5 & \underline{|-4} & \end{array} \qquad \text{then } y = -4$$

If $x = 3$:

$$\begin{array}{rrrrl} 1 & -2 & -5 & 6 & \underline{|3} \\ & 3 & 3 & -6 & \\ \hline 1 & 1 & -2 & \underline{|0} & \end{array} \qquad \text{then } y = 0$$

If $x = -3$:

$$\begin{array}{rrrrl} 1 & -2 & -5 & 6 & \underline{|-3} \\ & -3 & 15 & -30 & \\ \hline 1 & -5 & 10 & \underline{|-24} & \end{array} \qquad \text{then } y = -24$$

After a little experience the whole process can be done at sight and with considerable ease. Let us consider one other illustration. Suppose that it is

desired to have the value of the polynomial $2x^4 - 5x^3 - 5x^2 - 12$ when $x = 3$.
(Note the 0 coefficient for the missing x term.)

$$\begin{array}{rrrrr|l} 2 & -5 & -5 & 0 & -12 & \underline{3} \\ & 6 & 3 & -6 & -18 & \\ \hline 2 & 1 & -2 & -6 & \underline{|-30} & \end{array} \qquad \text{polynomial} = -30 \text{ when } x = 3$$

Synthetic division provides us with a particularly simple and valuable method in connection with the finding of roots of algebraic equations. We recall that, by means of the graph, we found the zeros of the function defined by $y = x^3 - 2x^2 - 5x + 6$. These were $x = -2$, $x = 1$, and $x = 3$. This is equivalent to saying that these are the roots of $x^3 - 2x^2 - 5x + 6 = 0$.

On many occasions it is desirable and even necessary to determine the real roots of an equation in one unknown if such exist. Such roots are not always rational. The synthetic division method is excellent for integral roots and even for rational roots. There are a number of methods of approximating irrational roots, one of which we shall take up when the background information for it has been developed.

We have seen that the roots of a polynomial are those that make the polynomial identically zero. Suppose that we have a polynomial or a polynomial equation, how can we find the roots? Synthetic division provides us with a method of quickly getting values for the polynomial. The real number domain, however, is too vast a canvass from which to make random selections of possible roots. Fortunately there are clues.

Let us go on a search for some general clues. Suppose that $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ has a rational root p/q, where p and q are integers, as are the coefficients of the polynomial. We assume too that the fraction is in its lowest form, so that p and q are relatively prime to each other. If p/q is a root, then

$$(1) \qquad a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0$$

We simplify the equation by multiplying both members by q^n, the least common denominator. This yields

$$(2) \qquad a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0$$

Further, we divide through by p and get

$$(3) \qquad a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1} + \frac{a_nq^n}{p} = 0$$

The last term, $\frac{a_nq^n}{p}$, provides us with our first clue. Since each of the preceding terms in (3) is an integer (each term is a product of integers),

this last term must be an integer too, for otherwise the sum could not possibly be 0. This means that p must be a factor of a_n, since by hypothesis it is prime to q. This is our first clue.

Specifically, in $2x^3 + 5x^2 - 23x + 10 = 0$, p must be a factor of 10 if the equation has a rational root.

Our second clue is obtained by a sort of reverse twist of the foregoing. We divide both members of the equation in (2) by q instead of by p. We get

$$\frac{a_0p^n}{q} + a_1p^{n-1} + \cdots + a_{n-1}pq^{n-2} + a_nq^{n-1} = 0$$

The first term is now the place for attention. Since all the other terms are integers, this term too must be an integer to obtain an algebraic sum of 0. Then q must be a factor of a_0, since it is prime to p.

In $P(x) = 2x^3 + 5x^2 - 23x + 10 = 0$, any rational root p/q must be such that q is a factor of 2 and p a factor of 10. Consequently the following sets are the permissible values:

$$p:\{\pm 1, \pm 2, \pm 5, \pm 10\}$$

and

$$q:\{\pm 1, \pm 2\}$$

and so

$$\frac{p}{q}:\{\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm 5, \pm 10\}$$

By trying the various possibilities in the set p/q, we find that 2, -5, and $\frac{1}{2}$ are the roots of the equation. We illustrate two of the three synthetic divisions,

$$\begin{array}{rrrr|l} 2 & 5 & -23 & 10 & \underline{2} \\ & 4 & 18 & -10 & \\ \hline 2 & 9 & 5 & \underline{|0} & \end{array} \qquad \begin{array}{rrrr|l} 2 & 5 & -23 & 10 & \underline{\frac{1}{2}} \\ & 1 & 3 & -10 & \\ \hline 2 & 6 & -20 & \underline{|0} & \end{array}$$

For the purpose of indicating other values of synthetic division, let us suppose that we know only that $x = 2$ is a root of the last equation. In this case $(x - 2)$ must be a factor of $P(x)$. The coefficients of the other factor are always found in the last line of the synthetic division, which in this case consists of 2, 9, and 5. These, the 2, 9, and 5, are the coefficients of a second-degree polynomial in the same variable. We have then

$$(2x^2 + 9x - 5)(x - 2) = 0$$

That this product gives the original polynomial can be verified by actual multiplication. From the product equation, we deduce also the possibility that

$$2x^2 + 9x - 5 = 0$$

This quadratic can now be solved by factoring, or by the formula, or even by synthetic division. In any case, we get the other two roots, $\frac{1}{2}$ and -5.

We recapitulate the whole process through another illustration.

$$3x^3 - 2x^2 - 7x - 2 = 0 \qquad p:\{\pm 1, \pm 2\}$$

$$\begin{array}{rrrrl} 3 & -2 & -7 & -2 & \underline{|-1} \\ & -3 & 5 & 2 & \\ \hline 3 & -5 & -2 & \underline{|0} & \end{array} \qquad q:\{\pm 1, \pm 3\} \qquad \frac{p}{q}:\left\{\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}\right\}$$

$$3x^2 - 5x - 2 = 0$$
$$(3x + 1)(x - 2) = 0$$
$$3x + 1 = 0 \qquad x - 2 = 0$$
$$x = -\tfrac{1}{3}, \qquad x = 2, \qquad \text{and} \qquad x = -1$$

Roots are: $\{-\frac{1}{3}, 2, -1\}$

CONVERSE: If $x = 2$, then

$$\begin{aligned} 3x^3 - 2x^2 - 7x - 2 &= 3(2)^3 - 2(2)^2 - 7(2) - 2 \\ &= 24 - 8 - 14 - 2 \\ &= 0 \end{aligned}$$

EXERCISES (VI-5)

1. Graph the following:

   a. $y = x^3 - 4x$ c. $y = -x^3 + 3x^2 - 4$
   b. $y = x^3 - 9x^2 + 23x - 15$ d. $y = x^4 - 5x^2 + 4$

2. If $f(x) = 2x^3 - 3x^2 + 4x - 5$, find:

   a. $f(3)$ b $f(-2)$ c. $f(0)$ d. $f(4)$ e. $f(\frac{1}{2})$

3. If $g(x) = 3x^4 - 2x^3 + 7x - 2$, find:

   a. $g(1)$ b. $g(-3)$ c. $g(4)$

4. Find the roots of the following equations:

   a. $x^3 - 2x^2 - 5x + 6 = 0$
   b. $2x^3 - 13x^2 + 22x - 8 = 0$
   c. $x^3 - 5x^2 + 11x - 15 = 0$
   d. $y^4 + 2y^3 - 13y^2 - 38y - 24 = 0$

5. Determine the cubic equation whose roots are:

   a. 3, 1, -2 b. $\frac{1}{2}$, 4, -1 c. 2, $2 + i$, $2 - i$

6. If the roots of $ax^3 + bx^2 + cx + d = 0$ are r_1, r_2, and r_3, determine the relationships between the coefficients and the roots.

7. a. If $g(y) = y^3 + 4y^2 - 4y + k$ and $g(-1) = 3$, find k.
   b. If $f(x) = x^3 - 3x^2 + kx + 6$ and $f(2) = 4$, find k.

VI-5 REVIEW

1. Find a polynomial of lowest degree and with integral coefficients which has the roots:

   a. 1, -1, 2 c. 2, 3, -5
   b. 2, $1 - i$, $1 + i$ d. 2, $-1 + i\sqrt{3}$

2. Find the roots of the following equations:

a. $x^3 + x^2 - 4x - 4 = 0$
b. $x^3 + 1 = x^2 + x$
c. $x + 2 - \frac{1}{x^2} = 0$
d. $6x^3 + 7x^2 = x + 2$
e. $x^4 - 13x^2 + 36 = 0$
f. $\csc x \cos x + 2 \cos x = \csc x + 2$
g. $2 \sin^2 x - 1 = 3 \sin x$

3. Find the roots of $x^2 - ix + 2 = 0$.

4. Draw the graph of:

a. $y = x^3 + x^2 - 4x - 4$
b. $y = x^4 - 10x^2 + 9$

5. a. If $P(x) = (1 + i)x - 1 + i$, find $P(i)$.
b. Show that $P(-i)$ is a root of $P(x) = 0$.

6. $P(x)$ is a polynomial with integral coefficients. If x is defined only for the set of rational numbers, then $P(x)$ is a rational number. Explain and illustrate.

7. a. A rational function R is defined as the quotient P_1/P_2, where P_1 and P_2 are polynomial functions and $P_2 \neq 0$. Illustrate.
b. Define a rational function value.

8. Show that a polynomial with complex coefficients can be expressed as $A + Bi$, where A and B are polynomial with real coefficients.

9. Graph the equation $y = x^3$ and its inverse for real values of x and y.

10. The inverse of a function is sometimes a relation. Explain and illustrate.

11. Solve the equations:

a. $by^2 - by = y - 1$
b. $3z - 5\sqrt{z} - 2 = 0$ (Let $\sqrt{z} = x$)
c. $x^2 - \sqrt{2}x - 2 = 0$

12. In the quadratic equation $ax^2 + bx + c = 0$, where the coefficients are rational numbers:

a. If $b^2 - 4ac > 0$, then the roots are real numbers.
b. If $b^2 - 4ac = 0$, then the roots are real and equal.
c. If $b^2 - 4ac < 0$, then the roots are imaginary.

Explain and illustrate each proposition.

13. Use De Moivre's theorem to find the roots of $x^2 - i = 0$.

14. The following are factorable in the domain of complex numbers. Find their factors.

a. $x^2 + 1$
b. $x^4 + 1$
c. $x^3 + ix^2 + x + i$

(Synthetic division may be used.)

6. BINOMIAL THEOREM

There will be a need later on to know a bit more about binomials, particularly about $(a + b)^n$, where n is any non-negative integer. Well, we do know that

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$

and by continuing another step or two by multiplication, we get

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The task of multiplication is getting tedious. We can begin to organize our efforts in a search for some possible inherent pattern. In going on to the next power, we shall have to multiply $(a + b)^4$ by $(a + b)$. In the light of the distributive postulate, each term of the former will have to be multiplied by each term of the latter. Let us seek the end result from this viewpoint. The work is organized with this in mind.

$$\begin{array}{rcl}
(a+b)^4 & = & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
a(a+b)^4 & = & a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 + \\
+b(a+b)^4 & = & + \quad a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\
\hline
(a+b)(a+b)^4 = (a+b)^5 & = & a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}$$

A number of tentative generalizations are possible. In all the cases thus far, the a's are descending in power, while the b's, starting as a factor with power 1 in the second term, increase in power by 1 in each successive term. The highest power is the power of the binomial. The sum of the exponents of a and b in any term is the exponent of the binomial.

The coefficients in the first and last terms are 1. The other coefficients are obtained by adding together two coefficients of the expansion of the binomial to one power less. This last point must be seen in the development of the last case above. The third coefficient of $(a + b)^5$, for example, comes from $6 + 4$, which are the second and third coefficients of $(a + b)^4$. Similarly the fourth coefficient is the sum of the third and fourth coefficients of the previous power, and so forth.

This relationship among the coefficients was emphasized by Pascal in a triangular arrangement known as *Pascal's triangle*. Every number within the triangular form is equal to the sum of the two numbers directly above it in the preceding line.

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

The sixth line contains the coefficients for $(a + b)^6$. This, together with the observations regarding the literal factors, permits us to write at sight

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

There are other relations involving the determination of the coefficients. However, we call attention to one that is useful in writing an expansion

without reference to Pascal's triangle. The integers appearing in any one term in any expansion can be used unequivocally to determine the coefficient of the following term by the simple rule

$$\text{Next coefficient} = \frac{(\text{present coefficient}) \times (\text{present exponent of } a)}{(\text{exponent of } b) + 1}$$

We illustrate this with one of the previous cases:

$$(a + b)^4 = \quad a^4 \quad + \quad 4a^3b \quad + \quad 6a^2b^2 \quad + \quad 4ab^3 \quad + \quad b^4$$

$$\frac{1 \cdot 4}{1} = 4 \qquad \frac{4 \cdot 3}{2} = 6 \qquad \frac{6 \cdot 2}{3} = 4 \qquad \frac{4 \cdot 1}{4} = 1$$

In the first term, there being no b factor, we could imagine the presence of b^0, which is 1. Or, better, we can note that in all cases the second coefficient is the same as the power of the binomial. In this fashion, inductively, one can arrive at the general case

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}b^3 + \cdots + b^n$$

This is known as the **binomial theorem.**

EXERCISES (VI–6)

1. Noting that the a and the b in $(a + b)$ represent any algebraic terms whatsoever, find:

a. $(x + 2)^4$
b. $(x + 2y)^5$
c. $(x - 1)^6$
d. $\left(\frac{m}{2} + 4\right)^7$
e. $(a - b)^n$
f. $(1 + t)^3$
g. $(a + bi)^4$
h. $(\tan x + \cot x)^3$

2. Obtain results of the following correct to three significant figures:

a. $(1.02)^7 = (1 + 0.02)^7$ b. $(1.01)^{14}$ c. $(0.99)^5$

3. We make some further observations. Consider the coefficients in the expansion of $(a + b)^6$. The third coefficient, 15, came from $(6 \cdot 5)/2$. Now, we note that

$$\frac{6 \cdot 5}{2} = \frac{6 \cdot 5 \cdot (4 \cdot 3 \cdot 2 \cdot 1)}{1 \cdot 2 \cdot (4 \cdot 3 \cdot 2 \cdot 1)} = \frac{6!}{2!4!}$$

where $$6! = \text{factorial } 6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and, in general,

$$n! = n(n-1)(n-2)(n-3) \cdots 1$$

This building up of a simple fraction hardly seems the path toward simplification. Yet it is exactly that. The first illustration is for the third coefficient of $(a + b)^6$. The full term is $[6!/(2!4!)]a^4b^2$. We note that the numerator of the coefficients, is

the factorial of the power of the binomial and that the denominator is the product of the factorials of the two exponents appearing in the term. For consistency, we shall have to define $0! = 1$. *Why?*

We are in a position to write any term of a binomial expansion in isolated form. Suppose that we needed the sixth term of $(a + b)^8$. In that term one factor would be b^5, since the b's start a term late. Then we would have a^3 as another factor, since the sum of the exponents of a and b in any term must be the power of the binomial. These are enough to give us the coefficient too. Thus the sixth term is $[8!/(3!5!)]a^3b^5 = 56a^3b^5$.

Find the indicated term in each of the following:

a. $(a + b)^{12}$, 9th

b. $(x - 3)^{14}$, 6th

c. $\left(y + \frac{1}{2}\right)^{20}$, 7th

4. There is still another convenient symbol available by which to represent the binomial coefficients. By $C_{8,5}$ we mean $8!/[(8 - 5)!5!] = 8!/(3!5!)$ which is, of course, the coefficient of the sixth term in $(a + b)^8$. Similarly the 9th-term coefficient of $(a + b)^{15}$ is $C_{15,8} = 15!/(7!8!)$. In general

$$C_{n,r} = \frac{n!}{(n - r)!r!}$$

and this is the coefficient of the $r + 1$ term of $(a + b)^n$. This term in full is $C_{n,r}\, a^{n-r}\, b^r$.

Using this notation, find the indicated terms of the following:

a. $(a + b)^7$, 4th

b. $(2m - 3)^9$, 5th

c. $(k + 1)^{13}$, 8th

d. $\left(b - \frac{1}{b}\right)^{18}$, 9th

5. Find the values of each of the following:

a. $C_{8,2}$ b. $C_{12,10}$ c. $C_{12,2}$

d. Parts (b) and (c) suggest a generalization. What is it?

7. RATIONAL AND EXPONENTIAL FUNCTIONS

A rational function is one whose set of elements is defined by an equation of the form

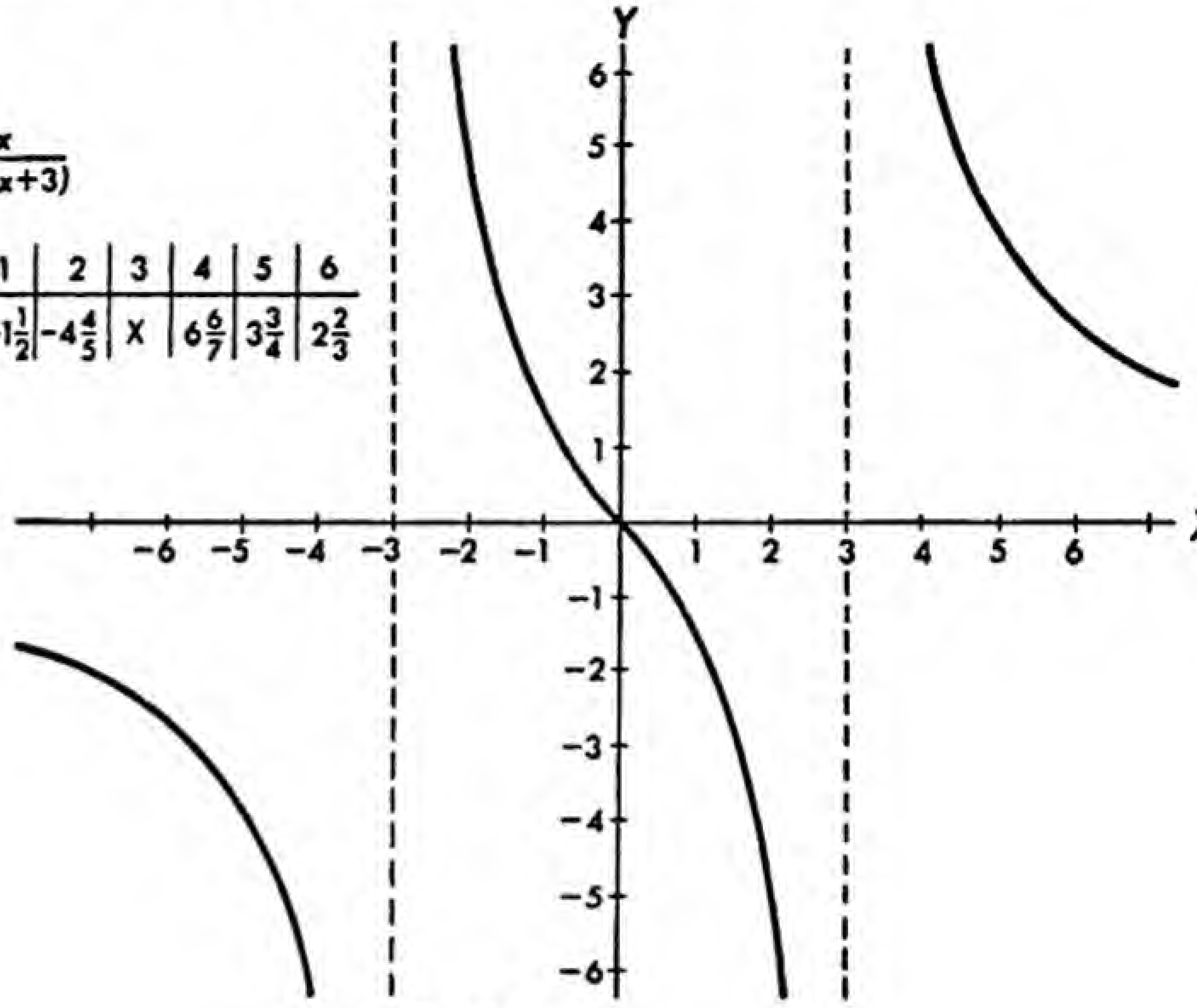
$$y = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$. The following equation and its graph (Fig. VI–11) is an illustration of this class:

$$y = \frac{12x}{(x - 3)(x + 3)}$$

The presence of two asymptotes may have been anticipated. The equa-

tions of the asymptotes are $x = -3$ and $x = 3$, which are excluded values in the function, since division by 0 is meaningless.



$y = \frac{12x}{(x-3)(x+3)}$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
y	$-3\frac{3}{4}$	$-6\frac{6}{7}$	X	$4\frac{4}{5}$	$1\frac{1}{2}$	0	$-1\frac{1}{2}$	$-4\frac{4}{5}$	X	$6\frac{6}{7}$	$3\frac{3}{4}$	$2\frac{2}{3}$

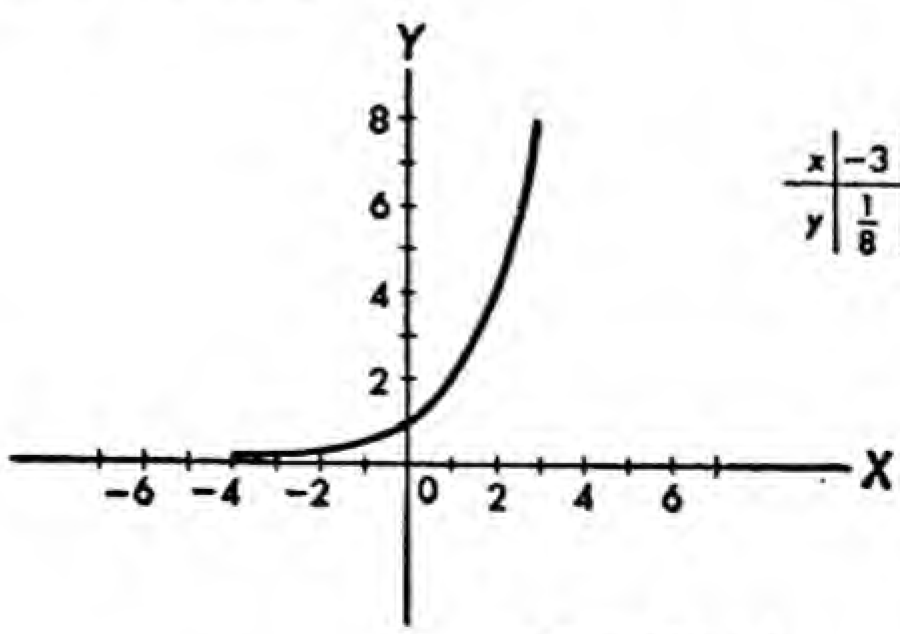
Fig. VI–11

It turns out that $y = 0$ is also an asymptote. To see this, one would have to solve the equation for x to find that $y = 0$ is an excluded value. (The solution leads to

$$x = \frac{6 \pm 3\sqrt{4 + y^2}}{y}$$

by means of the quadratic formula.) This fact may also have been surmised if one noted that as $|x|$ gets larger and larger than 3, $|y|$ gets continually closer to 0.

Consider the function determined by the set of values $\{x, 2^x\}$. The expression 2^x is known as an **exponential** expression because of the position of the variable. The equation $y = 2^x$ is an *exponential equation.* (Refer to Figs. VI–12 and VI–13.)



$y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Fig. VI–12

$y = 2^{-x}$

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

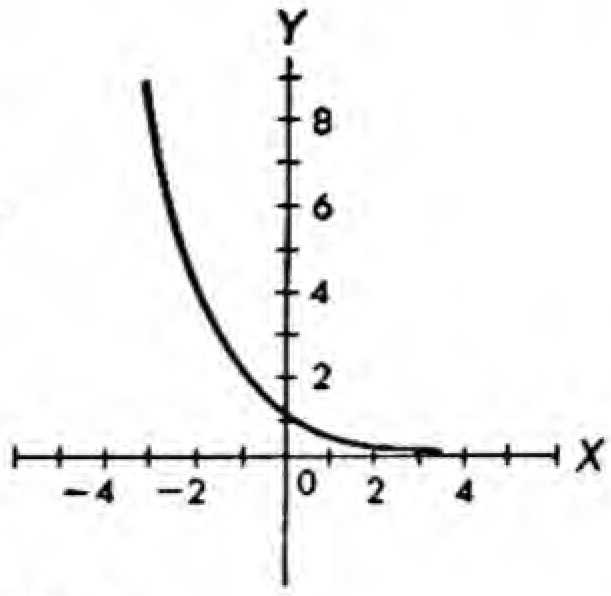


Fig. VI–13

EXERCISES (VI–7)

Indicate undefined values, domain, range, and equations of asymptotes, and draw the graphs of each of the following:

1. $y = \dfrac{x}{x-2}$
2. $y = \dfrac{1}{x^2}$
3. $y = \dfrac{1}{x(x-2)}$
4. $(x+1)^2 y = 1$
5. $y = \dfrac{4x-3}{2x}$
6. $y(x^2+1) = 4$
7. $y = \dfrac{2x}{(x+1)(x+3)}$
8. $y = \dfrac{1}{2^x}$
9. $y = \left(\dfrac{1}{3}\right)^x$
10. $y = 4^{-x}$
11. $y = 10^x$
12. $y = 2^{x-1}$
13. $y = \sin^2 x$
14. $y = 3^{-x^2}$

15. Which of the graphs may be judged as having symmetry about the Y-axis merely by inspection of the equation?

8. EXPONENTIAL INVERSE, OR THE LOGARITHM

We noted earlier that the quantity a^b is transcendental if a is an algebraic number other than 0 or 1 and b is an algebraic irrational. Thus, the set $\{x, a^x\}$ defines a *transcendental function* where x is any real number and a is neither 0 nor 1. Of course an infinite number of elements of this function will be algebraic; for instance, when x takes on algebraic rational values. But for the rest, and for a larger infinity, the elements contain transcendental numbers.

The graphs of 2^x and 2^{-x} represent, then, graphs of transcendental functions, Figs. VI–12 and VI–13. Of this class of functions, two play a particularly important role: $y = 10^x$ and $y = e^x$.

The number represented by $e = 2.71828183\cdots$, unlike other instances given before, is itself a transcendental number like π and plays an important

role in advanced mathematics and the sciences. We shall have occasion later to define it precisely and to use it extensively.

The new functions bring with them the need for additional attention, particularly with regard to their inverses. We shall see that the inverses have been the basis for one of the more remarkable discoveries in mathematics.

To obtain the inverses of $y = 10^x$ and $y = e^x$, we begin this time with the interchange of the variables x and y so that x may remain, as before, the independent variable. We get $x = 10^y$ and $x = e^y$, respectively. To solve these for y, we need both a *name* and a *symbol*, for we have neither as yet. The name is **logarithm** and the symbol is the abbreviation of the name, **log.** If the base is 10 in the original function, we indicate this in the inverse by "$\log_{10}$". If it is e, we write "$\log_e$" or, soon, "ln."

So, from $x = 10^y$ we get $y = \log_{10} x$

and from $x = e^y$ we get $y = \log_e x$

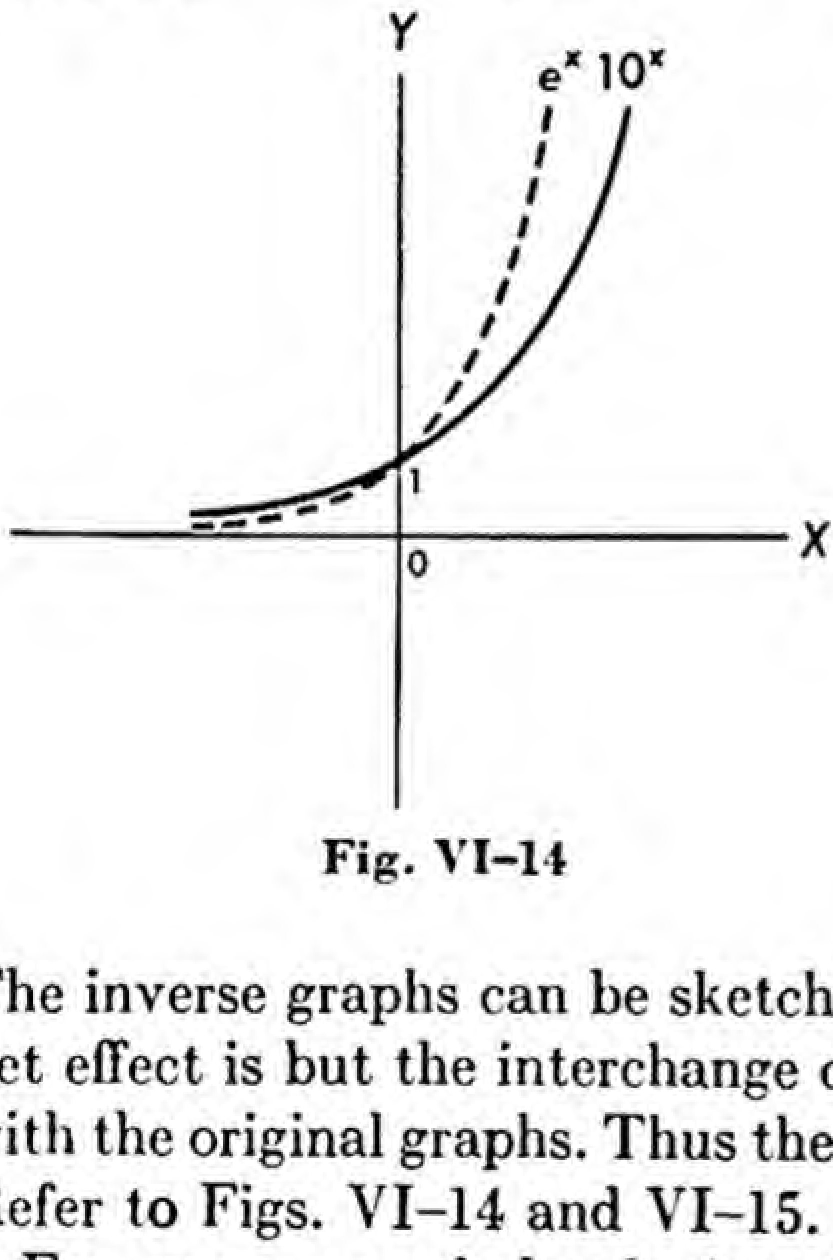


Fig. VI-14

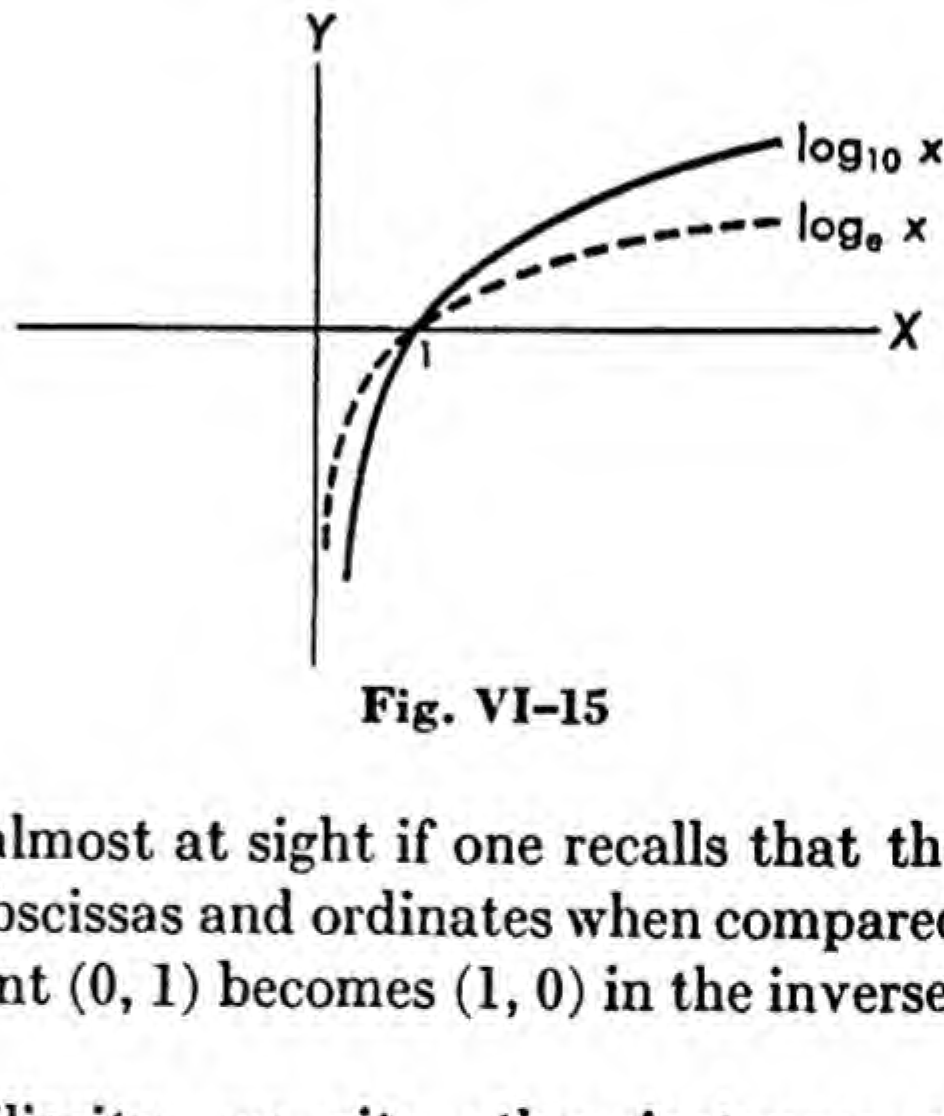


Fig. VI-15

The inverse graphs can be sketched almost at sight if one recalls that the net effect is but the interchange of abscissas and ordinates when compared with the original graphs. Thus the point (0, 1) becomes (1, 0) in the inverse. Refer to Figs. VI-14 and VI-15.

For purposes of developing familiarity, we cite other instances of exponential equations and their logarithmic counterparts.

Exponential Equation	Logarithmic Equation
$8 = 2^3$	$3 = \log_2 8$
For base 2, the exponent 3 yields 8.	For base 2, the log of 8 yields 3.
$81 = 3^4$	$4 = \log_3 81$
$1 = n^0 \quad (n \neq 0)$	$0 = \log_n 1$
$b = b^1 \quad (b \neq 0)$	$1 = \log_b b$
$N = b^y$	$y = \log_b N$

The illustrations emphasize the fact that the value representing a logarithm is an exponent. It follows, therefore, that the rules of operating with logarithms are the rules of exponents. A simple numerical example, trivial to be sure but nevertheless useful, will indicate the significance and potential value of logarithms. Note the substitutions in the following example. The only reference needed is a list of powers of 2 to about the 12th power.

$$\frac{128 \cdot 4096}{1024} = \frac{2^7 \cdot 2^{12}}{2^{10}} = \frac{2^{19}}{2^{10}} = 2^9 = 512$$

The most important observation to make is that arithmetic multiplication is replaced by exponential addition, and division takes the form of exponential subtraction. Thus exponential addition (meaning logarithmic addition) can be substituted for arithmetical multiplication. And subtraction of logarithms can replace arithmetic division.

Suppose that we make some changes in the preceding illustration as follows:

$$\frac{(128)^2 \cdot \sqrt[4]{4056}}{\sqrt{1024}} = \frac{(2^7)^2 \cdot (2^{12})^{1/4}}{(2^{10})^{1/2}} = \frac{2^{14} \cdot 2^3}{2^5} = \frac{2^{17}}{2^5} = 2^{12} = 4056$$

We note that squaring a number becomes doubling an exponent, which means doubling a logarithm. Taking the fourth root or square root of a number is equivalent to taking a fourth or a half respectively of the corresponding exponent or logarithm.

The two illustrations are indicative of the potential worth of the theory of logarithms. To be sure, the illustrations were carefully selected to highlight the theory effectively. Suppose that instead of 128, we had used 135. Then, since $128 < 135 < 256$, we have $2^7 < 2^n < 2^8$, where 2^n represents the value of 135. That is, since 135 lies between 128 and 256, its logarithm to the base 2 lies between 7 and 8. We might have anticipated the fact that n cannot be a rational value, for if it were, we would have

$$135 = 2^{m/p}$$

For the exponent to be rational and in its simplest form, we take it so that m and p are each integers and that m/p is in its lowest terms. If we raise both members of the equation to the pth power, we get

$$135^p = 2^m$$

This is impossible because 2 is not a factor of 135, and so, no product of an integral number of 2's can ever be equal to a product of an integral number of 135's. Consequently the original exponent n must be an irrational number.

If we divided the set of real numbers $\{n\}$ into two classes A and B, such that A contained all those n's for which $2^n < 135$ and B all those n's for which $2^n > 135$, we would have a Dedekind cut where $2^n = 135$. This defines the irrational exponent n. Consequently the logarithm of 135 to the base 2 may be approximated to any degree of accuracy. The technique of approximation must await the development of the calculus. The base 2 was selected for illustration purposes only. Any positive base, other than 0 or 1, would do as well at this point.

Two bases for logarithms are in common use. There is **base 10**, developed by Briggs, which has certain practical convenience because ours is a decimal number system. Then there is the **base e**, discovered at about the same time by Napier and independently of Briggs. The first one is called *common logarithms* and the other is *natural logarithms*. The value of the latter, which may seem remote at the moment, resides in the fact that the rate of change of e^x is precisely e^x; the measure of change is the quantity itself. For example, the rate of change of a chemical reaction may depend on the amount of the chemical substance present at the moment of the change in question. The rate of change in numerous instances involving growth, decay, vibration, and other phenomena may depend on the condition as it exists at the instant in question. These are ever-present situations in mathematics and science. This makes plausible the merit of e as a base in analytic and theoretical situations.

It is time to translate the exponential laws into their equivalents in logarithmic form. The letter b will be a positive number other than 1 and will represent the base. That is, $(b > 0) \wedge (b \neq 1)$.

We let the letters M and P represent any numbers of the set of positive real numbers, and the letters u, v, and k are any real numbers.

	Exponential Law		Logarithmic Law
If	$M = b^u$	then	$\log_b M = u$
and	$P = b^v$	then	$\log_b P = v$
Since	$MP = b^u b^v = b^{u+v}$	then	$\log_b MP = u + v = \log_b M + \log_b P$
Since	$\dfrac{M}{P} = \dfrac{b^u}{b^v} = b^{u-v}$	then	$\log_b \dfrac{M}{P} = u - v = \log_b M - \log_b P$
Since	$(P)^k = (b^v)^k = b^{vk}$	then	$\log_b P^k = vk = k \log_b P$

It is very important to examine the right-hand side again and again to get the import of the conclusions. The logarithm of a product of two numbers is equal to the sum of logarithms of the two numbers. The logarithm of a quotient of two numbers is equal to the difference of the logarithms of the numbers. And the logarithm of a number raised to a power is equal to the power multiplied by the logarithm of the number.

We can illustrate these important laws and make the conclusions more

specific. For convenience the indication of the base is omitted. The results would be the same for any base b as defined above.

$$\log(378 \cdot 57.3) = \log 378 + \log 57.3$$

$$\log \frac{378}{57.3} = \log 378 - \log 57.3$$

$$\log \sqrt[3]{(378)^2} = \log (378)^{2/3} = \tfrac{2}{3} \log 378$$

$$\log \frac{(562)^3 \sqrt[3]{73.1}}{(69.2)^4} = 3 \log 562 + \frac{1}{3} \log 73.1 - 4 \log 69.2$$

It has already been indicated that there are two bases in general use. We make the following agreements for convenience: If no base is indicated, then the base 10 is understood ($\log 17 = \log_{10} 17$); if the base e is being employed, we use the abbreviation "ln" for the logarithm (**ln 17 = $\log_e 17$**). Should any other base be used, we have to indicate that explicitly.

In connection with base 10, all numbers can be written or understood to be in scientific notation. For example, $562 = 5.62 \times 10^2$, and so, $\log 562 = \log 5.62 + 2 = 2.7497$. This follows from the law of multiplication with logarithms and the power law. By the power law, again, it follows that

$$\log(562)^3 = 3 \log 562 = 3(2.7497) = 8.2491$$

The integer 2 in 2.7497 is called the **characteristic**, and the approximate number 0.7497 is called the **mantissa** and is obtained from the table of mantissas. The table of mantissas gives the decimal portion of the logarithmic number. The numbers 562, 5.62, 5620, 56200, 56.2, 0.562, 0.000562, and so forth, may all be expressed as $5.62 \cdot 10^n$. In each case n will be the characteristic and the mantissa will be the log 5.62. When the characteristic is 0, as in $5.62 = 5.62 \cdot 10^0$, the mantissa is identical with the logarithm of the number. Indeed the characteristic must be 0 whenever the number has just one significant figure in front of the decimal point. After a little practice, you can discover a rule for finding the characteristic for any number. By way of further illustrations, consider:

$$\log \sqrt[3]{73.1} = \log (7.31 \cdot 10^1)^{1/3} = \tfrac{1}{3}(1.8639) = 0.6213$$

$$\log (69.2)^4 = \log (6.92 \cdot 10^1)^4 = 4(1.8401) = 7.3604$$

The preceding three illustrations may be involved in a single exercise as the following:

$$\log \frac{(562)^3 \sqrt[3]{73.1}}{(69.2)^4} = 8.2491 + 0.6213 - 7.3604 = 1.5100 = (0.5100 + 1)$$

and so

$$\frac{(562)^3 \sqrt[3]{73.1}}{(69.2)^4} = 3.24 \cdot 10^1 = 32.4$$

In the last step, the process is inverted from the logarithmic or exponential number 1.5100, in base 10, to its arithmetic decimal number equivalent of 32.4. The 0.5100 is the mantissa which we refer to the table of mantissas. The nearest mantissa to this in the table is 0.5105 for the number 3.24. This gives us the nearest value to three significant figures, which is all that we can use. The characteristic *1* which resulted indicated that the multiplier of 3.24 was 10^1. The entire inverse process is called the **antilogarithm.** Thus antilog $(1.5100) = 32.4$.

Numbers between 0 and 1 present a minor difficulty that require special attention. Thus,

$$\log(0.0764) = \log(7.64 \cdot 10^{-2}) = 0.8831 - 2$$

Should we combine the mantissa and the negative characteristic, we would get the negative result -1.1169. This negative result can be written as $-0.1169 - 1$. However, we cannot emphasize too strongly that this includes a negative mantissa and cannot be referred to our table of mantissas, which (by its method of construction) consists exclusively of positive mantissas. It is perfectly correct to work with negative mantissas, but they are inconvenient in connection with the antilogarithm. Therefore we must convert somehow to a positive mantissa. There are many devices for achieving this end. In fact, if the reader clearly sees the problem, he can invent a device of his own. We offer the following suggestion: Leave the original result as a binomial logarithm. That is, $\log(0.0764) = 0.8831 - 2$.

To achieve simplicity in the construction of tables, particularly the logarithms of trigonometric functions, the integral part of the binomial needs to be standardized. This is always taken to be -10. Consequently -2 is replaced by $8 - 10$, -3 by $7 - 10$, and so forth. Thus $0.8831 - 2$ becomes

$$0.8831 + 8 - 10 = 8.8831 - 10$$

With a little experience, this can be accomplished entirely at sight.

The binomial approach helps to avert the problem of a negative mantissa.

$$\begin{aligned}\log\frac{0.00723}{0.0568} &= 0.8591 - 3 - (0.7543 - 2) = 7.8591 - 10 - (8.7543 - 10)\\ &= 17.8591 - 20 - (8.7543 - 10) = 9.1048 - 10\\ \frac{0.00723}{0.0568} &= 0.127\end{aligned}$$

Now, a negative mantissa may also arise as a result of the subtraction process. In the last illustration, if we had subtracted the first results, we would have had a negative mantissa. This was avoided by using $17 - 20$ instead of $7 - 10$ for -3.

One other condition needs attention. Suppose that we had to take one-third of $(0.7543 - 2)$ or one-third of its equivalent, $8.7543 - 10$. One-third of the positive portion is not likely to come out even. The remainder may not be carried over to the second term. Instead, we round off the result, with the usual stipulation of adding one if the fraction left over is one-half or more. However, one-third of the second term, -10, presents a serious problem, for a characteristic refers to an integral number of decimal places. Its origin in 10^n indicates that in our scientific notation. Further, the characteristic cannot be rounded off because we cannot make something meaningful out of something that is inherently meaningless.

The problem then is to see whether it is possible to alter the form, not the value, of our logarithm so that in division the characteristic portion will always remain an integer. The solution that has already been developed for the mantissas is adequate here too. This time we concentrate on the second term of the binomial logarithm. If the characteristic is -2, we substitute $18 - 20$ if we plan to divide by 2, $28 - 30$ if by 3, and so forth. In $\frac{1}{3}(0.7543 - 2)$, we want the second term in the parentheses to be a multiple of 3. So, we use $28.7543 - 30$. Now the problem is solved:

$$\tfrac{1}{3}(28.7543 - 30) = 9.5848 - 10.$$

$$\begin{aligned}\log \sqrt{0.00769} &= \tfrac{1}{2}\log(7.69 \cdot 10^{-3}) = \tfrac{1}{2}(0.8859 - 3) = \tfrac{1}{2}(17.8859 - 20)\\ &= 8.9430 - 10\end{aligned}$$

$$\sqrt{0.00769} = \operatorname{antilog}(8.9430 - 10) = 8.77 \cdot 10^{-2} = 0.0877$$

The reader is aware of the fact that we have dealt exclusively with numbers of three significant digits. Less digits present little difficulty, for numbers like 7.2 and 0.9 may be taken as 7.20 and 0.900, where the zeros are treated temporarily as significant. When the final answer is rounded off to the appropriate number of figures, due compensation will have been made. For numbers with four significant digits, a process of **interpolation** (the insertion between given numbers) can be used to get their logarithms from tables constructed especially for three significant figures. For numbers with more significant figures, more detailed tables are essential.

The logarithmic values in the tables are calculated with the aid of infinite series, an important subject at which we shall have a look later. It may be interesting to take a look at a relevant one now.

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

When $x = \frac{1}{3}$,

$$\frac{1+x}{1-x} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2.$$

If we calculate the sum of the first six terms of this series, we get $\ln 2 = 0.6931$. Similarly, by substituting $x = \frac{1}{2}$, we get,

$$\ln \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \ln 3 = 2\left(\frac{1}{2} + \frac{1}{24} + \frac{1}{160} + \cdots\right) = 1.0986$$

if six terms of the series are summed. While there are other logarithmic series that can serve the same ends, this one yields more significant figures in less terms than do others.

While we are at it, we may just as well anticipate the infinite series from which e itself is computed:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

EXERCISES (VI-8)

1. Explain the purpose of interchanging the variables in writing an inverse function equation.
2. Evaluate the following and write each statement in inverse form:

a. 2^5
b. 10^3
c. $4^{1/2}$
d. 3^2
e. $8^{2/3}$
f. $\log_{10} 1000$
g. $\log_{10} 10^2$
h. $\log_e e$
i. $\log_5 125$
j. $\log_9 3$
k. $\log_b b$

3. Let $M = b^x$ and $N = b^y$. Find:

a. $\log_b M$
b. $\log_b N$
c. MN
d. $\log_b MN$
e. $\dfrac{M}{N}$
f. $\log_b \dfrac{M}{N}$
g. M^k
h. $\log_b M^k$

4. If $3^n = 30$, show that n is an irrational number.
5. Evaluate the following by logarithms:

a. $\sqrt[3]{5.79}$
b. $(0.671)^2$
c. $\dfrac{52\sqrt[3]{61.8}}{(0.671)^2}$
d. $\dfrac{\sqrt[4]{273}(6.23)^2}{(84.2)^3}$
e. $\dfrac{4.7 \times 83.2}{0.89 \times 73.4}$
f. $\dfrac{4^{5.1}}{3^{6.2}}$
g. $\sqrt{\dfrac{73.9}{87.6}}$
h. $\dfrac{352 \sin 27°}{\sin 53°}$
i. $\dfrac{17 \tan 54°}{69}$

6. Use logarithms to solve the triangles for the indicated parts:

   a. $A = 58°$, $B = 42°$, $c = 26.7$; a
   b. $B = 72°$, $C = 43°$, $c = 1.67$; a
   c. $C = 142°$, $B = 10°$, $a = 379$; c.

7. Right triangle solutions and solutions involving the Law of Sines are amenable to the logarithmic approach without difficulty. However, the Law of Cosines, because of the additive character of its terms, cannot be related to logarithms. For such occasions, two additional formulas are available. For the *sss* case, there is the formula

$$\tan \frac{1}{2}A = \frac{r}{s-a}$$

where

$$s = \frac{1}{2}(a + b + c) \qquad \text{and} \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

As with other triangle formulas, the sense of the relationship is the important thing. So, if A and a were replaced by B and b, or C and c, the relationship would still be true.

For the *sas* case, there is the formula

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

where the exchange of letters, as before, is entirely possible.

Using the appropriate formulas, find the value of the indicated letter:

   a. $a = 46.1$, $b = 27.3$, $c = 32.9$; A.
   b. $a = 1.73$, $b = 9.62$, $c = 8.91$; B
   c. $b = 384$, $c = 476$, $A = 74°$; B, a.
   d. $a = 46.2$, $b = 38.7$, $C = 67°$; A.

8. a. Suppose that $N = 10^x$. Show that $\log N = \ln N/\ln 10$. (First take the logarithm of both members of the equation to base 10 and then to base e.)
   b Suppose that $N = e^y$. Show that $\ln N = \log N/\log e$.
   c. Using the fact that $\ln 10 = 2.3026$, find $\ln N$.
   d. Using the fact that $\log e = 0.4343$, show that the two formulas are equivalent.

9. Calculate the volume of a cube whose edge is 2.71 inches. ($V = e^3$.)

10. The volume of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height. Find h, if $V = 212$ cubic inches and $r = 4.69$ inches.

11. The volume of a sphere is given by $V = (4/3)\pi r^3$. Assuming the radius of the earth, taken as an approximate sphere, to be 3960 miles, find the volume.

12. Find the (approximate) surface of the earth if the formula for the surface of a sphere is $S = 4\pi r^2$.

13. The formula $A = P[1 + (r/k)]^{kn}$ gives the amount A of money that is obtained from a principal P which is deposited at the rate of interest r. The interest is compounded k times a year for n years.

Find the amount that will accrue from a deposit of \$125 at 4%, compounded quarterly for 6 years.

14. Using the formula in exercise 13, determine the number of years that it would take a sum of money to double itself if compounded quarterly at 4%?

VI-8 REVIEW

1. Find the fourth term of $[(1/x + x^2)]^5$.
2. Find the coefficient of a^5b^3 in $(a + 3b)^8$.
3. Express

$$\frac{\sec x \sin x}{\cot x + \tan x}$$

in terms of $\cos x$.

4. Solve each of the following for positive values less than 2π:

   a. $2 \sin^2 2x + 3 \cos 2x + 3 = 0$.
   b. $2 \tan^2 (x/2) - 3 \tan (x/2) - 1 = 0$.

5. a. Sketch the graph of $y = (2/x) - 1$ and write the equations of the asymptotes.
   b. Write the inverse equation and sketch its graph. Write the equations of the asymptotes too.

6. The volume of a cone is given by $V = (1/3)\pi r^2 h$, where r is the radius of the base and h is the height. Find the height of a cone if the radius of its base is 7.46 inches and the volume is 195 cu. in.

7. Find the middle term of $(2a + 3b)^8$.

8. The number of bacteria, n, in a culture at the end of t hours is given by the formula $n = 10^{2+0.05t}$. Find the number present after $4\frac{1}{2}$ hours.

9. Find the values of the following:

   a. $\log_3 7$ b. $\log_2 135$ c. $\log_4 3$

10. Find x if $5^x = 2^{x+1}$.

11. If $\ln 2 = 0.6931$, $\ln 10 = 2.3026$, $\log 2 = 0.3010$, find:

    a. $\log 20$ b. $\ln 20$ c. $\ln 40$ d. $\ln 200$

12. The *velocity of escape* from the earth's gravitational field is given by $v = \sqrt{2gR}$. Find the velocity if $g = 32.2$ feet per second square and $R = 3960$ miles.

9. DISCONTINUITIES

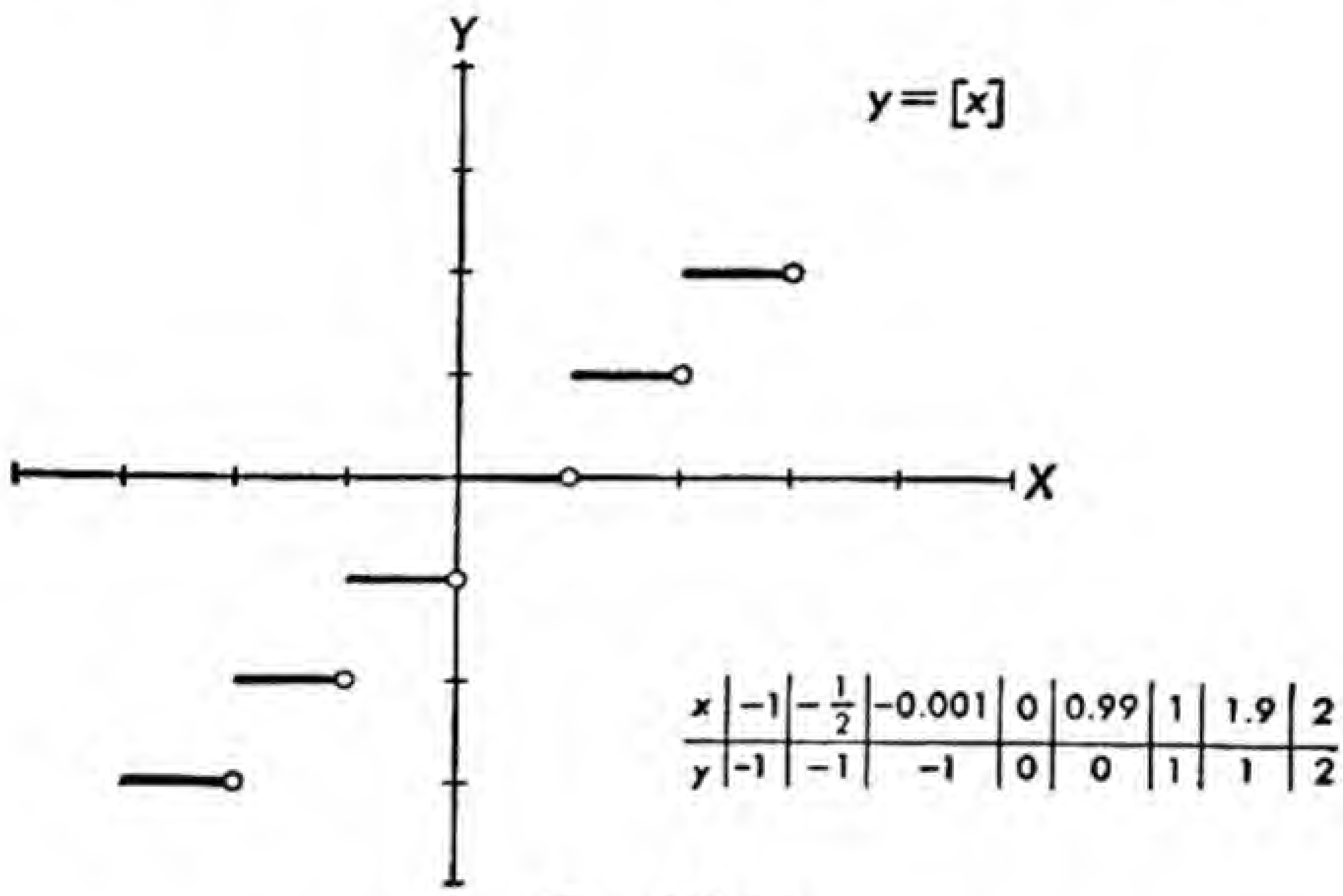
We can say that the graph of the $\tan x$ is composed of an infinite sequence of distinct one-dimensional branches. The discontinuities are easily spotted, for they signalize the improper mathematical operation (division by zero). Thus $\tan x = \sin x/\cos x$ and $\cos x = 0$ for $x = \pi/2, 3\pi/2, \cdots, [(2n + 1)\pi]/2$ for integral values of n. Consequently x may not be assigned any of these values, thereby creating the discontinuities.

For similar reasons, discontinuities arise also in connection with rational algebraic equations, as we have seen earlier. The equation $y = 1/x$ will be

discontinuous at $x = 0$ and will have an asymptote there. If solved for x, the equation yields $x = 1/y$, and so $y = 0$ is also an asymptote.

Well, there are discontinuities of another sort. Let us first consider a new symbol, [ ], which will be defined as representing *the greatest integer contained in the expression it encloses*. Thus, $[2\frac{3}{4}] = 2$, $[-5\frac{1}{2}] = -6$, and $[n] = 2$ when $n = 2, 2\frac{1}{2}, 2.9$, and so forth. For, when $[n] = -2$, n may be $-2, -1\frac{1}{2}$, and -1.999.

The graph (Fig. VI–16) of the function determined by the set of points which is defined by the equation $y = [x]$ gives rise to a series of discrete steps.



x	-1	$-\frac{1}{2}$	-0.001	0	0.99	1	1.9	2
y	-1	-1	-1	0	0	1	1	2

Fig. VI–16

EXERCISES (VI–9)

1. Find the vertical and horizontal asymptotes:

   a. $x^2y = 6$

   b. $(x + 1)y = 12$

   c. $x^2y - y = 5x^2$

   d. $y = \log_b x$

   e. $y = \dfrac{5}{(x - 2)(x + 2)}$

2. Graph each of the following:

   a. $x^2 = \dfrac{12}{y} + 9,\ y \neq 0$

   b. $y = |x|$

   c. $y = -[x]$

   d. $y = \left[\dfrac{1}{x}\right],\ x \neq 0$

   e. $y = \left|\dfrac{1}{x}\right|,\ x \neq 0$

   f. $y = 2^{[x]}$

   g. $y = x + |x|$

   h. $y = x + [x]$

   i. $y = \dfrac{1 - \sqrt{x}}{1 - x},\ x \neq 1$

10. SEQUENCES

Let us examine the function defined by $y = [x]$ more closely in our search for a definition of continuity. The function is defined for the entire domain of X. The range of Y is the set of integers. But something happens at places where X is an integer. Let us move carefully up to $x = 2$ by considering in X a sequence of values which approach 2 as a limit. For example, we take

$$1, 1\tfrac{1}{2}, 1\tfrac{3}{4}, 1\tfrac{5}{6}; 1\tfrac{7}{8}, 1\tfrac{9}{10}, \cdots, 1\tfrac{998}{999}, \cdots, \to 2$$

It is possible to imagine an extremely small interval with 2 at its center. The left end of this interval may be, for example, one-trillionth of a unit less than 2. It is then possible to calculate a term in the sequence which will not only lie within this interval but following which all terms of the sequence will continue to lie therein. We are, as you can see, relentlessly sneaking up on 2.

As x takes on the values of the foregoing sequence, y takes on related values determined by the equation $y = [x]$. For each and every value of this sequence, y has the value of 1.

$$\{y\}: 1, 1, 1, 1, 1, 1, 1, 1, \cdots, \to 1$$

This sequence can have no limit other than 1. Thus, when x approaches the limiting value of 2 from the left, y approaches the limiting value of 1. Yet the definition of our function is such that when $x = 2$, $y = 2$. The encroachment is futile. By definition y takes a discrete *jump* at $x = 2$. Hence the name **jump discontinuity**.

Well, we now have a number of additional subtle concepts such as *sequence* and *limit* which, if further refined, will aid us in determining an adequate definition of continuity.

The notion of **sequence** has been used a number of times without strict formulation. The idea may be sufficiently familiar to warrant an abstraction. *We mean by "sequence" an ordered set of numbers, or equivalently, of points.* To be ordered, the sequence must have a first term, a second, a third, and so forth. The expression *and so forth*, which is symbolized by three dots $(\cdots)$, means that we must have the unequivocal means by which the sequence can be continued. This is usually expressed literally by an nth term. Consider,

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n-1}{n}, \cdots$$

The general term, $(n - 1)/n$, is the rule permitting the determination of any term. Unless otherwise stated, $n = 1$ will yield the first term; $n = 2$, the second term; and $n = k$, the kth term, $(k - 1)/k$. Thus

> **a sequence is a real valued function whose domain of definition is the set of natural numbers.**

Other illustrations follow:

(a) $1, 4, 9, 16, \cdots, n^2, \cdots$ $\qquad (8^2 = 64$ is the 8'th term)

(b) $\frac{1}{3}, -\frac{4}{5}, +\frac{9}{7}, -\frac{16}{9}, \cdots, (-1)^{n+1}\frac{n^2}{2n+1}, \cdots$ $\left[(-1)^9 \frac{8^2}{17} = -\frac{64}{17}\right.$ $\left.\text{is the eighth term}\right]$

(c) $$1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots, \frac{1}{\sqrt{n}}, \cdots$$

(d) $\{a_n\}: a_1, a_2, a_3, \cdots, a_n, \cdots$

(This represents a general sequence whose law of formation is not specified.)

Consider now the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \cdots, \frac{2n-1}{2n}, \cdots$$

with the corresponding geometrical representation as shown in Fig. VI–17. Both arithmetically and geometrically it appears that the terms of the sequence are getting closer and closer to the value of 1. We shall say that 1 is a **limit** of this sequence as the number of terms becomes infinitely great *if the distance between 1 and any of these terms approaches 0.* In such an eventuality the geometrical separation or the arithmetical difference is disappearing. Symbolically, this can be indicated by

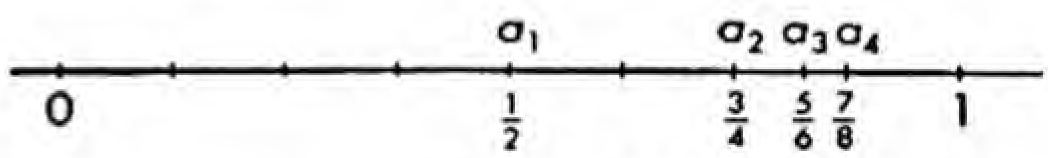


Fig. VI–17

$$\delta(1, a_n) \rightarrow 0 \qquad \text{as} \qquad n \rightarrow \infty$$

which is read as *the distance, or difference, between a_n and 1 approaches 0 as n becomes infinitely great.*

In keeping with an earlier concept of distance along the X-axis, we take the difference or distance between two values to be the absolute value of their difference. That is

$$\delta(x, x') = |x - x'|$$

For the sequence illustrated in Fig. VI–17, we have

$$\delta(1, a_n) = |1 - a_n| = \left|1 - \frac{2n-1}{2n}\right| = \left|\frac{1}{2n}\right|$$

We have already seen that a fraction with a constant numerator approaches zero when the denominator becomes infinitely great. That is,

$$\lim_{N \to \infty} \frac{c}{N} = 0$$

Consequently

$$\delta(1, a_n) = \left|\frac{1}{2n}\right| \to 0 \qquad \text{as} \qquad n \to \infty$$

and so, the sequence under consideration approaches the limit of 1. We describe this simply by saying that the sequence **converges to 1.**

More generally the infinite sequence $\{a_n\}$ converges to a limit a, if and only if

$$\delta(a, a_n) \to 0 \qquad \text{as} \quad n \to \infty$$

We describe this variously as:

$$\lim a_n = a \qquad \text{as} \quad n \to \infty$$

or

$$\lim_{n\to\infty} a_n = a$$

or even

$$\lim a_n = a$$

where $n \to \infty$ is understood.

For further illustration, we examine

$$\frac{4}{3}, \frac{6}{6}, \frac{8}{9}, \frac{10}{12}, \ldots, \frac{2n+2}{3n}, \ldots$$

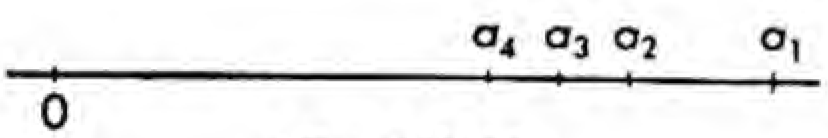


Fig. VI-18

This is a decreasing sequence proceeding leftwards on the line. (See Fig. VI-18.) Let us denote the limit, if one exists, by the letter a.

$$\delta\left|a, \frac{2n+2}{3n}\right| = \left|a - \frac{2n+2}{3n}\right| = \left|a - \frac{2}{3} - \frac{2}{3n}\right|$$

This distance approaches zero if $a = \frac{2}{3}$, since that leaves just the fraction $2/3n$, which approaches zero as n approaches infinity. Hence the limit of the indicated sequence is $\frac{2}{3}$.

In considering the limit of $|a - (\frac{2}{3}) - (2/3n)|$, we have considered that the limit of the trinomial is identical with the sum of the limits of the respective terms. While this seems intuitively reasonable, it nevertheless needs justification. This will be found in the exercises that follow this article.

Now consider $\quad 0, \frac{1}{3}, \frac{4}{4}, \frac{9}{5}, \frac{16}{6}, \frac{25}{7}, \ldots, \frac{(n-1)^2}{n+1}, \ldots$

Arithmetically it appears that the terms of this sequence are getting larger and larger, becoming infinitely large. By way of contrast to the earlier sequences, we say that this sequence **diverges.** Of course we can examine the distance relationship. Again we suppose that the limit is a.

$$\delta\left(a, \frac{(n-1)^2}{n+1}\right) = \left|a - \frac{n^2 - 2n + 1}{n+1}\right| = \left|a - \frac{1 - (1/n) + (1/n^2)}{(1/n) + (1/n^2)}\right|$$

It was desirable to make a complex fraction of one of our terms, since in doing this, we introduced the form c/N, which we recognize as approaching zero when N approaches infinity. We see now that the numerator of the complex fraction approaches 1 and the denominator approaches 0. This means that the fraction as a whole is becoming infinitely large as n approaches infinity. Thus the distance relation cannot approach zero and the sequence has no limit; it diverges.

We can think of a sequence such as:

$$\sin 0, \sin\frac{\pi}{4}, \sin\frac{\pi}{2}, \sin\frac{3\pi}{4}, \cdots, \sin\frac{(n-1)\pi}{4}, \cdots$$

which yields

$$0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, \cdots$$

This sequence oscillates between 1 and -1. Because of the lack of a limit, we consider this type of sequence to be a divergent one too.

EXERCISES (VI-10)

1. Find the limit of each of the following sequences, if one exists. After the first case, only the nth term is indicated for the remainder.

a. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$

b. $a_n = \left(\frac{1}{2}\right)^n$

c. $a_n = 1 - \frac{1}{n}$

d. $a_n = \frac{n+1}{n^2}$

e. $a_n = (-1)^n\frac{(n+1)^2}{n}$

f. $a_n = \frac{1+(-1)^n}{n}$

g. $a_n = \sin \pi n$

h. $a_n = \frac{n^2+n}{3n^2-n}$

i. $a_n = \frac{n}{(n+1)^2}$

j. $a_n = \frac{b^n}{c^n}$ $\left\{\begin{matrix}(b \neq c) \text{ and } (c \neq 0)\\ \text{(two possibilities)}\end{matrix}\right\}$

k. $a_n = \frac{1}{2^n+n}$

l. $a_n = c$

m. $a_n = \frac{n!}{(n+1)!}$

2. If the sequence $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, then

a. $\{a_n + b_n\}$ converges to $a + b$.
b. $\{a_n - b_n\}$ converges to $a - b$.
c. $\{a_nb_n\}$ converges to ab.
d. $\{a_n/b_n\}$ converges to a/b, provided $b_n \neq 0$ and $b \neq 0$.

These important limit theorems are listed without proof. Each may be stated otherwise. For example, the first case may also be listed as:

$$\text{If } \delta(a, a_n) \to 0 \text{ and } \delta(b, b_n) \to 0, \text{ then } \delta(a + b, a_n + b_n) \to 0$$

or

$$\text{If } \lim a_n = a \text{ and } \lim b_n = b, \text{ then } \lim(a_n + b_n) = a + b$$

In any case we have in words that

the limit of the sum of two converging sequences is the sum of the limits of the two sequences.

Write each of the other theorems in the three ways given here.

3. By choosing or creating sequences of your own, provide illustrations of each of the four cases in exercise 2.

4. Show that the sequences whose nth terms are $1/n$, c/n, $1/cn$, and $1/n^c$, where c is an integer not 0, all converge to the same limit.

5. Prove that $\lim\limits_{n\to\infty} \dfrac{1}{n^n} = 0$. $\left(\text{First prove that } \dfrac{1}{n^n} \leq \dfrac{1}{n}.\right)$

11. CONTINUITY

We have met some functions, as those defined by $y = [x]$, $y = 1/x$, and others, which contained *gaps*. We described such functions as *discontinuous*, and we saw that their graphs lacked connectedness at the points of discontinuity.

Let us look a bit closer at a function with a view of determining a definition of **continuity at a point.**

Consider the set of points defined by the equation $y = x^2$. We can select some abscissa (say, $x = 2$) and any sequence on the X domain whose limit is 2. For example, we can take

$$3, 2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \cdots, 2 + \frac{1}{n}, \cdots$$

That this sequence has a limit 2 is seen by the distance test:

$$\delta(2, 2 + \frac{1}{n}) = \left| 2 - 2 - \frac{1}{n} \right| \to 0 \qquad \text{as } n \to \infty$$

Each term of the sequence determines uniquely a term of another sequence on the Y range by the equation $y = x^2$. Thus, when $x = 3$, $y = 3^2$, and when $x = 2.5$, $y = 2.5^2$. The derived y sequence is then

$$3^2, \left(2\frac{1}{2}\right)^2, \left(2\frac{1}{3}\right)^2, \cdots, \left(2 + \frac{1}{n}\right)^2, \cdots$$

Suppose that we let L represent the limit of this sequence.

$$\delta\left[L, \left(2 + \frac{1}{n}\right)^2\right] = \left|L - \left(2 + \frac{1}{n}\right)^2\right| = \left|L - 4 - \frac{4}{n} - \frac{1}{n^2}\right|$$

This distance will approach zero as $n \to \infty$ only if $L = 4$. So, the limit of the last sequence is 4. This is precisely the value predicted for y by the equation $y = x^2$ when $x = 2$.

Now, we approached $x = 2$ through a sequence of values that were all larger than 2. This can be symbolized by referring to the limit L as an L^+ limit.

We have had a little experience with discontinuities, which should put us on our guard. An approach from one direction may yield an entirely different result from the approach in the opposite direction. So, we take an increasing sequence for x with the limit 2, as in:

$$\left\{x\right\}: 1, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, \cdots, 2 - \frac{1}{n}, \cdots$$

The corresponding y sequence has as its nth term

$$y_n = \left(2 - \frac{1}{n}\right)^2$$

and

$$\delta\left[L, \left(2 - \frac{1}{n}\right)^2\right] \to L - 4 \qquad \text{as } n \to \infty$$

This distance will approach zero if L is 4 again. The left-hand limit, L^-, is also 4. If this remains true no matter how x approaches 2, then there is no break or gap in the curve at $(2, 4)$. We say that the curve is *continuous* at $x = 2$.

What does this amount to in more general terms? Consider a real valued function f consisting of the set $\{x, y\}$. What shall we mean by a *limit of a function* and by *continuity of the function at a point?* The foregoing illustrations have prepared us for the following formal definitions:

> *The values $f(x)$ of f: $\{x, f(x)\}$ approach the limit L as $x \to X$ if and only if L is the common limit of every sequence $\{f(x_n)\}$ when $\{x_n\}$ is a sequence whose limit is X.*

If this is true, we write

$$\lim_{x \to X} f(x) = L$$

For *continuity at a point*, we have

The function of f is continuous at X if and only if:

(1) *The function is defined at $x = X$; that is, that $[X, f(X)]$ is a member of the set.*

(2) $\lim_{x \to X} f(x)$ *exists and equals L.*

(3) $f(X) = L$.

As defined, continuity at a point depends on a conjunction of a number of factors. A condition of discontinuity at a point would result from a disjunction of these factors. Thus a function is discontinuous at a point if:
a. It is not defined for the X in question, or
b. There is no limit L, or
c. $L \neq f(X)$, the limit does not correspond to the function value for X.

We have seen equations that are not defined for certain values: $y = x/x$ and $y = (x - 2)/(x^2 - 4)$. These are not defined for values that make the denominator zero, and so their functions cannot be continuous at such points. This is an illustration of the condition referred to in condition (a). (Limits do exist here for $x = 0$ and $x = 2$, respectively.)

Equations such as $y = 1/x$ and $y = [x]$ have no y limit as x approaches zero. These graphs are discontinuous at these points. This is the situation in condition (b). Finally, $L \neq f(X)$ for $f(2)$ in

$$y = \begin{cases} \dfrac{x-2}{x^2-4} & x \neq 2 \\ 1 & x = 2 \end{cases}$$

If a function is continuous at each point of an interval, it is defined as being *continuous over the interval*. Frequently, it is possible to remove a discontinuity by an expanded definition of the function. For example, the function defined by the equation (see Fig. VI–19)

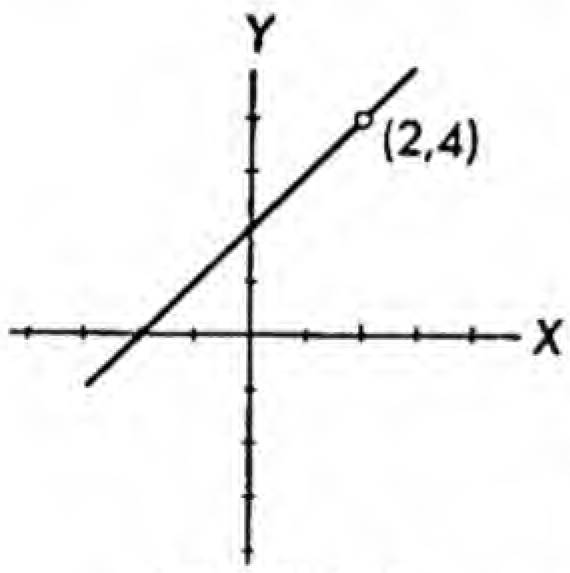


Fig. VI–19

$$y = \frac{x^2 - 4}{x - 2}$$

is not defined for $x = 2$, since y would be equal to a meaningless expression $0/0$. Yet, the limit, as we approach 2 from either side along any sequence whose limit is 2, is a finite number 4. The function value $(x^2 - 4)/(x - 2)$, is equal to $x + 2$ for all values of x excepting 2, since division by 0 is not permitted. So, for $x \neq 2$,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

In the light of this result, we broaden our original function by having as its defining equations:

$$y = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 4 & \text{for } x = 2 \end{cases}$$

The function is now continuous at $x = 2$ and, indeed, at every value of x. In this fashion a discontinuity is removable, providing the necessary limits exist. There is no removing of the discontinuities in $y = 1/x$ and $y = \tan x$.

EXERCISES (VI-11)

1. Show that $y = 4$ is a 4^- limit for $y = \dfrac{x^2 - 4}{x - 2}$.

2. Expand the definitions of the function values so that discontinuities can be removed. Graph the functions.

a. $y = \dfrac{x - 3}{x^2 - 9}$

b. $y = \dfrac{x^2 - 2}{x - \sqrt{2}}$

c. $y = 1 + 2^{-1/x^2}$

d. $y = \dfrac{x^3 - x}{x}$

12. MORE ON LIMITS

When dealing with the limits of functions in the preceding applications, we have gone beyond the definition of the limit of a function in that we have been dealing with the sum, difference, product, and quotient of functions. We have enough experience now to correct this situation. Part of the job was done much earlier in the chapters on trigonometric functions and the subsequent one on functions. There we saw what was meant by the sum, difference, product, and quotient of functions. Now let us examine these from the viewpoint of limits. It will suffice, to demonstrate the case of the sum.

Suppose that as $\{x_n\} \to X$

every $\{f(x_n)\} \to L_1$ and every $\{g(x_n)\} \to L_2$

Then, by the theorem on the limit of a sum of sequences, we have

every $\{f(x_n) + g(x_n)\} \to L_1 + L_2$

This is precisely the condition that is to be met for the limit of the function values of $f + g : \{x, f(x) + g(x)\}$ to approach the limit $L_1 + L_2$.

With this illustration, we can state the entire group of theorems on the limits of functions:

If $\lim\limits_{x \to X} f(x) = L_1$ and $\lim\limits_{x \to X} g(x) = L_2$

then

(1) $\lim\limits_{x \to X} [f(x) \pm g(x)] = L_1 \pm L_2$

(2) $\lim\limits_{x \to X} f(x) \cdot g(x) = L_1 \cdot L_2$

(3) $\lim\limits_{x \to X} f(x)/g(x) = L_1/L_2$ provided $L_2 \neq 0$.

Thus,

the limit of the sum (difference, product, or quotient) of two functions is equal to the sum (difference, product, or quotient) of their limits.

EXERCISES (VI–12)

1. a. Discuss the continuity of $y = [x]$ when x is not an integral value.
   b. Explain why $y = [x]$ is discontinuous for integral values of x.

2. In the light of the theorems on limits given in this article, discuss the validity of the following theorems on continuity: If $f(x)$ and $g(x)$ are both continuous at $x = a$, then:

   I. $f(x) + g(x)$ is continuous at $x = a$.
   II. $f(x) - g(x)$ is continuous at $x = a$.
   III. $f(x)\,g(x)$ is continuous at $x = a$.
   IV. $f(x)/g(x)$ is continuous at $x = a$, $g(x) \neq 0$.

3. Illustrate each of the cases in exercise 2 with functions and function values of your own choosing.

4. a. Show that $f(x) = c$, where c is a constant, is a continuous function.
   b. Show that $f(x) = x$ is a continuous function.
   c. Describe how the last two conclusions together with the results in exercise 2 can be used to prove that the real polynomial function P, whose equation is $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is everywhere continuous.
   d. Why would a rational function R, whose equation is $R(x) = [P_1(x)]/[P_2(x)]$ be continuous excepting where $P_2(x) = 0$?

5. For what values are the following discontinuous?

   a. $y = \dfrac{x}{x+2}$ b. $y = \dfrac{x^2}{4 - x^2}$ c. $y = \dfrac{5}{x^2 + 5x - 6}$

6. Show that $y = |x|$ is continuous at $x = 0$.

7. Expand the function defined by the equation $y = (3x^2 - 12)/(2x + 4)$, $x \neq -2$, so that the new function is everywhere continuous. Graph the function.

8. Why is $y = \sin 1/x$ discontinuous at $x = 0$? Is this discontinuity removable? Sketch the curve.

9. Show that

$$y = \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$.

10. It is given that $y = x^2$ for rational values of x, and $y = -1$ for irrational values of x. Discuss the continuity or lack of continuity of the function so defined.

11. Sketch graphs to illustrate each of the following theorems (which are not proved here):
   a. If $y = f(x)$ is a continuous function value for all x in the closed interval $a \leq x \leq b$, then there is an x_1 and an x_2 in the interval such that $f(x_1) = M$ and $f(x_2) = m$, where M and m represent respectively the maximum and minimum values so that $m \leq f(x) \leq M$.
   b. If N is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c) = N$.

12. The definition of the continuity of a function is frequently given in terms of two arbitrarily small quantities δ and ϵ, where δ depends on ϵ, so that we can write $\delta = g(\epsilon)$.

If $f(x)$ is continuous at $x = a$, then for any positive value of ϵ, no matter how small, a number δ may be found such that

$$|f(x) - f(a)| < \epsilon \quad \text{in the interval} \quad |x - a| < \delta$$

Conversely, if these conditions hold, the function is continuous.

Consider the function $f:(x, 1/x)$. Suppose that $\epsilon = 0.001$. What value should be selected for δ if the function is to be shown to be continuous at $x = 1$; 0.01; and 0.001?

13. EXPLICIT AND IMPLICIT

We shall deal almost exclusively hereafter with continuous functions. The equations of most of these have been in the form of $y = f(x)$. In $y = 3x^2$, for example, we follow specific directions for finding the value of y for any given value of x. The procedure is indicated explicitly in $3xx$. These are **explicit** function values.

On the other hand, we shall come across functions that represent an **implicit** relationship. A simple case in point is a function defined by the equation $x + y - 8 = 0$. Or, we could say that the set of points $\{x, y\}$ of the function f is indicated by an implicit rule $x + y - 8 = 0$.

Now we could decide that x is the independent variable, which we did by showing $\{x, y\}$, and solve the equation for y, getting $y = 8 - x$. In this way we have derived an explicit expression for the function. On the other hand, we could even have decided that y is the independent variable, and then we would proceed to express explicitly the value for x as $x = 8 - y$.

Unfortunately not every implicitly defined function can be made explicit. This is true particularly when the defining equation is of the fifth or higher degree. There is no general method of solving $x^5 - y^5 = 3x^2y^2$ for either y or x.

VI-13 REVIEW

1. Find the limits of the following:

a. $\lim\limits_{y \to 2} \dfrac{3y^2 + y - 10}{4 - y^2}$

b. $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x}$

c. $\lim\limits_{x \to \pi/2} \dfrac{\sin x + \cos x}{\csc x}$

d. $\lim\limits_{\theta \to 0} \dfrac{\sin 2\theta}{\sin \theta}$

2. Find the values of the following: $\left[\text{Assume that } \lim u^{\frac{p}{q}} = \left(\lim u\right)^{\frac{p}{q}}.\right]$

a. $\lim\limits_{x \to 1} \dfrac{2\sqrt{x} - 2}{x - 1}$

b. $\lim\limits_{x \to 3^+} \dfrac{\sqrt{x^2 - 9}}{\sqrt{x - 3}}$

c. $\lim\limits_{n \to \infty} \log 2^{1/n}$

3. Graph each of the following (indicate a removable discontinuity):

a. $y = \dfrac{(1-x)^2}{1-x}$ b $y = \dfrac{x}{x} + \dfrac{1}{x-1}$

c. $y = \dfrac{x-9}{\sqrt{x}-3}$ for $x \neq 9$, $x > 0$; and $y = 6$ for $x = 9$.

4. Write the first four terms of the sequences whose nth term is given by the following:

a. $1 + \dfrac{(-1)^n}{n}$ c. $1 + \dfrac{1}{10^n}$

b. $\dfrac{2^n - 1}{2^n}$ d. $\dfrac{n!}{(n+1)!}$

5. Examine the sequences in the preceding exercise for divergence and convergence, and in the case of the latter, determine the limit.

6. Graph the function defined by the equation

$$y = \frac{x^2}{|x|}$$

Expand the definition of the function to eliminate a removable discontinuity if one exists.

7. Graph the function defined by $y = x$ for $|x| \leq 1$ and $y = 1$ for $x > 1$.

8. Write the explicit equation for $f:\{s, t\}$ if $st - 2t = s$.

VII

ANALYTIC GEOMETRY

1. THE ALGEBRA OF THE STRAIGHT LINE

Through graphs and associated concepts and techniques, we have already made some beginnings in *analytic geometry*. This, as the name implies, is an integrated approach to mathematical subject matter through the media of algebra and geometry. The fusion, essentially by René Descartes, was a milestone in the development of mathematics after Euclid.

The simplest geometric element is the point, and this has already been treated analytically through the assignment of a unique system of rectangular coordinates whenever we *graphed* or *plotted* an (x, y).

What of the line? Geometrically we postulated the existence of the line when we assumed that two distinct points determine a line. This means that the points (2, 3) and (−4, 5) or, generally (x_1, y_1) and (x_2, y_2), determine unique lines. (Refer to Fig. VII–1.)

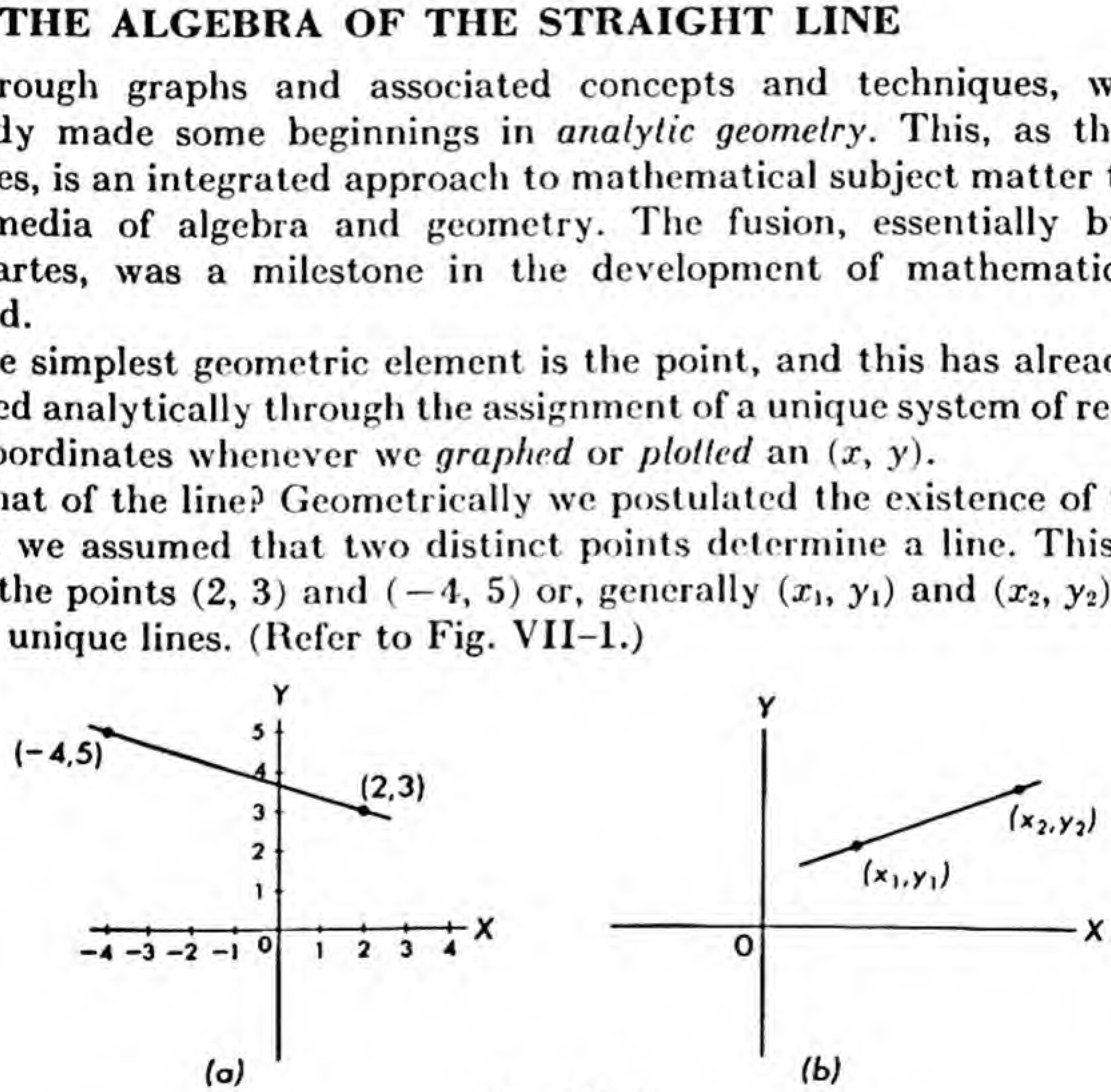


Fig. VII–1

In and by itself this geometric condition permits of no further progress algebraically. The word *determines* needs further exploration before it can be given algebraic connotation.

We have seen that any line contains innumerable points. Now, if parallels are drawn to the axes from these points, right triangles are formed, (Fig. VII–2), which are all similar. The corresponding angles, α, are all equal to each other. The angle α is a constant with respect to any one line m. We shall call this the *angle of inclination*, and in conformity with earlier commitments, measure it counterclockwise from the X-axis or from a line parallel to the X-axis. Figs. VII–3(a) and (b) indicate further that we take $0 \leq \alpha < \pi$.

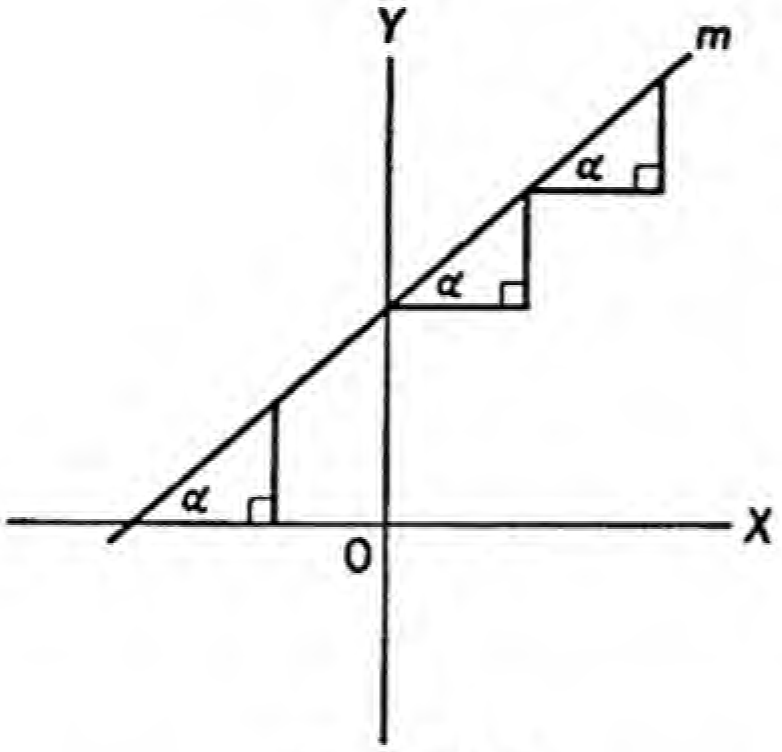


Fig. VII–2

Any one of the triangles that are formed also yields a number of constant trigonometric ratios for any one line. Of these, the tangent ratio takes on special interest because it involves sides whose lengths are most easily determined from the coordinates of any two points on the line.

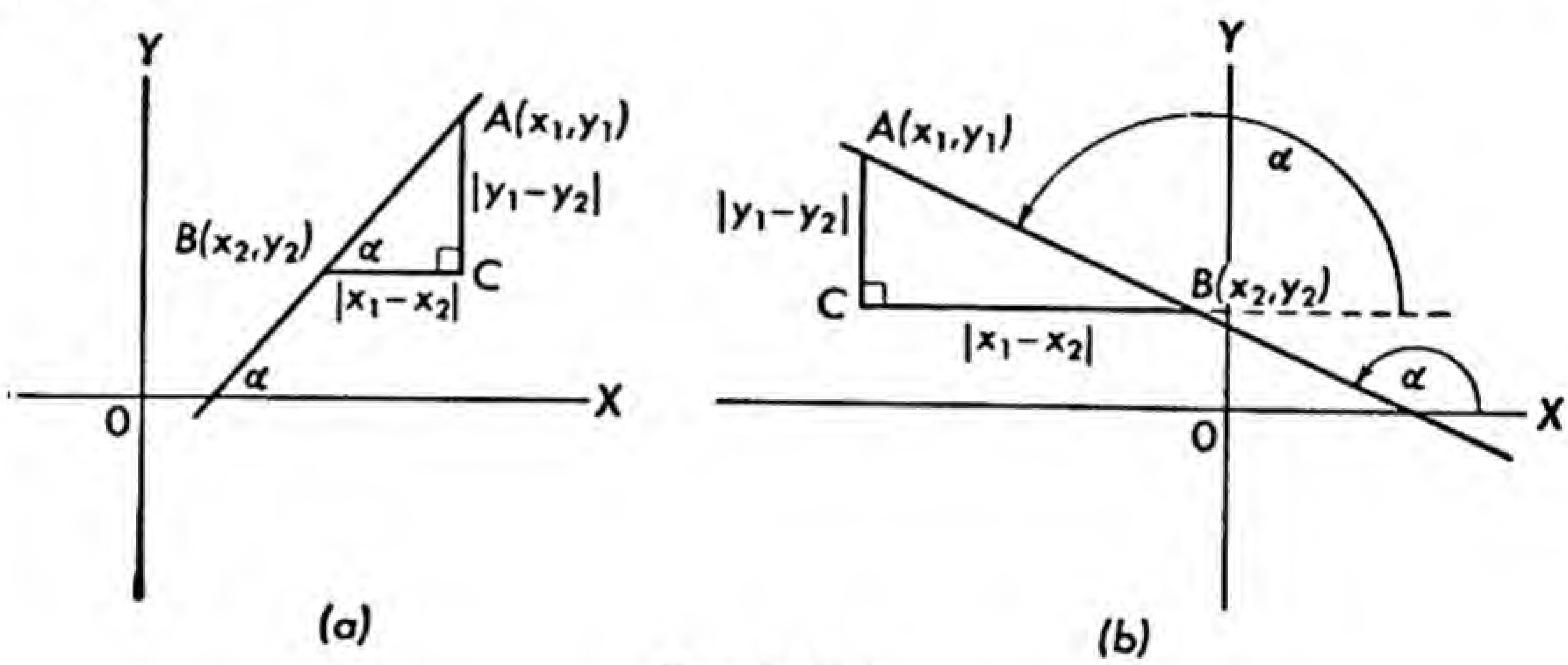


Fig. VII–3

If A and B (Fig. VII–3) have the coordinates (x_1, y_1) and $x_2, y_2)$, respectively, then $AC = |y_1 - y_2|$ and $BC = |x_1 - x_2|$. In the case where the angle of inclination is obtuse (Fig. VII–3b), tan α must be taken as a negative quantity in conformity with earlier decisions. From our trigonometric experience we know that the tangent ratio is sometimes negative and sometimes positive. We can achieve the right sign simply and automatically if, in getting the tangent ratio, we start with the same point (either A or B) for both differences and if we disregard the absolute value symbols.

This ratio is also called the *slope* of the line and is generally symbolized by the letter m. Thus,

$$\text{Slope} = m = \tan\alpha = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

For the slope of the line determined by (2, 3) and (−4, 5), the slope is

$$\frac{3-5}{2+4} \quad \text{or} \quad \frac{5-3}{-4-2}$$

which is $-1/3$ either way.

Through any given point (x_1, y_1) in the plane, only one line may be drawn with a given slope m (Fig. VII–4). A line is then uniquely determined by these two conditions. This means that for a given (x_1, y_1) and an m, the set of points $\{x, y\}$ on the line is uniquely determined. Let (x, y) be any member of the set. Then

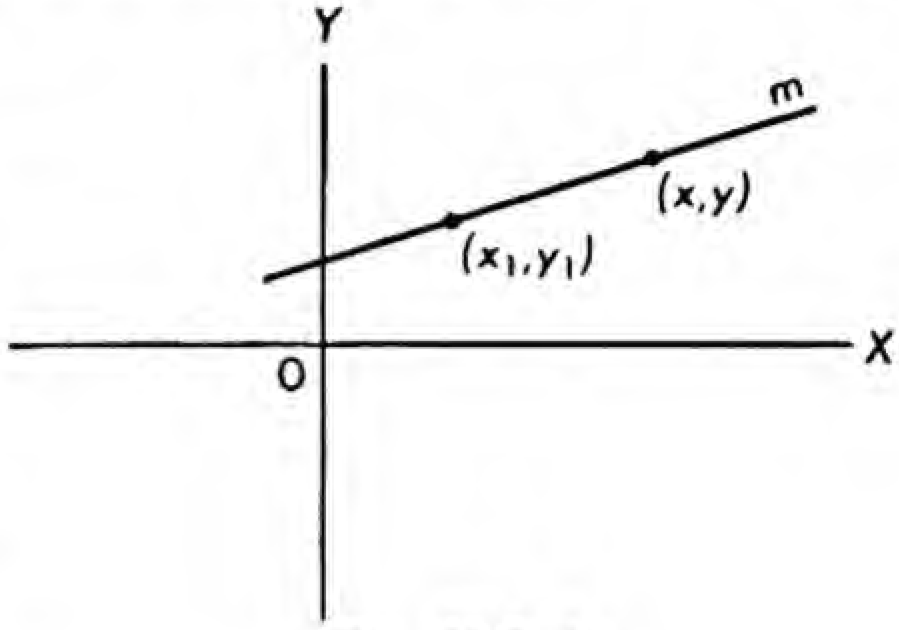


Fig. VII–4

$$\frac{y-y_1}{x-x_1} = m \qquad (x \neq x_1)$$

or $$y - y_1 = m(x - x_1)$$

This is noteworthy progress. We have achieved the algebraic description of a straight line. Because of the initial conditions that were employed, this is known as the *point-slope equation* of the straight line. The line that passes through (1, 6) and has the slope $2/3$, has the equation

$$y - 6 = \tfrac{2}{3}(x - 1)$$

The need arises on occasion to alter the form of an equation. This can be done by virtue of the techniques applicable to an equation. Using multiplication and addition with respect to equalities, the last equation can yield

$$3y - 18 = 2x - 2$$

and $$3y - 2x - 16 = 0$$

Had we started with a different point and slope, the last equation would merely have different coefficients. The nature of the algebra involved is such that the format of the last equation is always achievable for a straight line. This suggests another general form for the equation of the line.

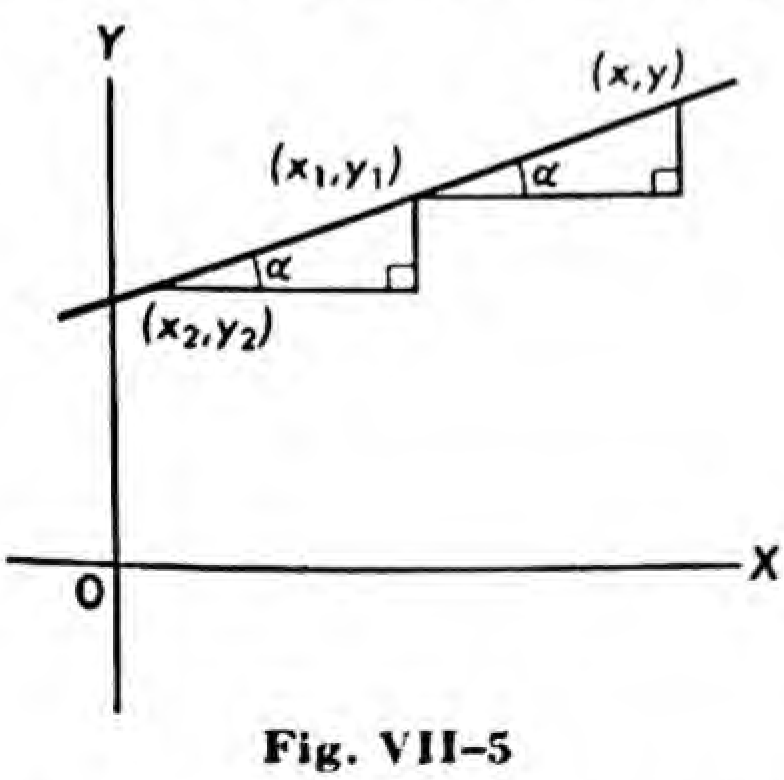


Fig. VII–5

$$Ax + By + C = 0$$

We shall see continually that general equations can always take on a variety of forms with (occasionally) distinctive advantages for each form. This can be shown right now.

At the very start we recalled that two points determine a straight line. We saw, too, that the slope is constant for any one straight line. Now, if (x_1, y_1) and (x_2, y_2) are two fixed points, and (x, y) is any point on the line determined

by them (Fig. VII–5), then by virtue of the constant slope, we can write a new general equation:

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

For obvious reasons this is known as the *two-point equation* of the straight line. The equation of the straight line through (3, −1) and (−2, −5) is

$$\frac{y + 1}{x - 3} = \frac{-1 + 5}{3 + 2} = \frac{4}{5}$$

This can be written as

$$5y + 5 = 4x - 12$$

and

$$5y - 4x + 17 = 0$$

Had we started to write the equation with the second point, (−2, −5), the first two equations would look different, but in the third form the result would be identical. A little reflection shows that in this new form, we are in effect using the point-slope concept.

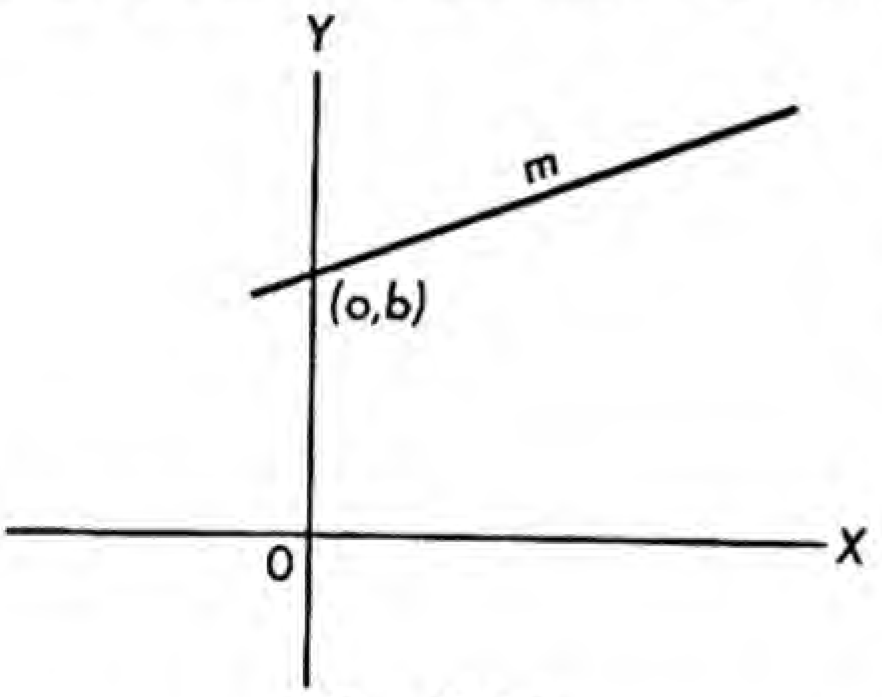


Fig. VII–6

We return to the point-slope case for a special application. Let us take as the point the y-intercept (Fig. VII–6), the point in which the line intersects the Y-axis. Let this point be $(0, b)$. From

$$y - y_1 = m(x - x_1)$$

we get $\quad y - b = mx$

and so $\quad y = mx + b$

This is known as the **slope y-intercept** general equation of the straight line. Thus, if the y-intercept of a line is 4 and the slope is 2, its equation is $y = 2x + 4$.

Another case of frequent worth results from the general equation of the straight line through two special points, that is, the x- and y-intercepts which are denoted by $(a, 0)$ and $(0, b)$, respectively (Fig. VII–7). By means of the two-point equation, we get

$$\frac{y - b}{x} = \frac{-b}{a}$$

$$ay - ab = -bx$$

$$ay + bx = ab$$

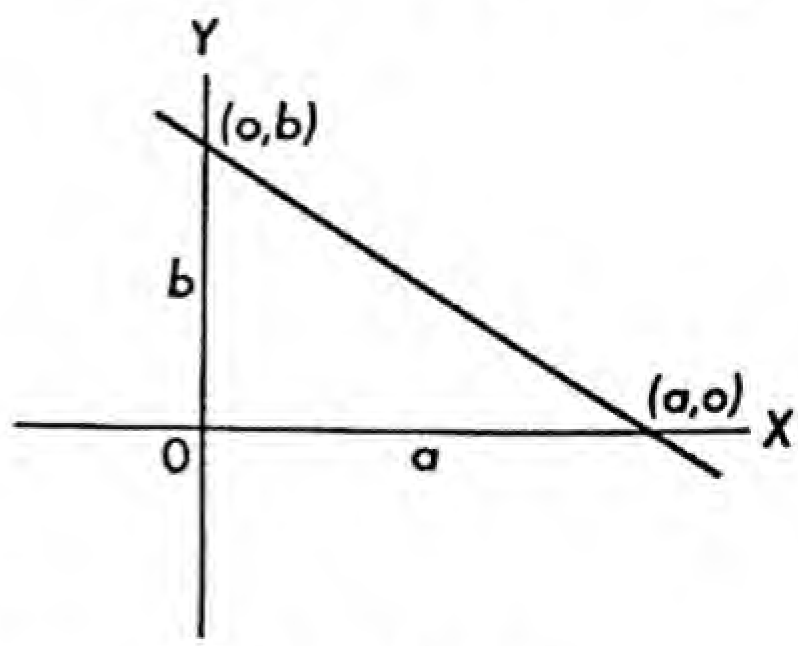


Fig. VII–7

By the occasionally useful but strange procedure of creating a fractional equation where none exists by dividing both members by the lone constant ab, we get

$$\frac{x}{a} + \frac{y}{b} = 1$$

The advantage here is one of proximity. The x-intercept is the denominator of the x term, and the y-intercept is the denominator of the y term. Should a line have the x- and y-intercepts of 3 and 4, respectively, its equation is

$$\frac{x}{3} + \frac{y}{4} = 1$$

This is the *two-intercept equation* of the straight line.

We have, then, a variety of forms for the straight line equation. Some are more convenient at times than others. Still other forms are possible, but enough have been shown to indicate the manner of evolving general forms. While the development was initiated with the point slope-case, any one of the others could have served almost as well.

It should be recalled that two special cases of the straight line have served us some time ago as the foundation of this rectangular coordinate system. Lines that are parallel to the X-axis (Fig. VII–8) have the equation

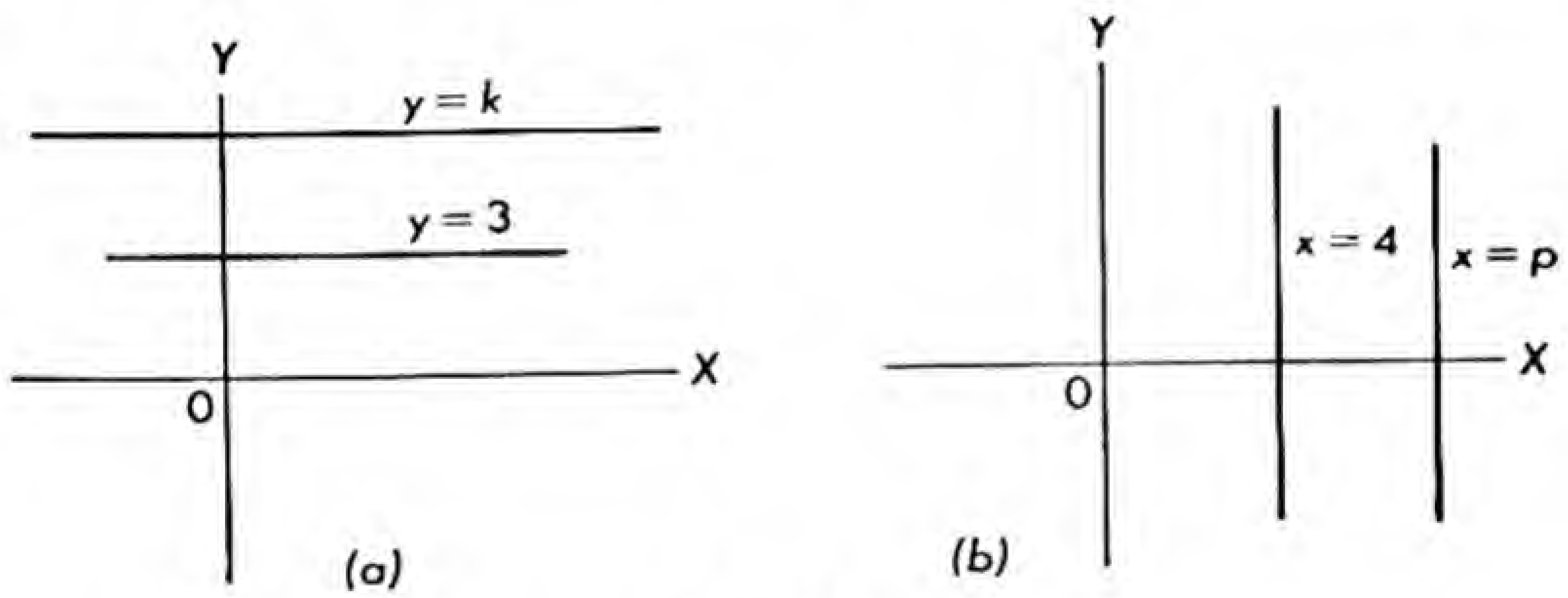


Fig. VII–8

form, $y = k$, and lines that are parallel to the Y-axis have the equation form $x = p$. The horizontal lines have an angle of inclination of 0°, and so a slope of 0. The vertical lines have an angle of inclination of 90°, and so the slope is undefined.

EXERCISES (VII–1)

1. Sketch lines with the following angles of inclination:

   a. 30° b. $\frac{\pi}{2}$ c. 120° d. π e. $\frac{\pi}{4}$ f. $\frac{3\pi}{4}$

2. a. Show that a line with an angle of inclination of 150° is the same as a line with an angle of inclination of $-30°$.
   b. Show that an alternative definition of an angle of inclination could substitute $-(\pi/2) < \alpha \leq (\pi/2)$ for $0 \leq \alpha < \pi$.

3. a. Find the slopes of the sides of a triangle that is determined by the points $A(3, 6)$, $B(-5, 8)$, and $C(2, -4)$.
   b. Find to the nearest degree the angles of inclination of each of the sides.

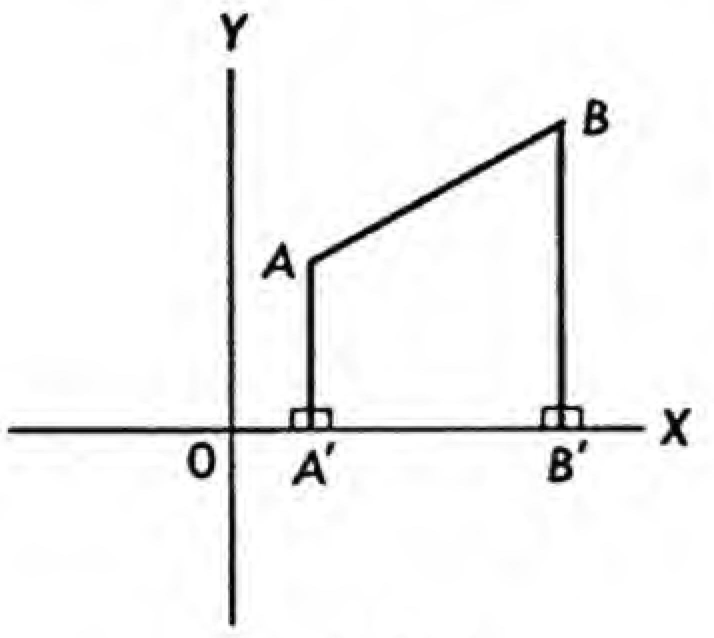


Fig. VII–9

4. If perpendiculars are dropped from A and B (Fig. VII–9), the end points of the segment AB, to the X-axis, A' and B' are the *projections* of A and B, respectively and $A'B'$ is the *x projection or horizontal projection* of AB. The y projections may be defined similarly.
   a. Find the coordinates of the x and y projections of each of the following points:

   $(2, 3)$, $(-5, 3)$, $(-6, 0)$, (x_1, y_1)

   b. Find the lengths of the x and y projections of the segments determined by each of the following pairs of points:

$(3, 1)$ and $(7, 8)$; $(-3, 4)$ and $(5, 7)$; (x_1, y_1) and (x_2, y_2)

5. It was mentioned in the text that the slope for any two points is the same irrespective of from which point the differences are taken for the ratio. Show that this is always true.

6. It was mentioned in the text that a line is uniquely determined by a given point and a given slope.
   a. Explain why (x_1, y_1) and m do indeed determine a line, and that
   b. Only one line is determined by these data.

7. Find the equations of the lines with the following data (in each case sketch the line):
   a. Passes through $(2, 5)$ and has a slope of $1/3$.
   b. Passes through $(-3, -4)$ and has the slope $-4/5$.
   c. Passes through $(2, -3)$ and $(-3, 2)$.
   d. Passes through $(-5, 4)$ and $(2, 8)$.
   e. Has a y-intercept of 3 and a slope of -2.
   f. Has a y-intercept of -4 and a slope of $4/5$.
   g. Has an x-intercept of 3 and a y-intercept of -5.
   h. The x- and y- intercepts are -4 and 2, respectively.
   i. Passes through $(-2, 4)$ and has no slope.
   j. Passes through $(-2, 4)$ and has a slope of 0.

8. If an equation such as $2y - 5x + 7 = 0$ is solved for y, we get $y = (5/2)x - (7/2)$, from which, by comparison with the general equation $y = mx + b$, we can see that the line has a slope of $5/2$ and a y-intercept of $-7/2$. Find the slope and y-intercept of each of the following:

a. $3y - 2x + 9 = 0$	d. $2y + 5x = 0$
b. $2y - 3x + 8 = 0$	e. $5y + 8 = 0$
c. $5x + 2y = 12$	f. $\frac{x}{2} + \frac{y}{5} = 7$

9. Find the coordinates of two points that lie on each of the lines in exercise 8.

10. a. Show that the x- and y-intercepts of any line $Ax + By + C = 0$ are given by $-(C/A)$ and $-(C/B)$, respectively.
    b. Find the intercepts for each of the following:

$$y = 3x - 5;\ 7x - 3y = 8;\ 5y - x = 10$$

11. If (x_1, y_1) is on a line of slope m, then $(x_1 + h, y_1 + mh)$ is on the same line for any value of h. Why?

2. DISTANCE IN COORDINATE GEOMETRY

It is time to consider another very important attribute of the straight line, or (better) the straight line segment. The notion of distance is of singular importance. In the coordinate system thus far, distance has been measured only along the axes and along segments parallel to the axes. What of oblique line segments?

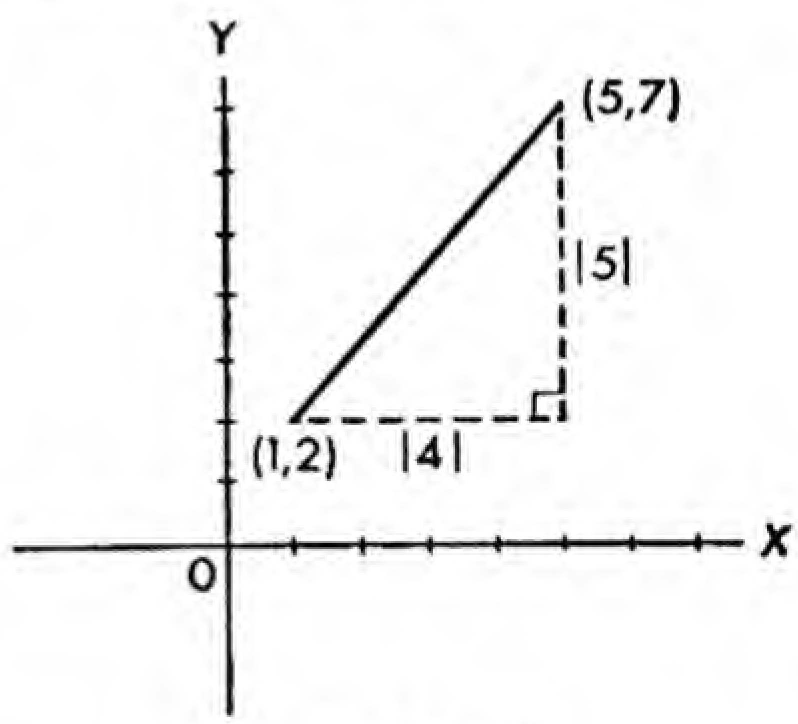


Fig. VII–10

The segment in Fig. VII–10 is a case in point. However, the problem is quickly resolved when the parallels to the axes are drawn in as shown. In the right triangle that is formed (Fig. VII–10), the lengths of the arms are seen at sight, and then, of course, the Pythagorean formula settles the rest.

$$d^2 = 4^2 + 5^2$$
$$d = \sqrt{41}$$

This suggests almost immediately the general formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Fig. VII–11

In the general case (Fig. VII–11), we take the coordinates of the given points as (x_1, y_1) and (x_2, y_2). The sides of the right triangle formed by the perpendiculars are, as shown, the absolute values of the differences of the abscissas and ordinates, respectively. Since the square of an absolute value of a real number is necessarily positive, there is no need to retain the absolute value symbol in the result obtained by applying the Pythagorean theorem. Thus

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

and

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

EXERCISES (VII-2)

1. Find the distances between the indicated points:

a. $(-2, 3), (10, 8)$ c. $(1, 5), (-3, 7)$
b. $(2, -4), (-13, 4)$ d. $(-1, -3), (4, 5)$

2. Show that the following sets of three points determine an isosceles triangle:

a. $(2, 3), (5, 8), (1, 7)$
b. $(6, 7), (-8, -1), (-2, -7)$

3. Given the points A $(1, 3)$, B $(5, 7)$, and C $(3, y)$. Show that it is impossible for $\triangle ABC$ to be isosceles when the vertex is at C.

4. The points in exercise 3 can determine two isosceles triangles if A is the vertex. Find the coordinates of C.

5. Show that the following sets of three points determine right triangles:

a. $(0, 4), (-2, 8), (-6, 1)$
b. $(-2, -1), (2, 3), (1, -4)$
c. $(-2, 3), (3, -2), (0, 4)$

6. Show that the points $(1, 3)$, $(3, 7)$, and $(-2, -3)$ lie on a straight line.

7. Show that the points $(2, 4)$, $(4, 8)$, $(8, 6)$, and $(6, 2)$ determine a parallelogram.

8. It is possible to develop, discover, and prove general relationships through the technique of analytic geometry. Let us illustrate a general proof by showing that the diagonals of a rectangle are always equal.

This requires a rectangle, meaning any rectangle with any dimensions. This could be achieved by the use of letters for the coordinates of the vertices. For convenience, and without any loss in generality, it would be well to find a simple orientation of the rectangle with respect to a set of axes. The Figs. VII-12 and VII-13 illustrate two possibilities.

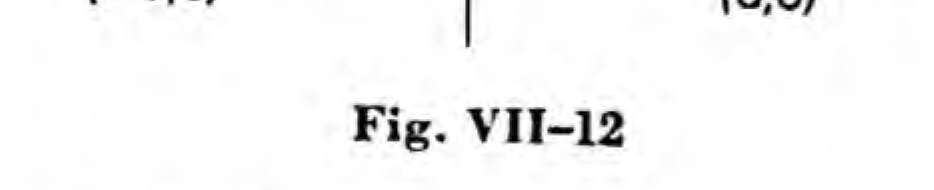


Fig. VII-12

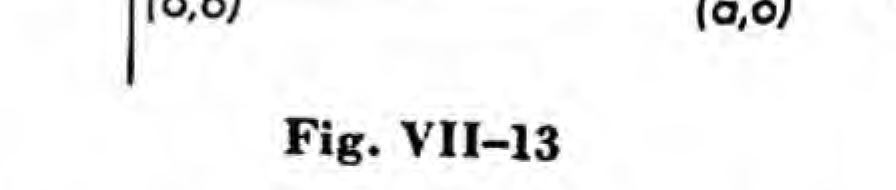


Fig. VII-13

In Fig. VII-12, either diagonal comes out to be $\sqrt{4a^2 + b^2}$. In Fig. VII-13, the diagonals are each $\sqrt{a^2 + {}^2b}$. In both cases the coordinates were selected so that the quadrilateral is indeed a rectangle. Once this was done, the distance formula was applied at sight to determine that the diagonals are equal

9. Prove analytically that the diagonals of an isosceles trapezoid are equal.

10. Prove that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

11. $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points, and $M(x_m, y_m)$ are the coordinates of the midpoint of the segment AB (Fig. VII–14). With the auxiliary lines drawn, and by means of the congruent triangles, show that

$$x_m = \tfrac{1}{2}(x_1 + x_2)$$
$$y_m = \tfrac{1}{2}(y_1 + y_2)$$

Also, state the conclusions in terms of averages.

12. Find the coordinates of the midpoints of each of the following segments:

a. (3, 5), (5, 7)
b. (−2, 4), (5, 7)
c. (−3, −5), (2, −6)
d. (6, −1), (−4, −5)

13. a. Find the midpoints of the sides of the triangle: (1, 4), (5, 6), and (3, 8).
b. Find the lengths of the segments determined by these midpoints.
c. Compare the lengths in (b) with the lengths of the sides.

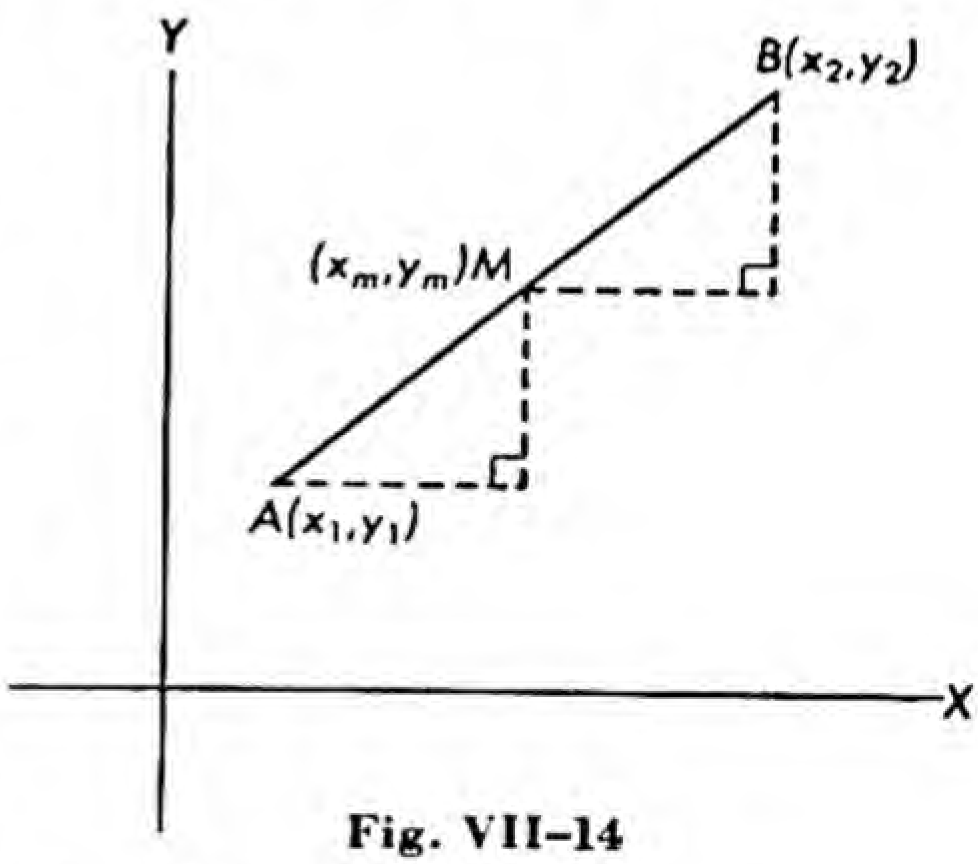


Fig. VII–14

14. The end of a diameter of a circle is at (−3, 2). The center is at (1, 4). Find the other end of the diameter and the length of the radius.

15. The center of a circle is at (2, 3) and the radius is 4. Find the coordinates of the points on the circle whose abscissa is 3.

16. Prove that the diagonals of a parallelogram bisect each other.

17. Prove that the line joining the midpoints of two sides of a triangle is one-half the third side.

18. Use the diagram in exercise 11, but suppose this time that M divides AB in some known ratio (as, for example, $BM{:}MA = m{:}n$). Develop formulas for the coordinates of M

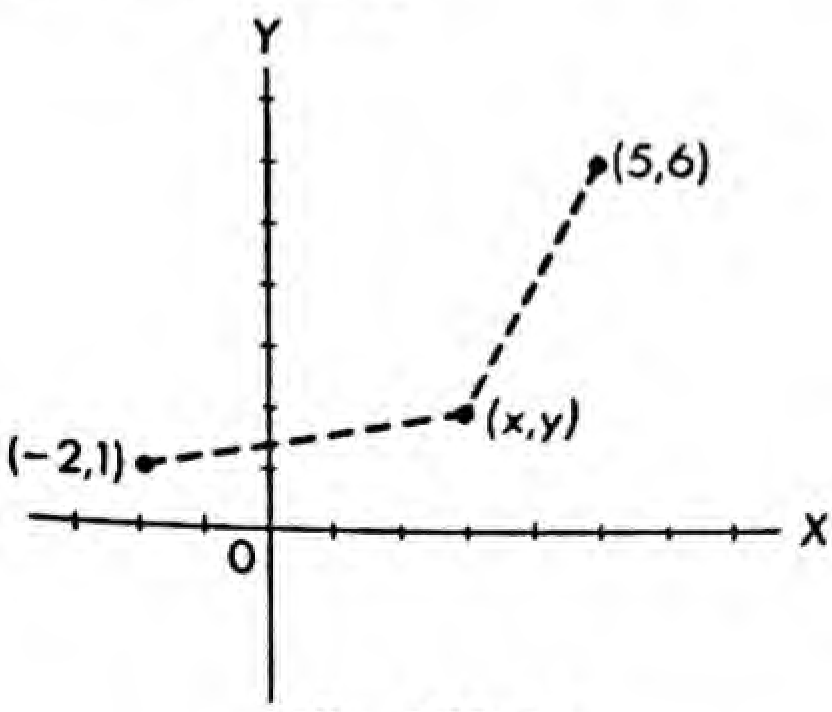


Fig. VII–15

19. Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

3. FITTING AN EQUATION TO A CONDITION

Suppose that it is desired to determine a set of points every member of which is equidistant from (−2, 1) and (5, 6) (Fig. VII–15). This is often referred to as finding the *locus of points.*

Let (x, y) represent any member of the set of points. The distances of

(x, y) from the two given points must be equal.

$$\sqrt{(x-5)^2+(y-6)^2} = \sqrt{(x+2)^2+(y-1)^2}$$
$$(x-5)^2+(y-6)^2 = (x+2)^2+(y-1)^2 \quad \text{(by squaring both members)}$$
$$x^2-10x+25+y^2-12y+36 = x^2+4x+4+y^2-2y+1$$
$$-14x-10y+56=0$$
$$7x+5y-28=0$$

This linear equation defines the infinite set of points, each of which is equidistant from the given points.

EXERCISES (VII-3)

1. a. Find the loci of points equidistant from $(-2, 3)$ and $(4, 5)$.
   b. Sketch the points and the loci.
   c. Find the coordinates of any point on the loci, and check to see that it fulfills the given conditions.
2. Graph each of the following by the indicated method:
   a. $3y - 6x = 9$, by tabular values.
   b. $2y + 3x = 12$, by finding the intercepts.
   c. $5y - 2x + 10 = 0$, by the slope intercept method.
3. Find the locus of a point which is:
   a. Equidistant from the axes.
   b. In the ratio of 2 to 3 from the X- and Y-axes, respectively.
   c. In the ratio of m to n from the X- and Y-axes, respectively.
4. If $2x + ky = 7$ passes through $(1, -5)$, find the value of k.

VII-3 REVIEW

1. Find the equations of lines parallel to $2y - 3x = 8$:
   a. With a y-intercept of 2.
   b. With an x-intercept of -2.

2. Find the equation of a line passing through $(-1, 4)$ which is parallel to the line passing through $(-1, 1)$ and $(5, 4)$.

3. Find the ordinate of a point whose abscissa is 10 and which lies on a line through $(-6, 4)$ and $(5, 0)$.

4. Sketch each of the following lines:

a. $\dfrac{x}{4} - \dfrac{y}{3} = 1$ c. $y + 3 = 2(x - 2)$

b. $\dfrac{1}{y} = \dfrac{1}{3}x - 2$ d. $3x + 4y - 6 = 0$

5. Find the coordinates of a point on the Y-axis that is equidistant from $(2, -3)$ and $(8, 7)$.

6. Find the coordinates of a point on the line $y = x$ that is equidistant from $(-4, 0)$ and $(1, 6)$.

7. How far from the origin is (a, b)?

8. Find the coordinates of a point P that divides AB in the ratio 2:3, where A is $(2, -4)$ and B is $(7, 6)$.

9. One vertex of a parallelogram is at $(2, 3)$, and the intersection of the diagonals is at $(5, 5)$. Find the coordinates of another vertex of the figure.

10. Prove by analytic methods, that two medians of an isosceles triangle are equal.

11. Consider any point $P(x, y)$, r units from the origin O, with the inclination of OP equal to θ. Show that the coordinates of P may be given by $(r \cos \theta, r \sin \theta)$.

12. Two vertices of any $\triangle ABC$ may be taken at $A(0, 0)$ and $C(b, 0)$.

   a Express the coordinates of B in terms of c and A.

   b. By use of the distance formula show that

$$a^2 = (b - c \cos A)^2 + c^2 \sin^2 A$$

   c. By simplifying the previous result, obtain the Law of Cosines.

13. The slopes of various lines are

   a. $\frac{1}{3}\sqrt{3}$ b. $\sqrt{3}$ c. $-\sqrt{3}$ d. -1

Find their angles of inclination.

14. Two lines whose slopes are $\frac{1}{3}\sqrt{3}$ and -1 form a triangle with the X-axis. Find the angle between the lines.

15. Find the loci of points equidistant between $(-2, 0)$ and $(4, 1)$.

16. Given $kx + 2y - 6 = 0$, where k is an undetermined coefficient. Find the value of k so that (a) the line shall pass through $(2, -1)$; (b) the line shall be parallel to the X-axis; and (c) the line shall have a slope of $-\frac{1}{3}$.

4. SYSTEMS OF EQUATIONS

Suppose that we had two linear graphs (Fig. VII–16), each of which satisfies some condition. If these lines intersect, they determine a point that is a member of the two sets of points which constitute the two lines. The coordinates of the point of intersection must satisfy the conditions of both sets. By way of illustration, let two such equations be

(1) $$2x + 3y = 18$$
and
(2) $$3x - 2y = 1$$

Fig. VII–16 indicates that the lines meet at the point (3, 4). The coordinates of this point do indeed (as they must) satisfy the two equations, for

(1) $$\begin{aligned} 2(3) + 3(4) &= 18 \\ 6 + 12 &= 18 \end{aligned}$$

(2) $$\begin{aligned} 3(3) - 2(4) &= 1 \\ 9 - 8 &= 1 \end{aligned}$$

Inasmuch as a pair of distinct lines cannot intersect in more than one point, (3, 4) is the only pair of numbers that satisfies the conditions contained in the two equations.

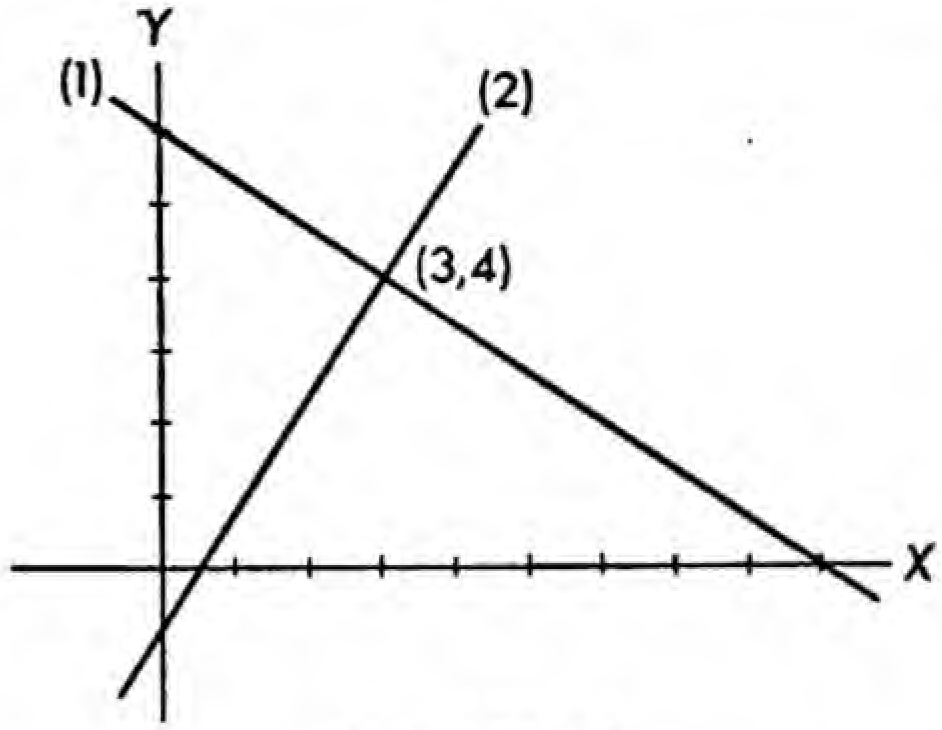


Fig. VII–16

It is not always convenient to obtain a common solution graphically Surely there must be an algebraic equivalent. There is. To begin with, let us particularize the scope of the (x, y) of the two equations to represent the common point of the two sets of points. This being so, we can now solve either equation for x or for y and substitute in the other. From (1) we get

$$y = \frac{18 - 2x}{3}$$

and substituting in (2) $3x - 2\left(\frac{18 - 2x}{3}\right) = 1$

$$9x - 36 + 4x = 3$$

$$13x = 39$$

$$x = 3$$

$$y = \frac{18 - 2x}{3} = 4 \qquad (3, 4)$$

Thus the common solution is (3, 4) for the equations considered simultaneously.

The pair of equations, so conceived, are called *simultaneous equations* In reviewing the method of solution, one should note that the turning point came at the moment, after substitution, when we achieved an equation in one unknown. Had we solved originally for x instead of y, we would have gotten, after substitution, an equation in y instead.

Once we recognize that the key step is the attainment of an equation in one unknown, then we can find other ways of getting there. If, for example, the coefficients of the y terms in the original equations were equal numerically, then by the addition of equalities (the addition of the two equations) or subtraction (if the signs are the same), we could eliminate the y variable from the scene. If the coefficients are not equal numerically, they can be made equal by the postulate concerning the multiplication of

equalities. We can illustrate these thoughts by using the same pair of equations.

$$\begin{array}{rll} 2x + 3y &= 18 & \text{(multiply both members by 2)} \\ 3x - 2y &= 1 & \text{(multiply both members by 3)} \\ \hline 4x + 6y &= 36 & \\ 9x - 6y &= 3 & \\ \hline 13x \qquad &= 39 & \text{(addition of equalities)} \\ x &= 3 & \\ y &= 4 & \text{(by substitution)} \end{array}$$

EXERCISES (VII–4)

1. Solve each of the following sets of simultaneous equations by two methods:

a. $3x - 2y = 3$
   $x + 2y = 17$

b. $5x - 2y = 11$
   $3x + 4y = -9$

c. $x + 2y = 2\frac{1}{2}$
   $3x - 5y + 9 = 0$

d. $7x = 9y - 1$
   $x - y - 1 = 0$

e. $mx - y = k$
   $x + my = 2k$

f. $\frac{x}{2} - \frac{y}{3} = 4$
   $\frac{x}{3} + \frac{y}{5} = 1$

g. $\frac{2}{x} + \frac{3}{y} = 4$
   $\frac{6}{x} + \frac{5}{y} = 8$
   (Solve first for $1/x$ and $1/y$.)

2. One of the methods of solution of simultaneous linear equations involved a step wherein the two equations were combined under certain conditions. Examine the consequences of adding or subtracting at random.

a. Suppose, for example, that the two equations were $2x + y = 0$ and $3x + 4y = 10$. Their sum is $5x + 5y = 10$. Their difference is $x + 3y = 10$. After doubling the first equation, tripling the second, and adding the results, we get $13x + 14y = 30$. Now graph all five equations on the same axes.

b. Can you explain the graphic result?

c. On the same axes, graph $x = -2$ and also $y = 4$. How does this fit into the preceding picture?

3. The situation in exercise 2 can be generalized in an algebraic sense. Suppose that (m, n) is a common solution of $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$. Prove that (m, n) satisfies

$$(a_1x + b_1y - c_1) + k(a_2x + b_2y - c_2) = 0$$

for any value of k.

5. GENERAL SOLUTION LEADS TO A NEW SYMBOL

Another glance at the last solution in Art. 4 shows that the value 3 for x came directly from the numbers 13 and 39, in $13x = 39$, and that these

two numbers came from the coefficients and constants of the two equations. Every number in the solution and at any point in the solution can be traced back to the source, the coefficients and constants of the original equations. These original numbers give rise in some unique way to the solutions. A change in only one of the original numbers would suffice to change the answers.

To see the manner of interaction of the coefficients and constants, it is well to use literal rather than numerical symbols. In this way the trace will be clear and distinct. We turn to that now.

$$\begin{array}{rll}
ax + by &= c & \text{(multiply by } b'\text{)} \\
\underline{a'x + b'y} &\underline{= c'} & \text{(multiply by } b\text{)} \\
ab'x + bb'y &= cb' & \\
\underline{a'bx + bb'y} &\underline{= c'b} & \\
(ab' - a'b)x &= cb' - c'b & \text{(by subtraction)}
\end{array}$$

$$x = \frac{cb' - c'b}{ab' - a'b} \qquad (ab' - a'b) \neq 0$$

Similarly,

$$y = \frac{ac' - a'c}{ab' - a'b}$$

The results can be used hereafter as formulas for the solutions of all linear simultaneous equations. It would help matters if the differences of the products of both numerators and denominators were conveniently symbolized. Indeed, an ancient symbol does exist for this. We define

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b$$

This is called a **second-order determinant.** A third-order determinant has three rows and three columns. In accordance with the definition, we have

$$\begin{vmatrix} 3 & 1 \\ 4 & 6 \end{vmatrix} = 18 - 4 = 14 \qquad \begin{vmatrix} 5 & 8 \\ -1 & 2 \end{vmatrix} = 10 - (-8) = 18$$

The computation is easily visualized by finding the product of the terms along arrow (1) and subtracting the product of the terms along arrow (2).

We return now to the general solution of the simultaneous equations. The denominators of both x and y are identical and may be represented by the same determinant. $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$. This may be described as the determinant of the coefficients in the original equations, taken in exactly the same order as they appear in the equations.

The numerator of the x value can be described by the determinant $\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}$. The only difference between this and the previous determinant, or the determinant of the denominator, is that the x coefficients have been replaced by the constants of the equations in the same order.

The numerator of the y solution is the determinant $\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}$. The parallel with the x case is exact. The determinant is like that of the denominator excepting that the y coefficients, the b's, have been replaced by the constants, taken in the same order as they appear in the equations. Thus

$$x = \frac{\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} \quad \text{where} \quad \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \neq 0$$

for the foregoing simultaneous equations. The reader should make note of the pattern of the determinants as described so that when using them, they can be written at sight, as in the following illustration:

$$\begin{aligned} x - 2y &= -9 \\ 3x + 5y &= 17 \end{aligned}$$

$$x = \frac{\begin{vmatrix} -9 & -2 \\ 17 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{-45 - (-34)}{5 - (-6)} = \frac{-11}{11} = -1$$

$$y = \frac{\begin{vmatrix} 1 & -9 \\ 3 & 17 \end{vmatrix}}{11} = \frac{17 - (-27)}{11} = \frac{44}{11} = 4$$

EXERCISES (VII–5)

1. Find the value of each of the following:

a. $\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}$ b. $\begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}$ c. $\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix}$

d. $\begin{vmatrix} x & x^2 \\ 1 & x \end{vmatrix}$ e. $\begin{vmatrix} 1/x & 1/y \\ 1 & 1 \end{vmatrix}$

2. Solve by means of determinants:

a. $x - 2y = 4$
$3x - 2y = 0$

b. $2x + 3y = 2$
$4x + 9y = 5$

c. $4x + 3y = 7$
$2x - 5y = -16$

3. Show that

a. $\begin{vmatrix} ma & c \\ mb & d \end{vmatrix} = m\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ b. $\begin{vmatrix} ma & mc \\ b & d \end{vmatrix} = m\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

c. $\begin{vmatrix} ka & mc \\ kb & md \end{vmatrix} = km\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

These illustrate the fact that a determinant is *factorable* in the sense that a common factor can be removed from a column or row or both.

4. Use the observation in exercise 3 to evaluate the following:

a. $\begin{vmatrix} 150 & 1 \\ 225 & 2 \end{vmatrix}$ b. $\begin{vmatrix} 14 & 21 \\ -3 & 5 \end{vmatrix}$ c. $\begin{vmatrix} 12 & 35 \\ 18 & -20 \end{vmatrix}$

5. Investigate the effect on the second-order determinant of the interchange of two rows or two columns.

6. Find the values of each of the following:

a. $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ b. $\begin{vmatrix} a+c & c \\ b+d & d \end{vmatrix}$ c. $\begin{vmatrix} a+b & c+d \\ b & d \end{vmatrix}$

d. $\begin{vmatrix} a-c & c \\ b-d & d \end{vmatrix}$ e. $\begin{vmatrix} a-b & c-d \\ b & d \end{vmatrix}$

f. Note how the last four cases are obtainable from the first case. Also compare all the answers. Do these suggest any general conclusion?

7. Without computing the values of any of the determinants, establish the validity of the following:

a. $\begin{vmatrix} 5 & 3 \\ 6 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$

b. $\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix}$

c. $\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$

d. $\begin{vmatrix} 5 & 3 \\ 6 & 2 \end{vmatrix} = 2\begin{vmatrix} 5 & 3 \\ 3 & 1 \end{vmatrix}$

e. $\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{vmatrix} = \frac{1}{72}\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}$

8. Find the value of each of the following:

a. $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$ b. $\begin{vmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 2\sin x \\ \cos x & \sin 2x \end{vmatrix}$

6. MORE GEOMETRY, ALGEBRAICALLY CONCEIVED

Attention has been called repeatedly to the necessity of preventing a 0 in any of our denominators. In the solution of simultaneous equations by determinants, this means that the determinant of the denominator must not be 0.

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b \neq 0$$

Under what conditions can this arise, and what is the significance of this occurrence?

If $ab' - a'b = 0$,

then

$$ab' = a'b$$

and so

$$\frac{a}{a'} = \frac{b}{b'} \qquad \text{or} \qquad \frac{a}{b} = \frac{a'}{b'}$$

We describe the alternative conclusions at the same time by noting that in

$$ax + by = c$$
$$a'x + b'y = c'$$

the coefficients of the corresponding variables are proportional when the

foregoing determinant is 0. However, we know that in any one equation, these coefficients determine the slope of the line. We see this again. In

$$ax + by = c$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

and so the slope is

$$m = -\frac{a}{b}$$

Similarly, the slope of the second line is

$$m' = -\frac{a'}{b'}$$

When the foregoing determinant is 0, the equations in question have no common solution. This means that their lines do not intersect and therefore that the lines are parallel or, if the c's are in the same proportion, that the lines are coincident.

Now the determinant will be 0 when the respective coefficients of the variables are in proportion. We see that this will happen when the slopes of the lines are equal. We come then to the conclusion that *when the slopes of two lines are equal, the lines are either parallel or coincident.*

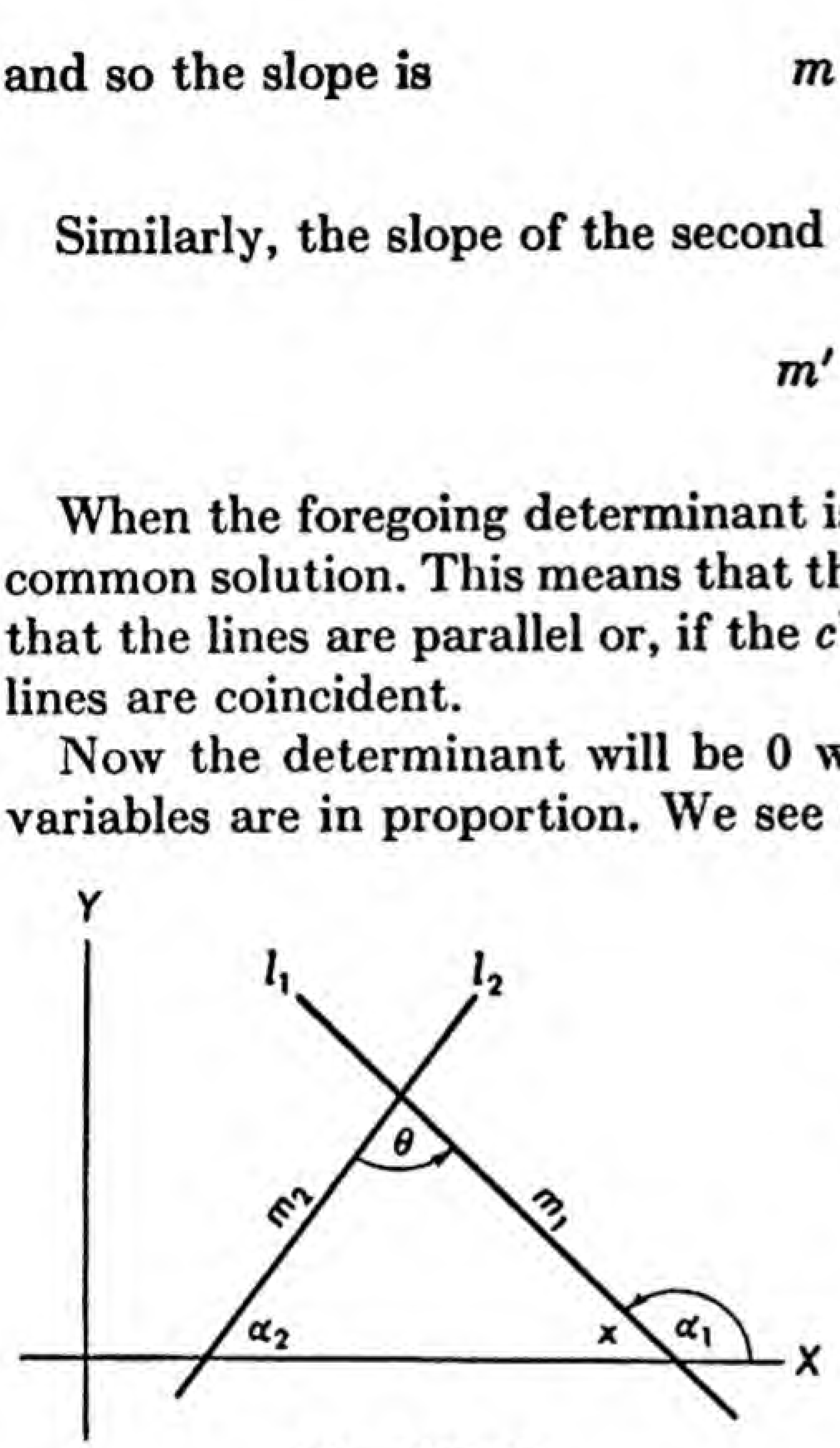


Fig. VII–17

This conclusion can also be seen through a geometric analysis of parallel lines that have the same angles of inclination, and consequently the same slope. Conversely, lines that have the same angles of inclination are parallel and have the same slope.

It is interesting to note that the forbidden situation in algebra leads to the exceptional case in geometry. We shall see such conjunctions of events again and again. In fact this occurs right now in studying the relationships concerned with the angles formed by intersecting lines. (Refer to Fig. VII–17.)

Let α_1 be the angle of inclination of a line whose slope is m_1; let α_2 and m_2 be the angle of inclination and slope, respectively, of a second line which intersects the first. Let θ be an angle, measured counterclockwise, formed by the two lines. Because α_1 is supplementary to x and because of

180° in the triangle, we have

$$\theta + \alpha_2 = \alpha_1$$

or

$$\theta = \alpha_1 - \alpha_2$$

Then

$$\tan \theta = \tan (\alpha_1 - \alpha_2)$$

So

$$\tan \theta = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2}$$

or

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

It should be noted that θ was measured counterclockwise from line l_2 to line l_1. If we were interested in the angle adjacent to this, such an angle would go from l_1 to l_2. All we would need to do would be to interchange the subscripts in the final formula.

This is a valuable formula for the determination of an angle between two intersecting lines. The applications of this are straightforward. The formula, however, has two special cases worth investigating. One occurs when the numerator is 0. If

$$m_1 - m_2 = 0$$

then

$$m_1 = m_2$$

This leads to the fact that $\tan \theta = 0$, or 180°. In both cases this means that the lines are either coincident or parallel. This agrees with the previous finding although the present finding is reached from an entirely different viewpoint.

The other special case is the nonpermissible one where the denominator is zero. If

$$1 + m_1 m_2 = 0$$

then

$$m_1 m_2 = -1$$

or

$$m_1 = -\frac{1}{m_1}$$

or

$$m_2 = -\frac{1}{m_1}$$

It will be recalled from our discussion of the tangent that when the denominator of the tangent ratio approaches 0 the angle approaches 90° or 270°. Of course lines that are 90° to each other also form an angle of 270°. So, we have the conclusion that *the slopes of perpendicular lines are the negative reciprocals of each other, or that the product of the slopes is* -1. The preceding three equations represent this conclusion. The converse of this case holds as well. Specifically, lines with the slopes 3/4 and $-4/3$ are perpendicular to each other.

EXERCISES (VII-6)

1. Select from the following list of equations those that are parallel, coincident, or perpendicular:

a. $y = \frac{2}{3}x - 5$
b. $y + x = 1$
c. $5y - 4x = 8$
d. $3x + 2y = 6$
e. $4y + 5x = 8$
f. $2y = 5 + 3x$
g. $7 - 4y = 6x$
h. $3y + 15 = 2x$
i. $x - 7 - y = 0$

2. a. Find the equation of the line that passes through (1, 4) and is parallel to $2y + 3x = 1$.
   b. Do the same with (−2, 3) and $3x - 4y = 8$.
3. a. Find the equation of a line which passes through (1, 4) and is perpendicular to $2y + 3x = 1$.
   b. Do the same with (−3, −2) and $5y + 10x = 0$.
4. Prove (by analytic methods) that the diagonals of a square are perpendicular to each other.
5. Prove that the diagonals of a rhombus are perpendicular to each other.
6. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to one-half of it
7. Prove that the line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and equal to one-half the sum of the bases.
8. If a and b are respectively the x- and y-intercepts of any line, show that the equation of any line parallel to it is given by $(x/ka) + (y/kb) = 1$, where k is any number other than zero
9 Through (1, 5) a line is drawn perpendicular to $x - 3y = 2$. Find the point of intersection of the two lines.
10. Find the point of intersection of the altitudes of the triangle whose vertices are at (−1, 2), (3, 5), and (6, −2).
11. Find the distance of the line $3x + 2y = 6$ from the origin.
12. Find the distance of $ax + by = c$ from the origin.
13. A set of lines possessing some particular common property is frequently called a "family of lines." Thus, the lines that pass through a common point, or are parallel to each other are examples of families of lines.
   a. Write the equation of the family of lines passing through the intersection of $3x - y = 5$ and $x + 2y = 4$. (See ex. 3, sec. 4.)
   b. Determine the member of this family which passes through (3, 4); (−1, 5); (5, 1); (2, 6).
14. a. Write the equation of the family of lines passing through (3, 5). (Obviously one point does not determine a unique line.)

   Try this by using the point slope method. (The final equation must still contain m, which cannot be found without more information. It is the presence of m and the possibility that m may take on any real value that permits the one equation to represent a family (an infinity) of lines.)
   b. Try this by starting with the slope intercept general equation.

15. Sketch a few members of each of the following families of lines: (k is any real number)

a. $y = 3x + k$
b. $2y - 3x = k$
c. $5y + kx = 10$
d. $y - 4 = k(x + 3)$
e. $\frac{x}{3} + \frac{y}{k} = 1$

16. Find the angle of inclination of each of the following and graph the equations

a. $3y = 2x + 4$ b. $2y - 4x + 1 = 0$ c. $3x + 7y + 4 = 0$

17. Find the angles formed by the following intersecting lines. Get the angle from the first to the second line.

a. $3x - 2y = 8$; $4y - x = 7$
b. $x + y = 1$; $3x - 2y = 6$
c. $5x - 3y = 6$; $3x + 5y = 7$

18. Prove that the figure formed by connecting consecutively the midpoints of the sides of any quadrilateral is a parallelogram.

19. If the quadrilateral in exercise 18 were a rhombus, the resulting parallelogram would be a rectangle. Prove this.

20. We start with the line $y = mx + b$.

a. Write the equation of any line through the origin and perpendicular to line $y = mx + b$.
b. Show that the coordinates of the point of intersection of these two lines is given by

$$\left(\frac{-mb}{1+m^2}, \frac{b}{1+m^2}\right)$$

c. Show that the distance of the first line from the origin is given by

$$d = \frac{|b|}{\sqrt{m^2+1}}$$

21. Using the result of the preceding exercise, find the distance of each of the following from the origin:

a. $3x - 2y + 7 = 0$
b. $y = 5x + 1$
c. $5x + 3y = 10$

22. We start with the line $y = mx + b$ and the point (X, Y) which is not on this line.

a. Show that the equation of the line through (X, Y) and parallel to $y = mx + b$ is $y = mx + (Y - mX)$.
b. Show that the distance from the origin to the new line is

$$d = \frac{|Y - mX|}{\sqrt{m^2+1}}$$

Use exercise 20(c), of course.

c. Now show that the distance from (X, Y) to $y = mx + b$ is given by

$$d' = \frac{|Y - mX - b|}{\sqrt{m^2 + 1}}$$

23. a. To find the distance from (5, 6) to $y = -x + 3$, we note that $X = 5$, $Y = 6$, $m = -1$, and $b = 3$. Substituting these values in the formula of the preceding exercise, we get

$$d' = \frac{|6 + 5 - 3|}{\sqrt{1 + 1}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

b. To find the distance from (1, 1) to $y = -x + 3$, we get

$$d' = \frac{|1 + 1 - 3|}{\sqrt{1 + 1}} = \frac{+1}{\sqrt{2}} = +\frac{1}{2}\sqrt{2}$$

24. Find the distance from the indicated point to the line:

a. $(6, 8)$, $3y - 2x + 6 = 0$
b. $(-3, 4)$, $x - 2y = 0$
c. $(-2, -3)$, $4x + 2y - 3 = 0$
d. $(-5, 0)$, $\frac{x}{2} + \frac{y}{3} = 1$

25. The lines $y = 2x + 1$, $2y + x = 17$, and $3y + x + 3 = 0$ form a triangle. Find the length of the altitude to the third side.

VII–6 REVIEW

1. Find the point of intersection of the two lines, each of which has the following x- and y-intercepts respectively: $a = 3$, $b = 4$, and $a = -2$, $b = 1$.

2. Write the equations of any two lines that intersect each other perpendicularly at $(-3, 3)$ and which are not parallel to the axes.

3. If the slopes of the three sides of a triangle are $\frac{3}{4}$, $-\frac{1}{2}$, and $-\frac{1}{4}$, find the angles of the triangle.

4. a. How many different triangles are possible with the data given in exercise 3.
b. What general theorem (or theorems) is suggested by exercise 3?

5. Three consecutive vertices of a trapezoid are at (0, 0), (5, 0), and (4, 4). If the fourth vertex is at $(x, 4)$, find the value of x when the diagonals are also perpendicular to each other.

6. Two sides of a triangle are inclined 45° to each other. If the slope of one of the sides is $\frac{1}{3}$, find the slope of the other.

7. The points (2, 1) and (8, 5) are opposite vertices of a square. Find the equation of the line determined by the other diagonal.

8. Find the member of the set of lines parallel to $3x - 2y = 8$ which passes through the point (1, 4).

9. The line $(x/a) - (y/4) = 1$ is parallel to the line $(x/3) - (y/a) = 1$. Find the value of a.

10. Solve the simultaneous equations $x - y = 7$ and $3x + 2y = 1$ algebraically, graphically, and by determinants.

11. Determine m so that the following three lines are concurrent: $3x - 2y = -1$, $4x - 5y = 1$, $2x - my = 5$.

12. Expand the following second order determinants:

a. $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ b. $\begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix}$ c. $\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix}$

13. Solve for X and Y:

$$X \cos \theta - Y \sin \theta = x$$
$$X \sin \theta + Y \cos \theta = y$$

14. Find the length of the altitude to BC in the triangle which has the vertices $A(-1, 0)$, $B(0, 6)$, and $C(8, 1)$.

15. Does the line through the intersection of $(2y - 3x = 6)$ and $(4x + y = 8)$ and the point $(4, 5)$ lie on an angle bisector of one of the angles formed by the lines?

16. The determinant in Art. 6 can be 0 also when at least one row, or one column, consists only of 0 elements. Discuss the lines $(ax + by = c)$ and $(a'x + b'y = c')$ from this viewpoint.

7. THE CIRCLE AS A LOCUS

The description of functions through the technique of locus suggests interesting possibilities. All we need do is to dream up some sufficient conditions, state them analytically, and then determine the equations. We have had a little glimpse of this in connection with linear equations. Let us go farther.

Consider the function determined by the locus of points (Fig. VII–18) each of which is exactly five units from the point $(2, 3)$. If we take $P(x, y)$ as any member of the set that satisfies the condition, we can write by the distance formula that

$$\sqrt{(x - 2)^2 + (y - 3)^2} = 5$$

or

$$(x - 2)^2 + (y - 3)^2 = 25$$

Of course we need not take the trouble, which would be considerable, to graph the equation to verify what we know from the start; that is, that the condition described yields a circle of radius 5 and with the center at $(2, 3)$.

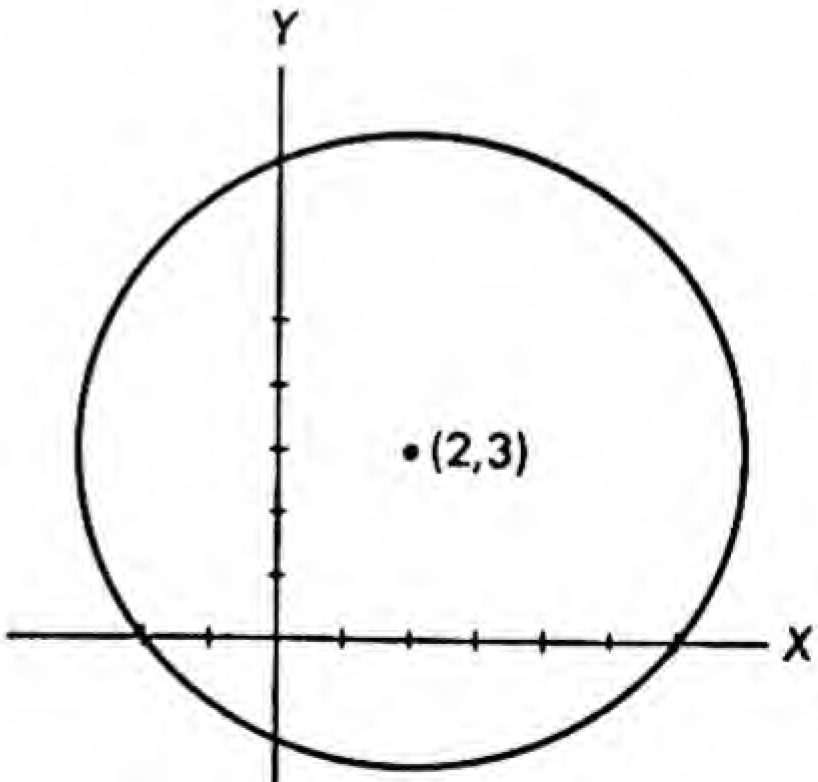


Fig. VII–18

It is but a short step to the general case (Fig. VII–19). We have by the same distance formula:

$$(x - h)^2 + (y - k)^2 = r^2$$

which is the equation of a circle with the center at (h, k) and a radius of r. If we take the origin as the center, then $(h, k) = (0, 0)$, and we have the special case

$$x^2 + y^2 = r^2$$

This implicit equation yields two explicit ones:

$$y = \sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

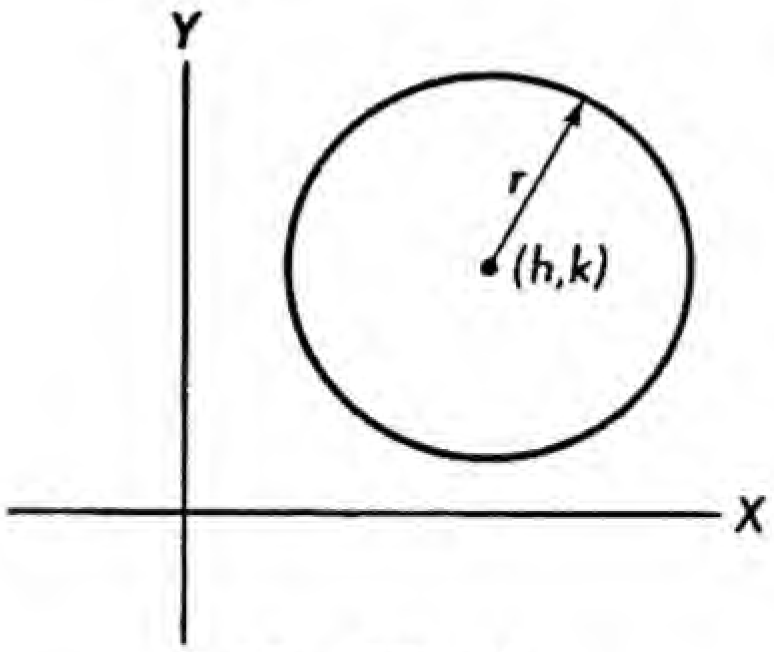


Fig. VII–19

Each represents a semicircle; together these constitute the circle at the origin described previously.

The domain of X depends on the radicand $r^2 - x^2$, which for real values of y must be positive. Consequently, $|x| \leq r$. Similarly we can see that the range of Y is $|y| \leq r$.

EXERCISES (VII–7)

1. Write the equation of the circle whose center is at $(3, -4)$ and whose radius is 6.

2. Write the equation of the circle that has the segment determined by $(-2, 5)$ and $(6, 3)$ as a diameter.

3. Write the equation of the circle that is tangent to the X-axis and has a center at $(5, -6)$. (A circle that is tangent to a line touches it at only one point. The radius to the point of contact, being the shortest line from the center to the tangent line, is perpendicular to the tangent.)

4. Prove via congruence that the two tangents to a circle from a point outside the circle are equal. (The implied reference to the lengths of the tangents is to be taken as from the outside point to the points of contact.)

5. Write the equation of a circle whose center is at $(-4, -5)$ and which passes through $(1, 4)$.

6. Determine two explicit equations for y from $(x - 2)^2 + (y - 3)^2 = 25$.

7. Find the equation of the circle circumscribing the triangle whose vertices are at $(-4, 0)$, $(0, 3)$, and $(2, -3)$.

8. Find the locus of a point such that its distance from $(1, 4)$ is twice as much as its distance from $(-3, 6)$.

9. a. What are the domain and range of the relation defined by $(x + 3)^2 + (y - 4)^2 = 36$?
   b. Does $(-7, 1)$ lie inside, on, or outside the circle?
   c. If $(2, k)$ lies on the circle, find the value(s) of k.

10. Write the equation of the family of concentric (with same center) circles with center at $(3, 4)$.

11. Determine the value of k in $(x - 4)^2 + (y - k)^2 = 25$ so that the circle passes through the origin.

12. The equation $(x - h)^2 + (y + 4)^2 = 49$ represents a family of circles. Describe and sketch some members.

13. Find the solution(s) common to $x^2 + y^2 = 25$ and $2x + y = 10$. Sketch graphs and discuss meaning of solution(s). (As with first-degree simultaneous equations, the second equation could be solved for y, or for x if that were simpler, and the result substituted in the first equation. In this way an equation in one unknown is obtained.)

14. Solve the following pairs of simultaneous equations and check your results:

a. $3x^2 + 4y^2 = 39$
$3x - y = 0$

b. $2x^2 + y^2 = 9$
$x + 3y = 1$

c. $x^2 + y^2 - 2x + 4y = -5$
$4x + y = 2$

15. Find the common solutions and interpret the results:

a. $x^2 + y^2 = 25$
$4y + 3x = 25$

b. $x^2 + y^2 - 4x = 6$
$3y + x + 8 = 0$

c. $x^2 + y^2 = 1$
$x + y + 6 = 0$

8. LOCATING THE CENTER

The general equation of the circle

$$(x - h)^2 + (y - k)^2 = r^2$$

can be written with the binomials expanded, as in

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$$

The very nature of this equation is such that no matter what the values of h and k, the coefficients of the second-degree terms, the x^2 and y^2, will be identical for the equation of the circle.

The equation of a circle with center at (2, 3) and radius 5 is

$$(x - 2)^2 + (y - 3)^2 = 25$$

or

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

Where the center is at $(\frac{1}{2}, -\frac{1}{3})$ and the radius is 4, we have

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = 16$$

or

$$x^2 + y^2 - x + \frac{2}{3}y - 15\frac{23}{36} = 0$$

or

$$36x^2 + 36y^2 - 36x + 24y - 563 = 0$$

The coefficients of the highest powers are and must be identical for the equation of the circle.

Now suppose that we have come across an equation of a circle. Can we reverse the steps and determine the location of the center and the value of the radius? Consider

$$x^2 + y^2 + 4x - 6y + 8 = 0$$

We know, from the preceding cases that the two x terms come from the square of some binomial $(x - h)^2$. The middle term of the expansion of this binomial will be $-2xh$, which is minus twice the product of the first and second terms of the binomial. This indicates that the sum $x^2 + 4x$ in our example here must come from $(x + 2)^2$. The square of this binomial will yield 4 more than the $x^2 + 4x$. Consequently $x^2 + 4x$ is the same as $(x + 2)^2 - 4$. Similarly, for $y^2 - 6y$, we could substitute $(y - 3)^2 - 9$. If we make both these substitutions in the foregoing illustration, we get

$$(x + 2)^2 - 4 + (y - 3)^2 - 9 + 8 = 0$$

Thus, we have

$$(x + 2)^2 + (y - 3)^2 = 5$$

from which we can tell that the center of the circle is at $(-2, 3)$ and the radius is $\sqrt{5}$.

EXERCISES (VII-8)

1. Find the coordinates of the center and the radius:
   a. $x^2 + y^2 - 4x + 6y + 4 = 0$
   b. $x^2 + y^2 + 8x + 10y - 8 = 0$
   c. $x^2 + y^2 - 6x = 0$
   d. $2x^2 + 2y^2 = 6x + 2y + 11$
   e. $x^2 + y^2 + 3y = 4$
   f. $3x^2 + 3y^2 - 5x + 6y = 3$

2. Find the coordinates of the center and the radius of $x^2 + y^2 + dx + ey + f = 0$.
3. Use the equation in exercise 2 to express d, e, and f in terms of h, k, and r.
4. Using the results of exercise 2, state the conditions for the real plane when $x^2 + y^2 + dx + ey + f = 0$:
   a. Has no graph.
   b. Is a single point.
   c. Is a circle.

5. a. Interpret and illustrate the situation in exercise 4(a).
   b. Interpret and illustrate the condition in exercise 4(b).
6. The point $P(x, y)$ moves so that its distance from $(4, 0)$ is always twice its distance from $(0, 8)$. Show that the locus is a circle, and find the center and radius.
7. Find the locus of a point such that the sum of the squares of its distances from $(2, 1)$ and $(-3, -2)$ is always 12.
8. Show that the locus of a point, the ratio of whose distances from two fixed points is a constant other than 1, is a circle or a degenerate case thereof.
9. Consider the circles $x^2 + y^2 + dx + ey + f = 0$ and $x^2 + y^2 + d'x + e'y + f' = 0$. What would be the nature of the graph of $x^2 + y^2 + dx + ey + f + k(x^2 + y^2 + d'x + e'y + f') = 0$ if:
   a. $k = -1$
   b. k is any real number other than -1 or 0.

10. Assuming a real circle $x^2 + y^2 + dx + ey + f = 0$, state the conditions such that:

a. The center is on the X-axis.
b. The center is on the Y-axis.
c. Center is at the origin.
d. The circle passes through the origin

9. THE CIRCLE IN GENERAL

Some of the recent efforts in connection with the circle suggest that we may take

$$x^2 + y^2 + dx + ey + f = 0$$

as another general equation of the circle. Were we to relate this to a particular circle, we would require sufficient information to fix numerical values for d, e, and f.

The definition of the circle as "the locus of points a fixed distance from a given point" was adequate to determine the equation of the circle involving the constants h, k, and r. But, as with the straight line, there should be other means of determining the figure. Indeed, such knowledge as will permit us to specify the values of d, e, and f must be also sufficient to determine a circle.

We have met many instances where knowledge of a member of a set is sufficient to determine a constant in the equation describing the set. For example, the fact that $(1, -4)$ lies on the line $(2y + kx = 6)$ is sufficient information to obtain the value of k.

Our present problem is a little more complicated in that we have to find the values of three unknown coefficients. There are many ways of looking at this. One, and a valuable viewpoint, is that we need as many relevant, independent facts as there are unknowns involved. In this case this means that we need three points on a circle to be able to specify the equation of the circle. Let us examine this. Suppose we desire to find the equation of a circle which passes through the points $(4, 2)$, $(1, 3)$, and $(-3, -5)$. Substituting these values in the general equation, we get the three equations:

$$\begin{aligned} 16 + 4 + 4d + 2e + f &= 0 \\ 1 + 9 + d + 3e + f &= 0 \\ 9 + 25 - 3d - 5e + f &= 0 \end{aligned}$$

Simplifying the three equations, we have

$$4d + 2e + f = -20 \tag{1}$$

$$d + 3e + f = -10 \tag{2}$$

$$3d + 5e - f = 34 \tag{3}$$

Well, then, we are faced with the problem of finding a unique solution (if one exists) for three equations taken simultaneously in three unknowns, all in the first degree. We can take a lead from the experience with two equations in two unknowns, in which, by substitution or elimination, we were able to derive one equation in one unknown which was then solvable. We can plan our present problem in the same direction. First, we can consolidate the equations by curtailing them to two unknowns and to two equations. Then the problem will be like the earlier cases which were referred to as a guide.

By subtracting equation (2) from equation (1), we eliminate the letter f from these two equations. By adding equations (2) and (3), we eliminate f again. In this way we get the equations (4) and (5), respectively, in the letters d and e. Now we are on familiar grounds.

$$\begin{array}{lrcll}
(4) & 3d - e & = & -10 & \\
(5) & 4d + 8e & = & 24 & \\
\hline
 & 28d & = & -56 & \text{Multiplication (4) by 8, add} \\
 & d & = & -2 & \\
 & -6 - e & = & -10 & \text{Substitution in (4)} \\
 & e & = & 4 & \\
 & -8 + 8 + f & = & -20 & \text{Substitution in (1)} \\
 & f & = & -20 & \\
 & (d, e, f) & = & (-2, 4, -20) &
\end{array}$$

The equation of the circle is then

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

and by writing this equation in the h,k,r form, we discover that the center is at $(1, -2)$ and that the radius is 5.

EXERCISES (VII-9)

1. Find the common solution for the simultaneous equations:

a. $2x + y - z = 5$
$x - y + 2z = -3$
$x + 2y - z = 4$

b. $a - 2b + 4c = 3$
$3a + 5b - 2c = -9$
$6a - b + 4c = -3$

2. Derive the equations of the circle passing through the points:

a. $(0, 4), (0, 0), (-5, 3)$.
b. $(3, -2), (4, 5), (1, -4)$.
c. $(3, 1), (5, -2), (-2, -3)$.

3. a. Determine three sets of integral solutions for $x + 2y - z = 4$.
   b. Determine three sets of integral solutions for the two equations $x + 2y - z = 4$ and $3x + y + z = 4$ taken simultaneously.
   c. For any solution found in exercise 3(b) write a third equation so that the solution will satisfy the three equations simultaneously.

10. DETERMINANTS AGAIN

The algebraic effort in Art. 9 recalls the introduction of determinants for the solution of simultaneous equations with two unknowns. Precisely the same approach is possible for three equations in three unknowns, with each in the first degree. We shall not undertake at this time to solve the equations with the literal coefficients by the elimination of variables. Suffice it to say that for x, for example, the solution will consist of a fraction with six terms in both the numerator and denominator. Half of each will have positive signs, and the remainder will have negative signs. All the terms will consist of products of three of the coefficients. The values of the other two variables are analogous. It is possible to construct a third-order determinant with such rules of computation as will give precisely those fractions.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

As in the case with the two unknowns, the denominators all have the same determinant, which is symbolized by D. The numerators also follow the same plan as in the earlier case. In solving for x, we eliminate the column of the x coefficients from D and replace it with the column of the constants, the d's. The analogous situation prevails for the solutions of y and z.

The rule of computation is along the diagonals. There are three left-to-right diagonals, L, each containing three terms. This is best seen by

writing or visualizing the first two columns repeated on the right, as shown in Fig. VII–21. Likewise there are three right-to-left diagonals, R. The terms along any diagonal are multiplied by each other. The value of the

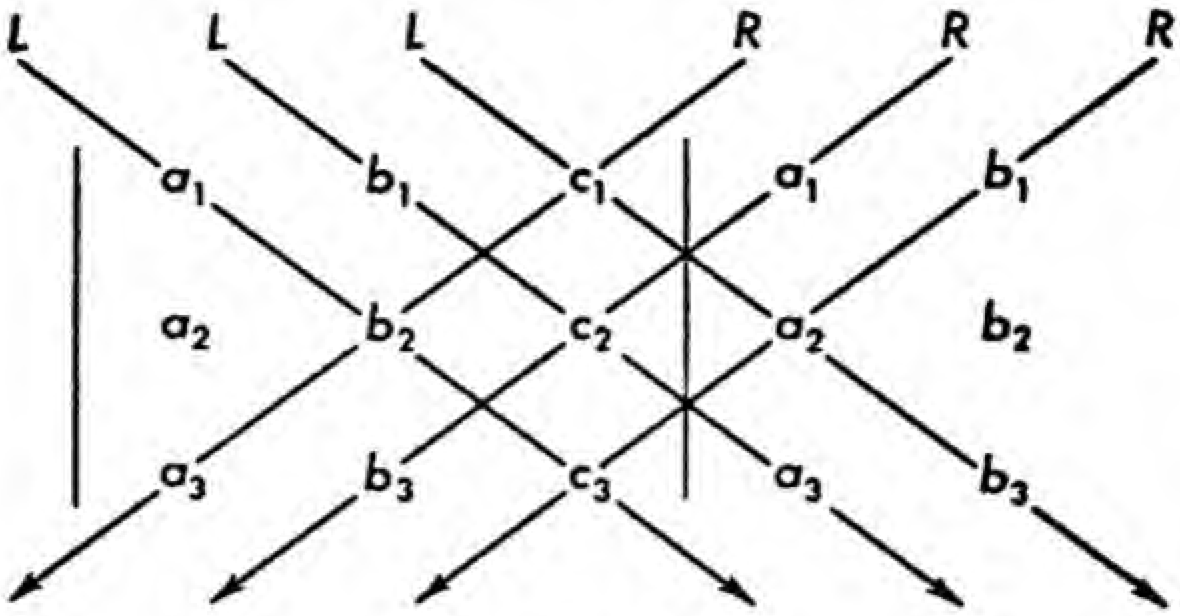


Fig. VII–21

determinant is equal to the sum of the three L diagonals less the sum of the three R diagonals.

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

We reconsider some earlier equations for the sake of illustrating this approach.

$$\begin{aligned} 4d + 2e + f &= -20 \\ d + 3e + f &= -10 \\ 3d + 5e - f &= 34 \end{aligned}$$

$$d = \frac{\begin{vmatrix} -20 & 2 & 1 \\ -10 & 3 & 1 \\ 34 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 1 \\ 1 & 3 & 1 \\ 3 & 5 & -1 \end{vmatrix}} = \frac{60 + 68 - 50 - (102 - 100 + 20)}{-12 + 6 + 5 - (9 + 20 - 2)} = -2$$

The exceptions to unique solutions for three equations in three unknowns may be pinpointed now. Since division by zero is the only inadmissible operation, the failure in a unique solution will occur when $D = 0$. That is, when

$$\begin{vmatrix} a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \end{vmatrix} = 0$$

We shall see later that a first-degree equation in three unknowns will be represented graphically by a plane in three dimensions. Three equations will yield three planes. A unique solution will occur when the three planes meet in a single point, like the corner of a room or the apex of a triangular pyramid. The planes may be parallel to each other. This means that there is no point in common, and that algebraically there is no common solution. In this case, as in the two-dimensional analogue with the parallel lines, the coefficients of the variables will be proportional to each other. There will not be any solutions either when two of the planes are parallel or when the intersection of two of the planes is parallel to the third plane. Also, the three planes may intersect in one line every point of which (meaning an infinite number of points) will be a common solution. The variety of exceptional cases has grown considerably.

EXERCISES (VII-10)

1. Evaluate each of the following:

$$\text{a. } \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 0 & 1 & 5 \end{vmatrix} \qquad \text{b. } \begin{vmatrix} 3 & 2 & 2 \\ 4 & -1 & 3 \\ 5 & -6 & 7 \end{vmatrix} \qquad \text{c. } \begin{vmatrix} 8 & -5 & 4 \\ -3 & -2 & 6 \\ 1 & 4 & 3 \end{vmatrix}$$

2. Find the common solutions where they exist:

a. $2x + y + z = 1$
$3x - 2y - z = 4$
$-x + 3y + 2z = -5$

b. $x - 5y = 7$
$3y + 4z = -4$
$2x + 6z = -3$

3. We have seen that the second-order determinants have various properties. They will be seen to hold for third-order determinants.

a. Prove that $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

b. Show that the factorability property holds if any row has the common factor k.

c. Consider the consequences of k being 0.

d. Examine the effects on the value of the determinant when two rows or two columns are interchanged.

e. What is the implication of (d) if two rows or two columns are equal to each other (term by term).

f. Study the effect of replacing a column or row by the sum or difference of the corresponding terms of that column or row with those of another column or row.

g. Suppose that all the rows were changed to columns in the same order. Consider the effect of such an interchange.

4. Show that the area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3), is given by

$$A = \tfrac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_1 - y_1 x_2 - y_2 x_3 - y_3 x_1)$$

(One approach is through the trapezoids formed by dropping perpendiculars from the vertices to the X-axis.)

5. Show that the area in exercise 4 is given by the determinant

$$K = \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

What is the meaning of the $\pm$ sign?

6. Find the areas of the following triangles:

a. (0, 5), (0, 0), (2, 0).
b. (3, −1), (2, −5), (−3, 4).
c. (−1, 4), (2, −3), (3, 6).

7. Explain why the equation of a line through the points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

8. Find by means of determinants the equations of the lines through:

a. (2, 3) and (−1, 4)
b. (3, −2) and (−1, −2)
c. (0, 5) and (9, 0)

11. FRAME OF REFERENCE

The equations of the circle that were developed earlier provide food for further thought. If we imagine ourselves at the origin of a rectangular network (Fig. VII–22), the equation of a circle with its center at the origin is $x^2 + y^2 = r^2$. The equation of the same circle with the center at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

If, on the other hand, we were stationed at (h, k) and took that to be our new origin, with axes passing through it parallel to the original axes, the equation of the second circle would be $X^2 + Y^2 = r^2$, where the capitals are used to designate the new axes. The two equations describe the same circle, and the different equations merely reflect the difference or relativity of the viewpoint. By comparing the two equations, we see that the key to the interrelationship can be stated as two equalities:

$$X = x - h \qquad \text{and} \qquad Y = y - k$$

Of course we could look from the old frame of reference to the new one and get, by solving the last two equations,

$$x = X + h \qquad \text{and} \qquad y = Y + k$$

These results are merely attributes of the relative positions of the two frames of reference and not of the circle or circles. The circle was used as a convenient vehicle for the contrast. We can diagram the relationships without reference to any figure whatsoever.

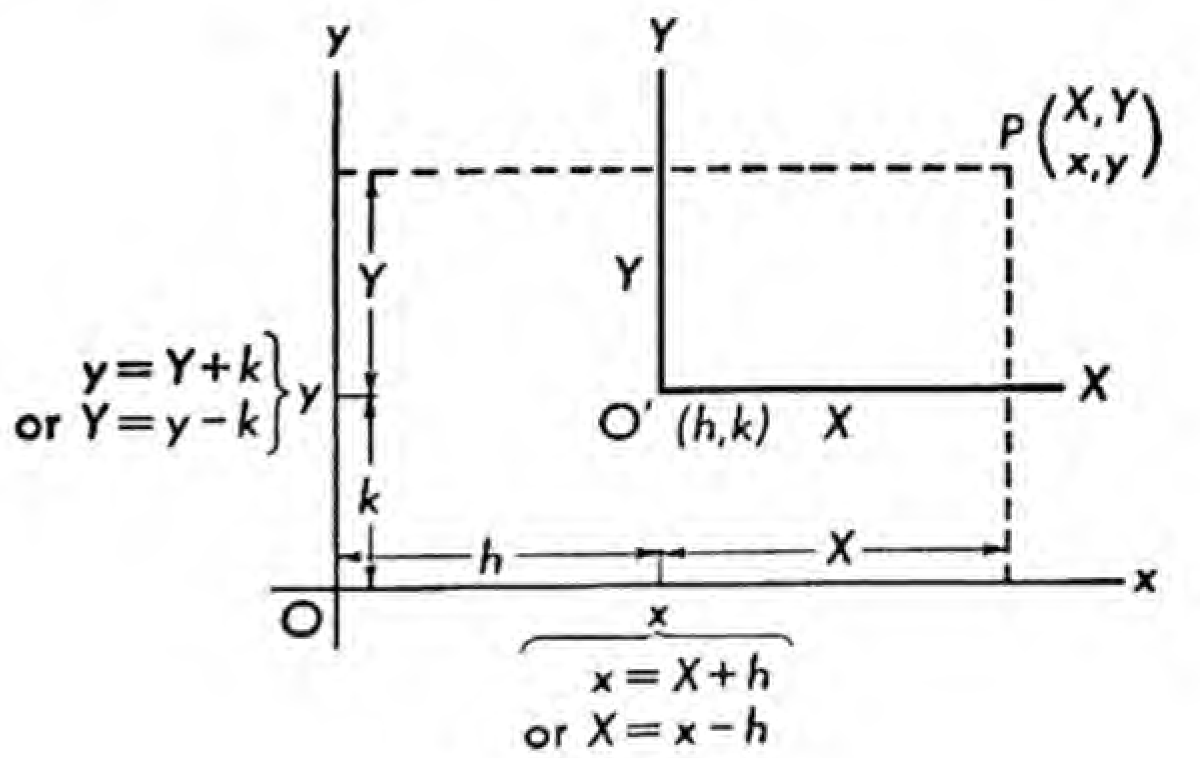


Fig. VII–22

The point P (Fig. VII–22) has two possible sets of coordinates, as indicated, depending on the set of axes to which P is referred. The equations follow a study of the relative lengths involved.

The change or the shifting of the axes in a mutually parallel direction is called a **translation of axes.** While our diagram referred to the first quadrant, the generality of the algebraic and geometric relationships is such that the foregoing *equations of translation* will hold for any quadrant.

EXERCISES (VII–11)

1. Determine the coordinates of the point (3, 5) if parallel axes are taken that intersect at

a. (1, 2) b. (6, 9) c. (−4, 6) d. (−4, −5) e. (6, −8)

2. Translate the origin to any point on the line $y = 3x + 4$, and write the equation of this line with respect to the new axes.

3. By translation of the axes, eliminate the first-degree terms of the following equations. Write the new equations and make a complete sketch for each.

a. $x^2 + y^2 + 4x + 6y = 12$
b. $x^2 + y^2 - 4x - 2y - 5 = 0$
c. $x^2 + y^2 - 10y + 9 = 0$

4. a. Write the equation of the straight line $2y = 3x$ in a translated network with the origin at $(3, -2)$.
   b. Do the same for $x^2 + y^2 = 9$.

5 The lines $y = 2x + 1$ and $y = 3x + 2$ are not obtainable, one from the other, by any translation. Explain.

VII–11 REVIEW

1. Find the equations of the tangents to the circle $x^2 + y^2 = 25$, where the abscissa is 3.

2. a. Write the equation of the line through $(0, 3)$ and the center of the circle $x^2 + y^2 - 4x = 12$.
   b. How is the given point situated with reference to the circle?

3. a. Solve simultaneously $x + y = 6$ and $x^2 + y^2 - 2x = 16$.
   b. Solve the same equations graphically.

4. Prove that every angle inscribed in a semicircle is a right angle. Let $P(x, y)$ be any point on the semicircle $y = \sqrt{r^2 - x^2}$.

5. Show by means of their equations that a line and a circle cannot meet in more than two points.

6. Find the equation of the locus of a point whose distance from the origin is always one-half its distance from $(0, 4)$.

7. Discuss the locus of

   a. $x^2 + y^2 + 6x - 4y - 23 = 0$ b. $4x^2 + 4y^2 - 8x - 20y = 71$

8. The points $(8, 5)$ and $(2, 1)$ are the ends of a diameter. Write the equation of the circle.

9. Find the equation of a circle through:

   a. $(6, 0)$, $(-1, -7)$, $(3, 1)$
   b. $(3, 1)$, $(2, 2)$, $(-5, -5)$

10. Find the equation of a circle concentric with $x^2 - 4x + y^2 = 8$ and passing through $(4, 5)$.

11. Show that it is not possible to get a circle through $(1, 4)$, $(3, 9)$, and $(-1, -1)$.

12. Find the equation of the circle whose center is on $2x + y = 0$ and which passes through $(4, 2)$ and $(-3, 1)$.

13. Find the common solution of

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y - 4z &= 2 \\ 3x + 3y - 5z &= 7 \end{aligned}$$

14. Transform into two equivalent determinants:

$$\begin{vmatrix} 3 & 1 & 3 \\ 4 & -4 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

15. Translate the axes so that the equation of the line $3x - 5y = 15$ will have its constant eliminated if the new origin is taken (a) on the X-axis and (b) on the Y-axis. State the coordinates of the new origin in each case.

12. THE PARABOLA

We move on and suggest new locus conditions, with particular attention at this time to conditions that yield second-degree equations.

Consider the set of points each of which is equidistant from a fixed point and a fixed line. The fixed point and line may be taken anywhere in the plane except that the point is not allowed to lie on the line. While the resulting equations will differ, they will differ only in appearance, for by a suitable change in frame of reference, any one equation may be transformed to any of the others.

Let us take the fixed point at $(0, p)$ (Fig. VII–23) and the fixed line as $y = -p$. As before we let $P(x, y)$ represent the set of points. The point P must satisfy the condition that its distance from the point $(0, p)$ and from the line $y = -p$ are equal to each other. Then

$$\begin{aligned}
\sqrt{x^2 + (y - p)^2} &= y + p \\
x^2 + (y - p)^2 &= (y + p)^2 \\
x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\
x^2 &= 4py
\end{aligned}$$

In the light of our discussion of translation of axes, the equation of the curve is

$$(x - h)^2 = 4p(y - k)$$

if the vertex of the curve is at (h, k) instead of at $(0, 0)$. (Refer to Fig. VII–24.) This is obtained by substituting $x - h$ for x and $y - k$ for k. There is no real need to distinguish the frames of reference by using capitals as we did earlier. The fixed point could be taken on the X-axis

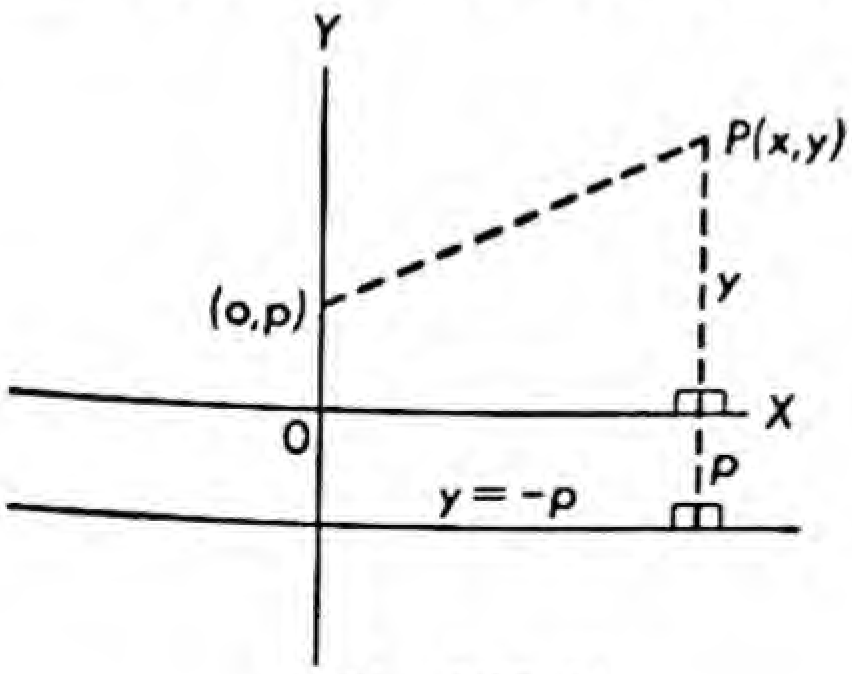


Fig. VII–23

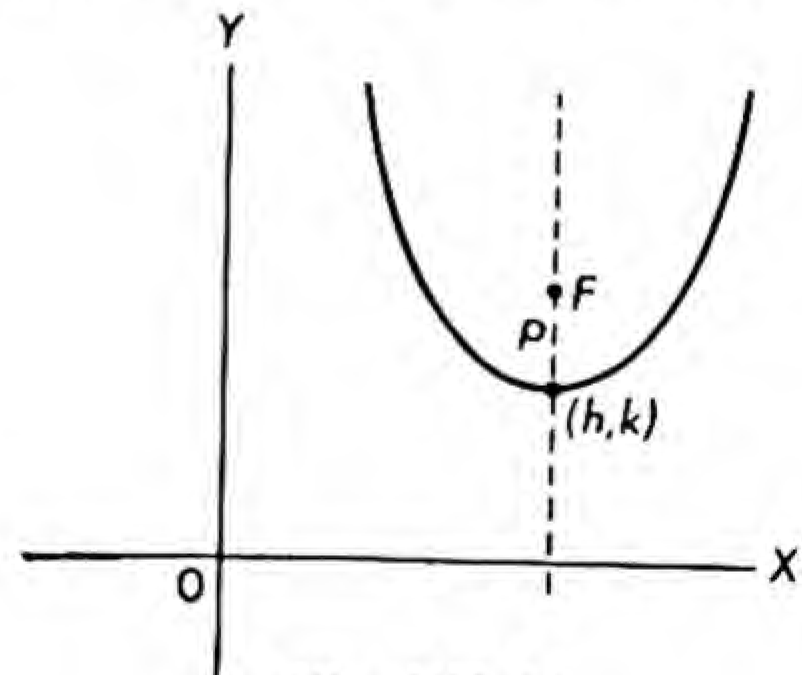


Fig. VII–24

and the fixed line perpendicular to it (Fig. VII–25). The net effect of this, assuming the basic units are kept the same, is an interchange of X for Y and

Y for X. It is as though the axes were rotated through 90°. This yields the equations

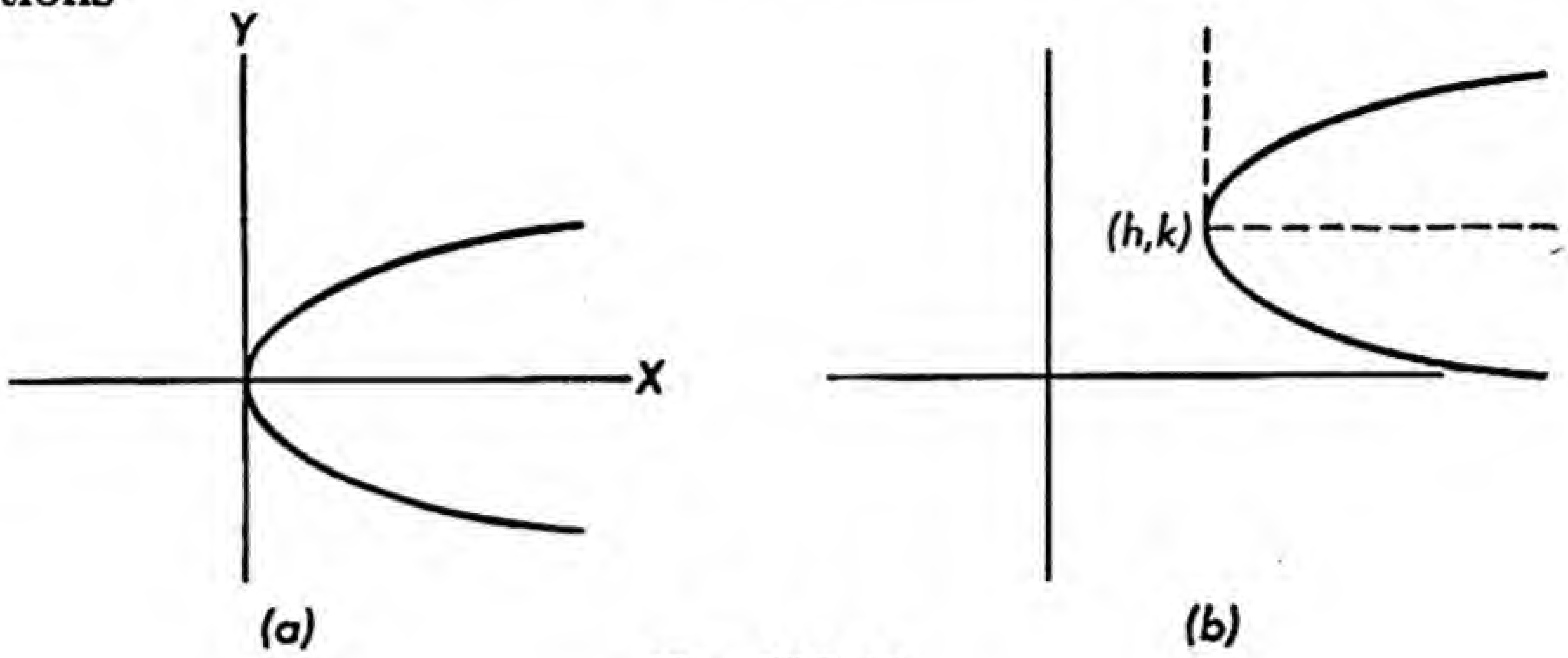


Fig. VII–25

$$y^2 = 4px$$
$$(y - k)^2 = 4p(x - h)$$

The curve we have defined and graphed is the well-known **parabola.** The word means *throw beside.* The Greeks were apparently aware of the fact that an object thrown out into space would pursue, with but minor variations due to interferences, a parabolic path.

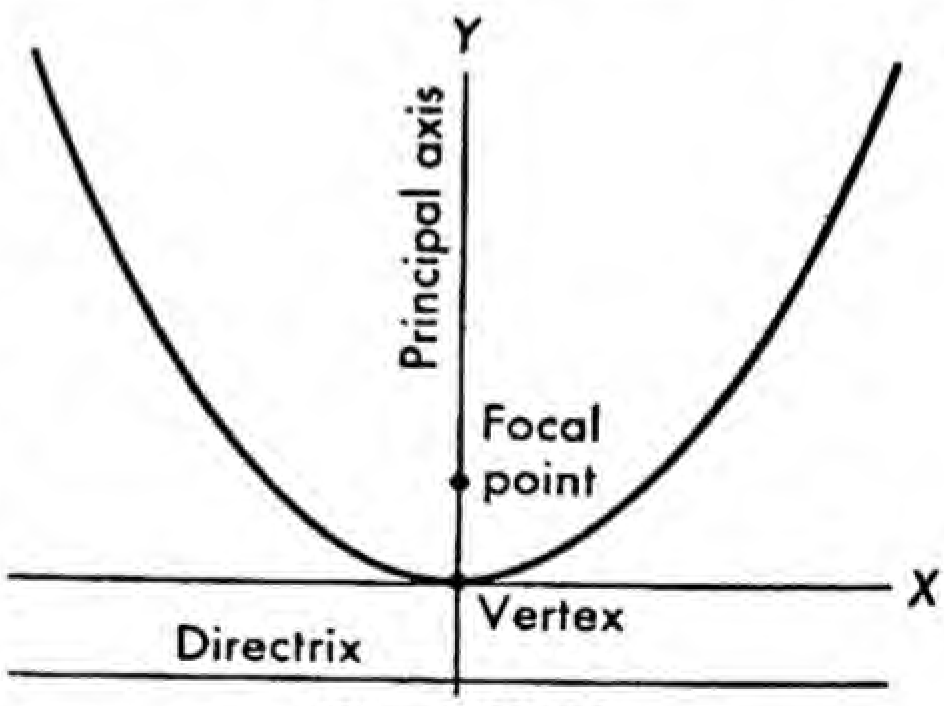


Fig. VII–26

The fixed point of the parabola (Fig. VII–26) is called the *focal point* and the fixed line is the *directrix.* The *principal axis* or *axis of symmetry* passes through the focal point and is perpendicular to the directrix. The point of the parabola that lies on the principal axis is the *vertex* of the curve.

Many of our bridges, like the George Washington Bridge in New York, have their cables suspended in parabolic arches which, for support, distribute the weight evenly throughout the entire length (Fig. VII–27).

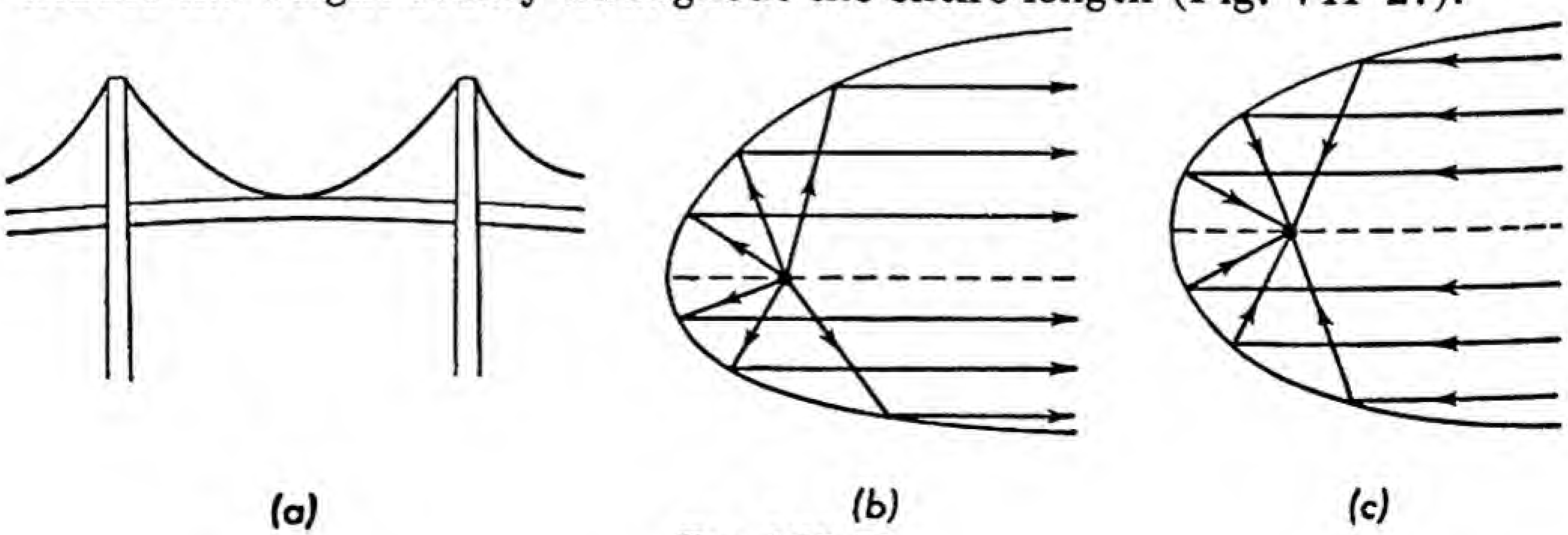


Fig. VII–27

Suppose that the parabola is revolved around its principal axis to form a three-dimensional surface. This is called a *paraboloid* whose equation we shall develop later. If the inside surface of this paraboloid is a polished, reflecting surface, we have the basis for illustrating one of the most useful and fascinating properties of all curves. Any radiating source of energy, be it sound, light, or heat, placed at the focal point will radiate waves to the paraboloid which will reflect each and every wave parallel to its axis, Fig. VII–27(b). This is a property that is exclusive with this kind of surface. Thus, in automobile headlights, flashlights, sound stages, and similar apparatus, there are parabolic reflectors that reflect a concentrated beam of energy. The principle works in reverse. Any parallel beam of radiant energy, parallel to the axis of an intercepting parabolic reflector, will reflect the energy to the focal point; hence the use of these reflectors in telescopes and in radar, Fig. VII–27(c).

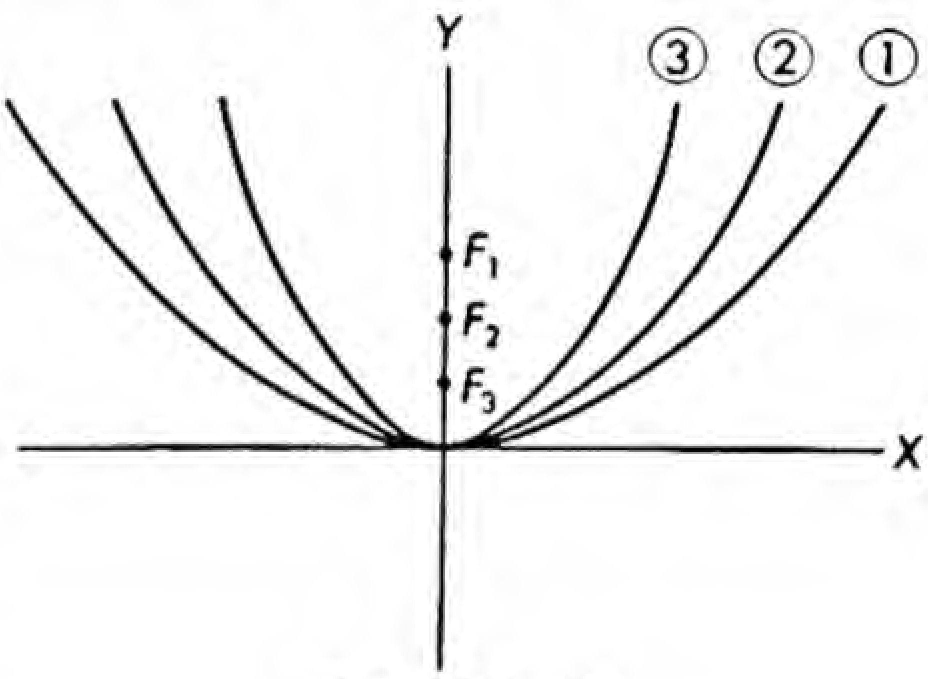


Fig. VII–28

The original equation of the parabola, $x^2 = 4py$, contains the letter p which locates the focal point at $(0, p)$. Specifically, if an equation were $x^2 = 8y$, then $4p = 8$, and so, $p = 2$. In this manner the focal point can be recognized at sight. A few sketches or an abstract analysis will show that the larger the focal distance, the *flatter* the curve (Fig. VII–28).

EXERCISES (VII–12)

1. For each of the following, give the coordinates of the vertex and the focal point. Also write the equation of the directrix; use these data for a reasonable sketch of the parabola.

a. $x^2 = 12y$	e. $x^2 = 2y$
b. $x^2 = -8y$	f. $(x - 2)^2 = 16(y - 3)$
c. $y^2 = 16x$	g. $(y + 3)^2 = 20(x - 4)$
d. $y^2 = 4x$	h. $(x + 3)^2 = y + 4$

2. The chord of a parabola that passes through the focal point and is perpendicular to the axis is called the *latus rectum*. For $x^2 = 10y$,
   a. Write the equation of the latus rectum.
   b. Find the length of the latus rectum.
3. By means of completing the trinomial square, we were able earlier to deter-

mine the center and the radius of a circle. The same technique can help us determine the salient facts for a parabola (see Fig. VII-29). The following is illustrative of this:

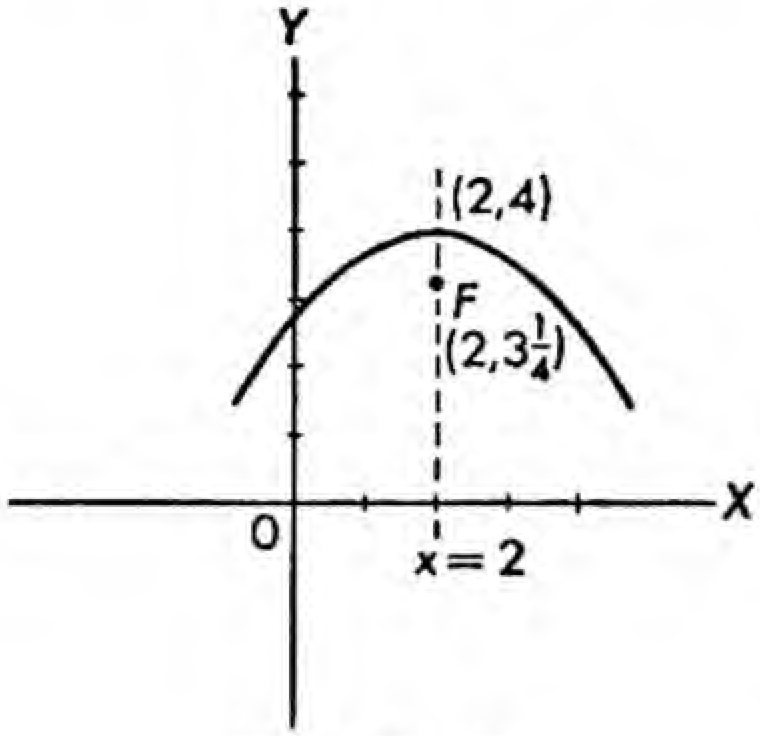


Fig. VII-29

$$x^2 - 4x + 3y - 8 = 0$$
$$(x-2)^2 - 4 + 3y - 8 = 0$$
$$(x-2)^2 = -3y + 12$$
$$(x-2)^2 = -3(y-4)$$

By comparison with $(x-h)^2 = 4p(y-k)$, we see that $4p = -3$, and so, $p = -\frac{3}{4}$. The vertex of the parabola is at (2, 4). The principal axis is $x = 2$, and the focal point is $\frac{3}{4}$ of a unit down from the vertex on this axis.

Analyze and sketch the following:

a. $x^2 + 2x + y - 3 = 0$
b. $y^2 - 3y - 4x + 1 = 0$
c. $2x^2 + 5x - 6y + 1 = 0$

4. Show that the length of a latus rectum is $4p$.

5. Some of the illustrations point to the fact that a general equation of the parabola may be written as

(1) $$x^2 + dx + ey + f = 0$$
or
(2) $$y^2 + dx + ey + f = 0$$

In either case, since three coefficients need to be determined for a particular parabola, this will require three equations. Consequently three known points on the parabola and its particular orientation, (1) or (2), will suffice to determine the parabola uniquely with respect to one or the other of the two general equations. Find the equation of the parabola passing through

a. $(-2, 3)$, $(1, 1)$, and $(4, 5)$ with its axis parallel to the Y-axis.
b. $(-3, 2)$, $(-1, 4)$, and $(3, 2)$ with its axis parallel to the X-axis.

6. Derive the equation of the locus of a point which moves so that its distance from $(-3, 2)$ equals its distance from $x - 4 = 0$.

7. A cable hangs in the form of a parabolic arch suspended from two vertical towers, each 80 feet high. The lowest point of the arch is 10 feet above the ground. The distance between the towers is 600 feet. Find an equation of the parabola, using the lowest point as the origin.

8. How high is a parabolic arch of 30-foot span which is 20 feet high at a distance of 6 feet from the center of the span?

9. The vertex of a parabola is at (2, 3), with its axis parallel to the Y-axis. The parabola also passes through (4, 5). Find its equation. Do the given data present a contradiction to that stated in exercise 5? Explain.

10. Prove that the ends of the latus rectum of a parabola and the point of intersection of the parabola's axis and the directrix determine a right triangle.

11. The trajectory (path) of water that spouts from a horizontal pipe will be in the form of a parabolic curve. Assume that in one case, the water arches out 10

feet at a point 10 feet below the spout. How far out will the water strike the ground if the pipe opening is 40 feet above the ground?

12. The trajectory of a shell is given by

$$y = -\frac{g \sec^2 \alpha}{2v^2}x^2 + x \tan \alpha$$

where α is the angle of inclination of the gun, v is the initial speed of the shell, and g is the acceleration due to gravity, which is 32 feet per square second, approximately.

a. If α is 45° and v is 1000 feet per second, sketch the resulting curve.
b. Find the x-intercepts. This gives you the *range* of the shell.
c. With the aid of the result in (b), find the maximum height reached by the shell in its trajectory.

13. THE ELLIPSE

We could consider another locus condition describing a set of points for each of which the sum of its distances from two fixed points is a constant. We take the fixed points at $(-c, 0)$ and $(c,)$ 0, equally distant from the origin (Fig. VII–30). The constant distance we take as $2a$, where $a > c$. If $P(x, y)$ is our variable point for the set, we have, by the distance formula and the locus condition,

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

We are faced with the need to develop a technique of solving radical equations containing two square root quantities. Since a radical represents a fractional power, we can simplify a radical term by raising it to an appropriate integral power. Thus a square root squared will yield its radicand to the first power. A cube root term, if cubed, will yield the radicand to the first power too, and so forth.

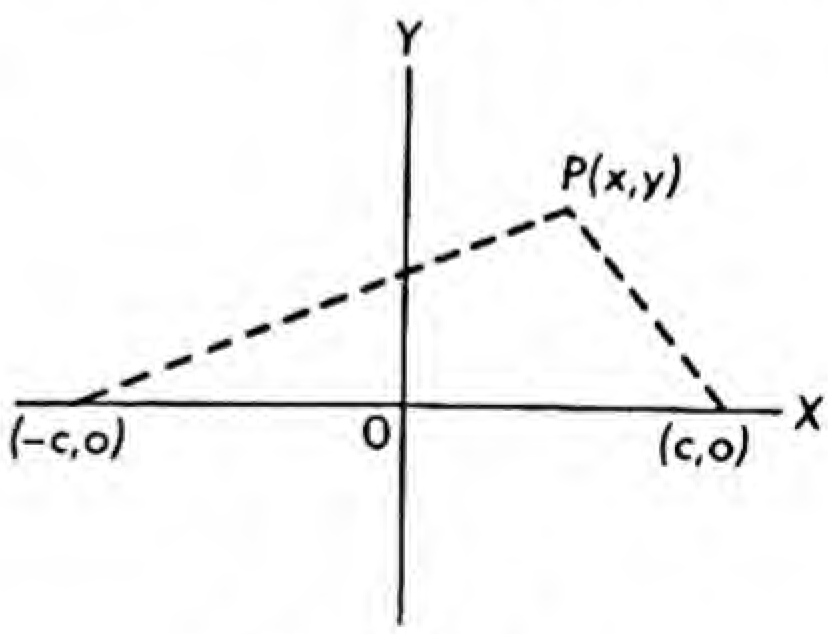


Fig. VII–30

$$\sqrt{M} = M^{1/2}; \qquad (M^{1/2})^2 = M$$

$$\sqrt{Q} = Q^{1/3}; \qquad (Q^{1/3})^3 = Q$$

In certain cases the equations have only the appearance of being irrational equations. The extraction of roots or the simplification of the radicals may remove the unknown from the radicands. In such cases it is not necessary to consider the present discussion about raising to a power.

It is necessary to recall that the raising of a polynomial to a power is not a distributive operation. For example, the squaring of a binomial sum does not consist of the sum of the squares of the terms. Instead we have $(a + b)^2 = a^2 + 2ab + b^2$. Of course, if the binomial were raised to the

nth power, we know by the binomial theorem that the result will consist of $(n + 1)$ terms for positive integral values of n.

This side reference to distributivity is relevant to the radical equation above. It is a fair presumption that the locus equation must be subjected to the operation of squaring in order to get an equation of a simpler sort. However, to square both members as it stands would be to obtain in the left member of the radical equation a middle term that would be a very complicated radical. We symbolize this condition as follows:

$$\left(\sqrt{M} + \sqrt{Q}\right)^2 = M + 2\sqrt{MQ} + Q$$

Now, we do know that a radical term raised to the appropriate power yields just the radicand. This suggests the plausible approach of having at least one member of the equation consisting of no more than a single radical term. This member, when raised to the appropriate power, will yield the radicand. If the other member also contains a radical term, it, too, can in time be isolated, and the equation again raised to a power. In brief, we isolate one radical term at a time and raise both members to the needed power, continuing this process as long as necessary. This is illustrated in the following equation solution:

$$\sqrt{x+5} - \sqrt{2x-4} = 1$$

$$\sqrt{x+5} = 1 + \sqrt{2x-4}$$

$$\left(\sqrt{x+5}\right)^2 = \left(1 + \sqrt{2x-4}\right)^2$$

$$x + 5 = 1 + 2\sqrt{2x-4} + 2x - 4$$

$$-2\sqrt{2x-4} = x - 8$$

$$\left(-2\sqrt{2x-4}\right)^2 = (x-8)^2$$

$$4(2x-4) = x^2 - 16x + 64$$

$$x^2 - 24x + 80 = 0$$

$$(x-20)(x-4) = 0$$

$$x = 20 \quad \text{or} \quad x = 4 \qquad \textit{Ans. } x = 4$$

In this case $x = 20$ does not check with the original equation; hence it is called an *extraneous* root. This situation occurs frequently when we raise members of an equation to a power. For example, from $x = m$ we can get as a result of squaring not only that $x = m$ but also that $x = -m$, contrary to the initial condition. Checking (that is, completing the solution) is particularly necessary here.

Let us return to the original equation in connection with the locus of a point, the sum of whose distances from two fixed points was $2a$. We had

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

By pursuing the method just developed, we get:

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2}=4a^2-4xc$$

$$a\sqrt{(x-c)^2+y^2}=a^2-xc$$

$$a^2[(x-c)^2+y^2]=a^4-2a^2xc+x^2c^2$$

$$a^2x^2-2a^2xc+a^2c^2+a^2y^2=a^4-2a^2xc+x^2c^2$$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

Here, the binomial factor $(a^2 - c^2)$ appears twice. This provides us with an opportunity to substitute a single quantity for the factor. However, an analysis of the geometry of the situation indicates a more purposeful situation.

We expect that some point P on the Y-axis will satisfy the locus condition of the present problem (Fig. VII–31). Because of the congruent triangles, $PF = PF'$, and each is equal to a, since the sum of the distances is $2a$. If we set $OP = b$, then by either right triangle we have

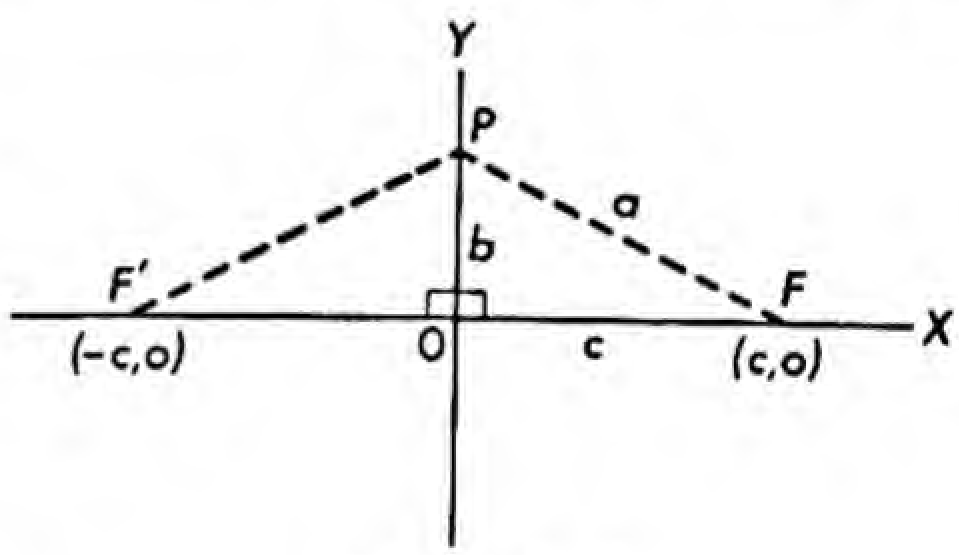


Fig. VII–31

$$b^2=a^2-c^2$$

Substituting this in the last locus equation, we have

$$b^2x^2+a^2y^2=a^2b^2$$

Once again we may do the unexpected as we did with a straight line equation; we divide both members by the constant term:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

By setting $y = 0$, we see that the graph (Fig. VII–32) intercepts the X-axis at $\pm a$, since x^2 becomes equal to a^2, and by setting $x = 0$, we get the y-intercepts $\pm b$, since $y^2 = b^2$.

The graph of the equation yields a well-known curve, the **ellipse.** The ellipse is said to have two *axes*. The one along the X-axis, of length $2a$, is called the *principal axis*. The one along the Y-axis is the *minor axis*, and its length is $2b$. By the geometry of the right triangle, a is larger than b, and the principal axis is the larger of the two. The halves of these axes are referred to as the *semi-axes*.

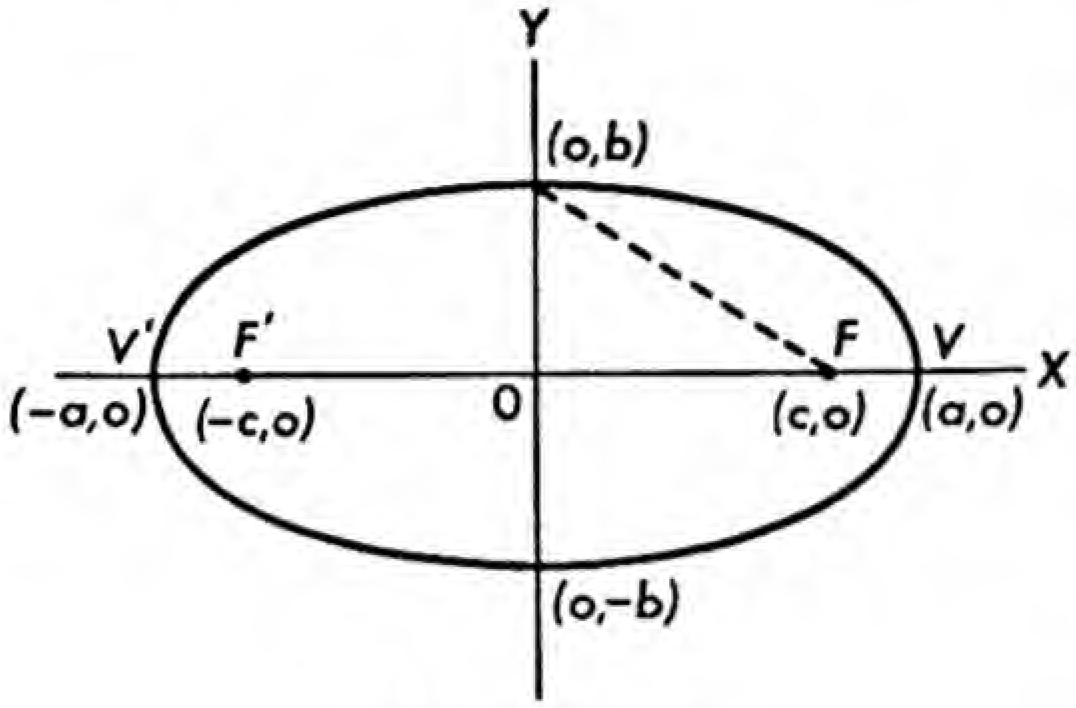


Fig. VII–32

The points F and F' along the major axes are called the *focal points* because of a reflection property. If the ellipse is rotated around the major axis, a three-dimensional solid is formed which is called an *ellipsoid* (Fig. VII–33). We suppose, as with the paraboloid, that the surface is highly polished. Any source of radiant energy placed at one focal point will have its rays reflected to the other point.

A building roof with an elliptical dome will have this reflecting property, which is responsible for the name *whispering gallery*. The statuary hall in Washington, D.C., the Tabernacle in Salt Lake City, and the Taj Mahal in India are instances of this.

A ray of energy emanating at F and striking the ellipsoid will be reflected to F'. The key to this phenomenon is the fact that for the ellipsoid, all angles such as $\angle 1$ and $\angle 2$, made with the tangent at P, must be equal. Should the center of the ellipse be at (h, k) instead of at the origin, we can accept the fact (in the light of our earlier discussion regarding translations) that the equation will be

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

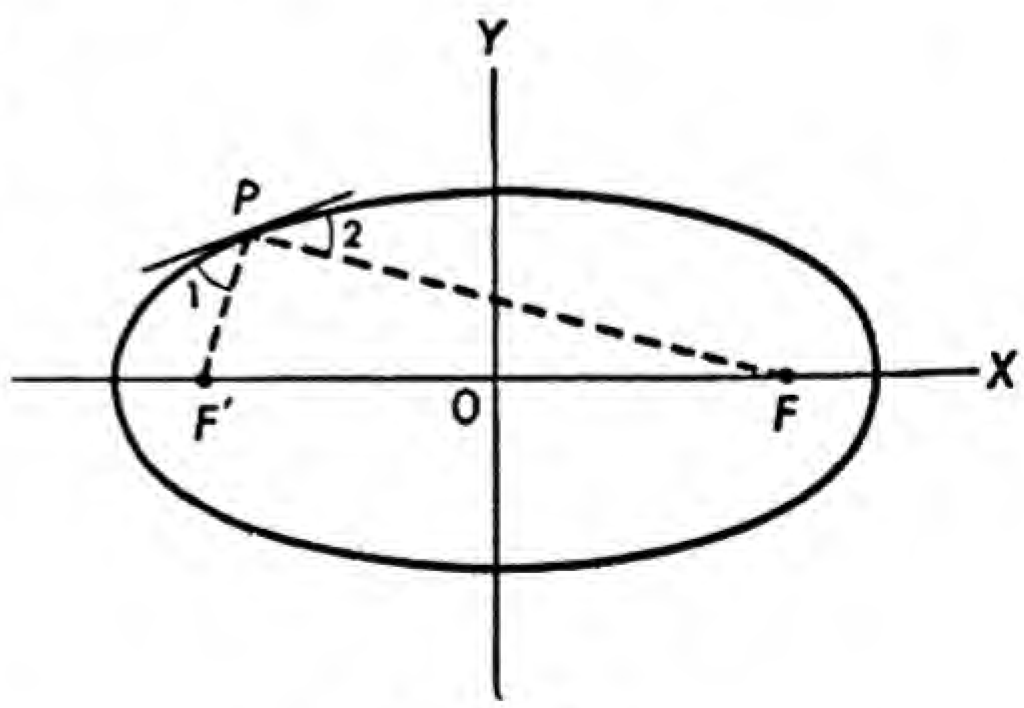


Fig. VII–33

If the focal points are taken on the Y-axis, it will be equivalent, as we saw earlier, to an interchange of the two axes (see Fig. VII–34). The new equations can then be obtained from the previous ones by an inter-

change of x and y or the x and y binomials. This leads to the equations

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

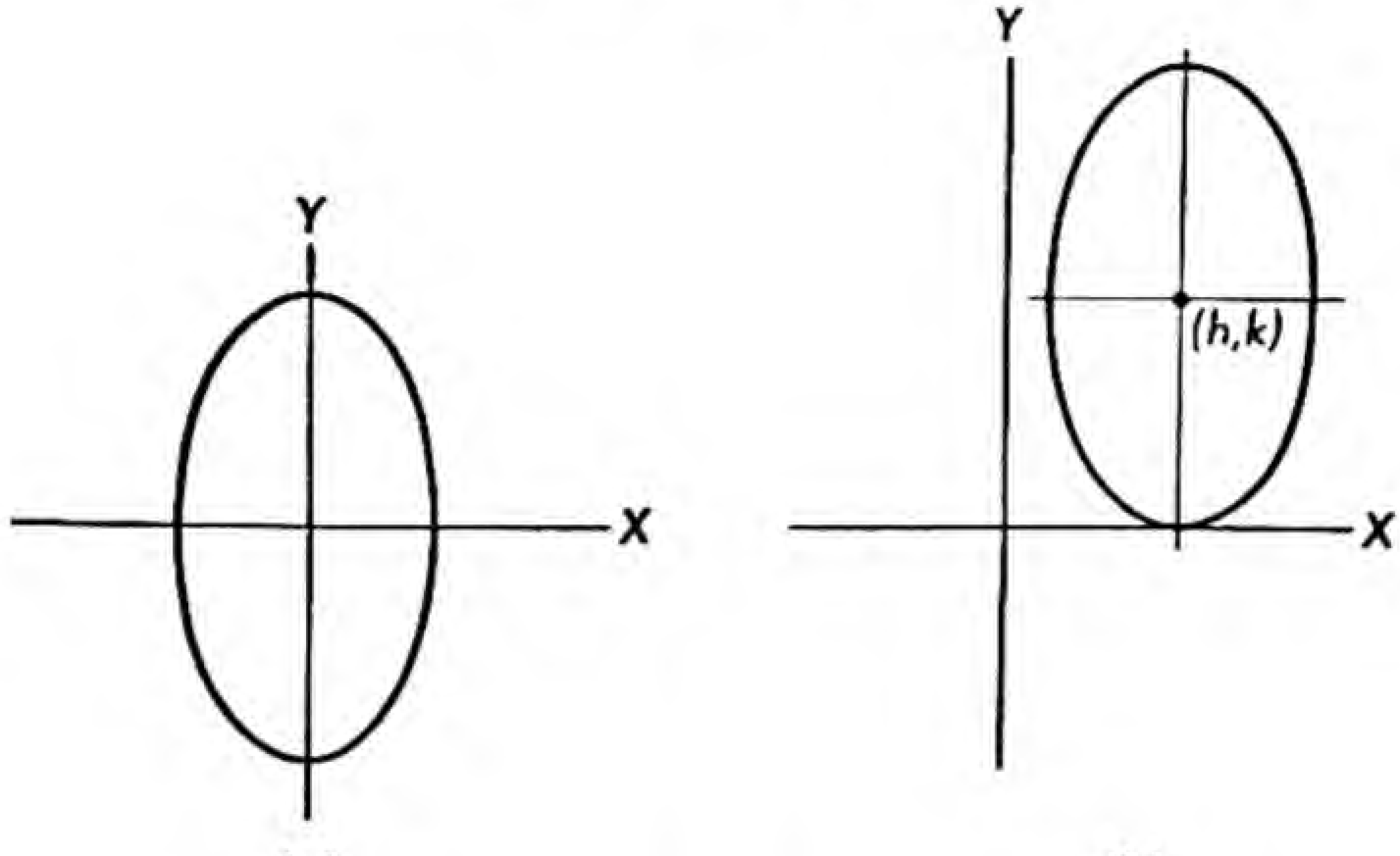


Fig. VII-34

The focal points may be brought closer to each other by the simple device of starting with smaller values for c. Since $a^2 = b^2 + c^2$ as c approaches the limiting value of 0, a will approach the value of b. The result will be that the ellipse will approach a circle with the radius equal to b.

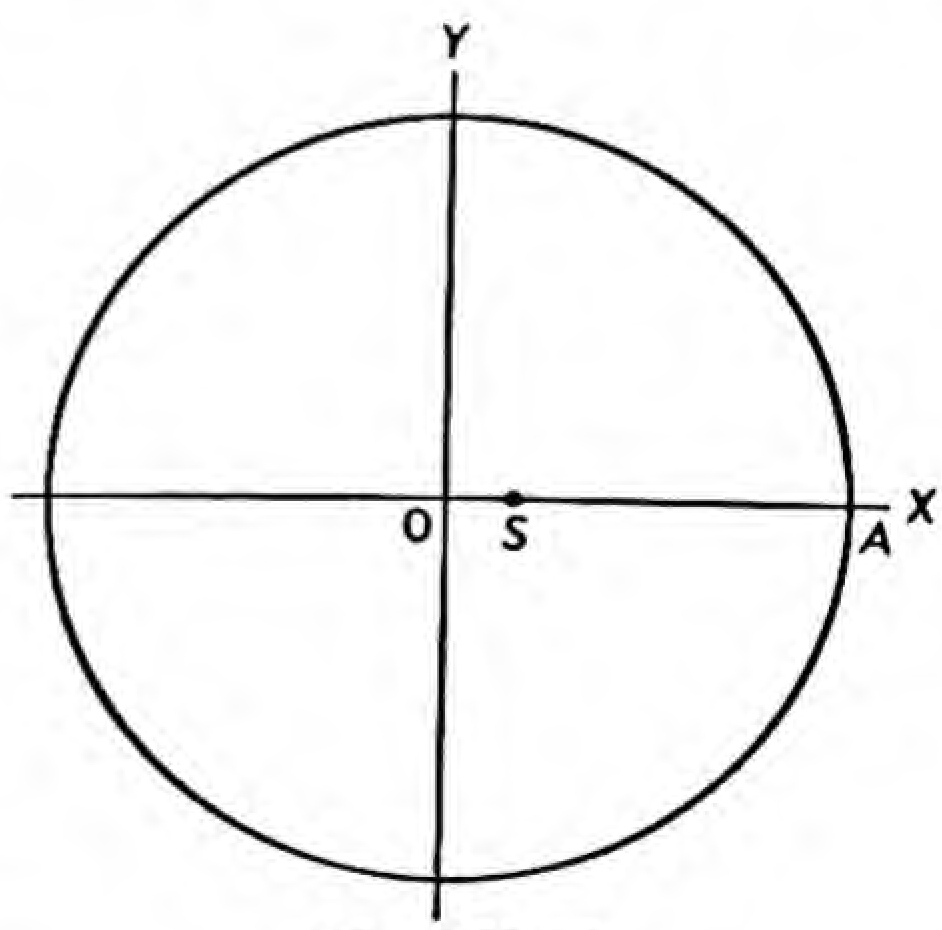


Fig. VII-35

Perhaps the most interesting ellipse is that of the orbit of the earth around the sun or of the orbits of the artificial satellites around the earth. In the case of the earth, the sun is at one of the focal points S, and the ratio of SO to OA is about 1:60 (Fig. VII-35). This means that the earth's orbit is so nearly circular that the distance between S and O if drawn to scale on this page, or even on a much larger page, would be invisible.

If we refer to our earlier denotations, $SO = c$ and $OA = a$, the ratio $SO:OA = c:a$. This ratio is called the *eccentricity* of the ellipse.

EXERCISES (VII-13)

1. Exponentiation is distributive with respect to certain operations. Illustrate.
2. Solve each of the following:

a. $\sqrt{3x} = 5$

b. $\sqrt[3]{4m} = 3$

c. $\sqrt{x+1} - 2 = 1$

d. $\sqrt{3x+7} + 3 = 8$

e. $\sqrt[3]{\frac{2}{a}} = 4$

f. $\sqrt{5-2y} = 5 + y$

g. $\sqrt{x+9} + 2x = 18$

h. $\sqrt{k-2} + \sqrt{2k+3} = 4$

i. $5\sqrt{4t+2} - 3\sqrt{3+2t} = 4$

j. $2\sqrt{x-3} + 3\sqrt{2x+1} = 11$

3. Develop the equation of the ellipse with the focal points on the Y-axis and the center at the origin.

4. Sketch some members of the family $(x^2/16) + (y^2/k^2) = 1$, where k^2 is less than 16.

5. Sketch each of the following:

a. $\frac{x^2}{100} + \frac{y^2}{36} = 1$

b. $x^2 + 4y^2 = 100$

c. $x^2 + 9y^2 = 36$

d. $4x^2 + 9y^2 = 36$

e. $9x^2 + 49y^2 = 441$

f. $x^2 + 2y^2 = 72$

g. $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$

h. $\frac{(x+3)^2}{4} + \frac{(y-4)^2}{16} = 1$

6. Write the equation of an ellipse with the center at the origin, given:
   a. One vertex at (10, 0) and a focus at (8, 0).
   b. Minor axis = 6 along the X-axis and major axis = 14.
   c. Minor axis = 10 along the X-axis and major axis = 20.
   d. Minor axis = 10, focus at (9, 0).

7. As with the parabola, the latus rectum is a chord of the ellipse through one of the focal points perpendicular to the (major) axis. Find its length for the ellipse.

8. The earth's orbit is an ellipse with the sun at one focal point. The semimajor axis is approximately 93,000,000 miles. The eccentricity = 0.0167. Find the shortest distance from the earth to the sun.

9. A point moves so that the sum of its distances from (2, 4) and (2, −4) is 12. Derive the equation of the locus.

10. A point moves so that the sum of its distances from (−1, 3) and (6, 3) is 10. Derive the equation of its locus.

11. A point moves so that its distance from (0, 2) is one-half its distance from $y = -3$. Derive the equation of the locus.

12. A point moves so that the product of its slopes to (−2, −2) and (5, 6) is always −3. Show that the locus is an ellipse.

13. Find the coordinates of the center and the lengths of the major and minor axes:

a. $4x^2 + y^2 - 16x - 6y - 43 = 0$

b. $9x^2 + 16y^2 - 36x + 96y = -36$

14. THE HYPERBOLA

A slight variation in the locus condition of the ellipse brings us to a new curve, the **hyperbola.** Here we seek the locus of a point, the difference of its distances from two fixed points being a constant. As before, we take the two points at $(-c, 0)$ and $(c, 0)$ and the fixed distance as $2a$, where $a < c$. We have

$$\left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a$$

Considering one of the possibilities, we get, further,

$$\sqrt{(x+c)^2+y^2} = 2a + \sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2 = 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

$$4cx - 4a^2 = 4a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = a\sqrt{(x-c)^2+y^2}$$

$$c^2x^2 - 2a^2cx + a^4 = a^2\{(x-c)^2+y^2\}$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2)$$

In the light of our earlier discussion in connection with the ellipse, it seems advisable again to make a substitution for the binomial coefficient. We take $c^2 - a^2 = b^2$, although at the moment there is no clear graphic image of the value of b. We get

$$x^2b^2 - a^2y^2 = a^2b^2$$

and so

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

By substituting $x = 0$, the values of y are imaginary, and so we have no y-intercepts in the real plane. On the other hand, by substituting $y = 0$, we get the x-intercepts $\pm a$ (Fig. VII–36).

Now $OV = a$, and if we erect a perpendicular to the X-axis at V and draw $OW = c$ (Fig. VII–37), then $WV = b$ by the Pythagorean formula. This provides a geometric image for the line that was previously introduced algebraically. Actually the line OW is one of the oblique asymptotes of the hyperbola.

We see that the hyperbola is a two-branched, two-valued curve. The fixed points are still called the *foci*, and V and V' are the *vertices*. The origin is

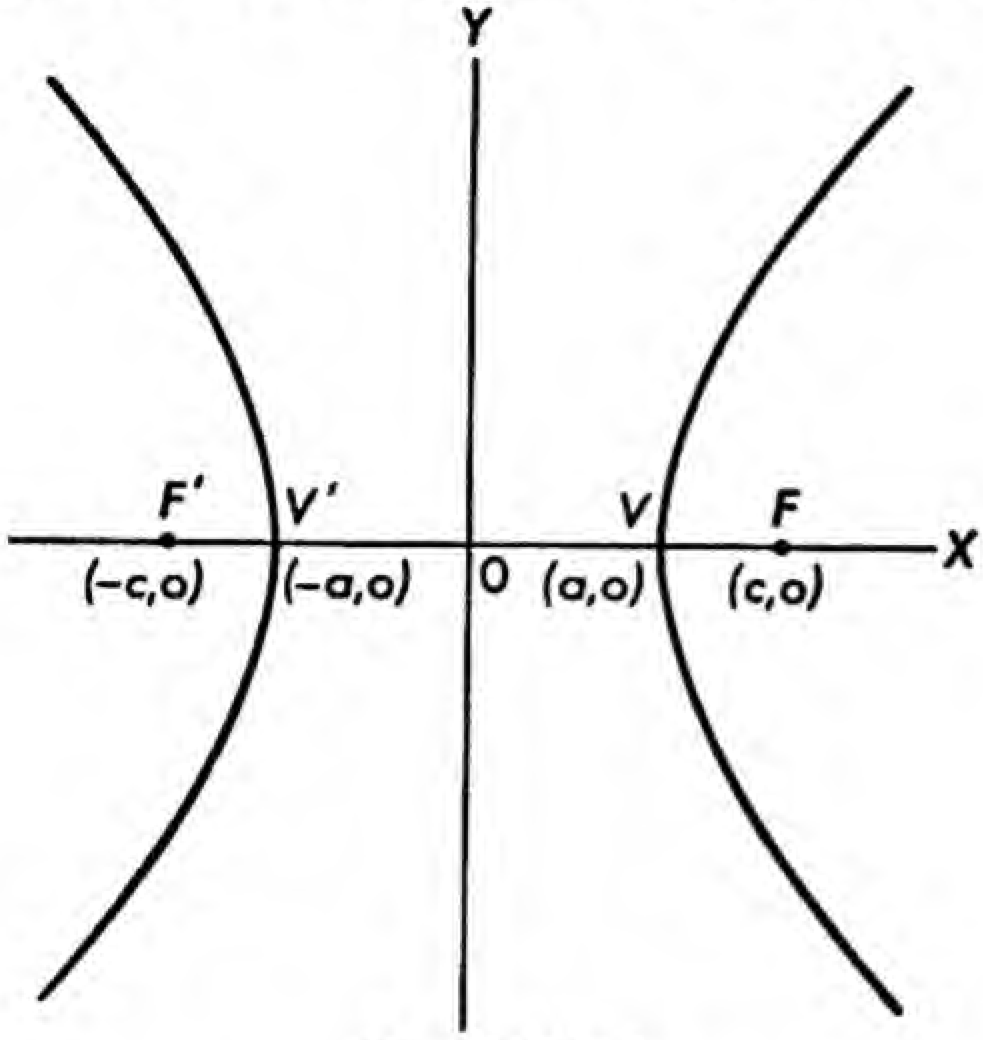


Fig. VII–36

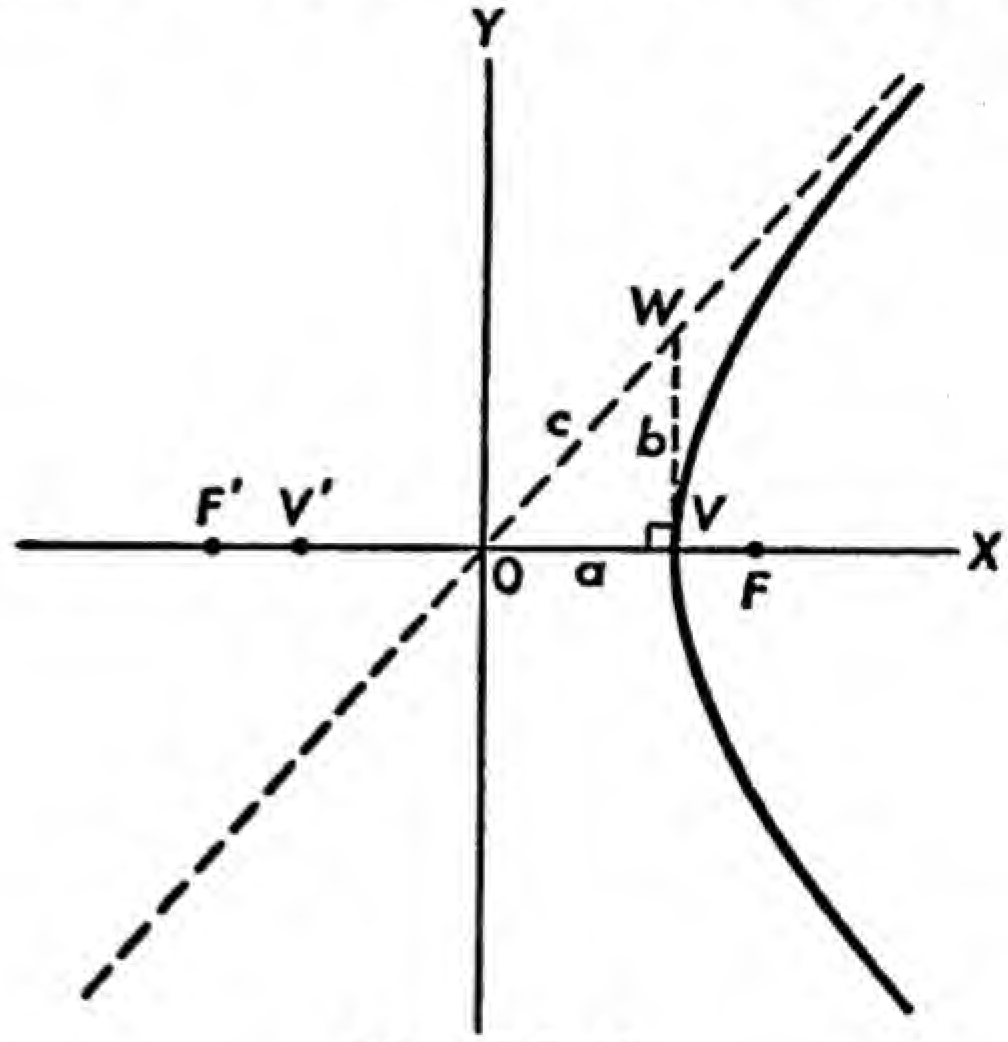


Fig. VII–37

the *center* of this hyperbola. Once again we anticipate the fact that if the center were at (h, k) the equation would be

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

EXERCISES (VII–14)

1. Develop the equation of the hyperbola, starting with

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$$

2. Sketch each of the following:

a. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ d. $x^2 - 25y^2 = 25$

b. $x^2 - y^2 = 1$ e. $y^2 - x^2 = 25$

c. $16x^2 - 9y^2 = 144$ f. $4y^2 - x^2 = 16$

3. Develop the equation of the locus of a point, the difference of whose distances from $(4, 0)$ and $(-4, 0)$ is 6.

4. In connection with Fig. VII–37, it was mentioned that OW is an asymptote. The slope of this line is b/a, the y-intercept is 0, and so the equation of this line is $y = (b/a)x$.

a. Show that OW does not intersect $(x^2/a^2) - (y^2/b^2) = 1$.

b. Take $y = (eb/a)x$, where $0 < e < 1$. Show that the abscissa of the intersection of this line with the hyperbola is given by $x = a/\sqrt{1 - e^2}$ in the first quadrant.

c. Suppose that the value of e in (b) approaches 1 as a limit. How does this demonstrate that OW is indeed an aymptote?

d. Indicate in a sketch that $y = -(b/a)x$ is also an asymptote.

5. Find the equations of the asymptotes for the equations in exercise 2.
6. Show that the length of the latus rectum is $2b^2/|a|$.
7. Find the equation of the hyperbola whose axes are the X- and Y-axes and whose center is at the origin and which passes through the points (3, 1) and (5, 3).

15. ROTATED FRAME OF REFERENCE

We have had occasion to refer our graphs to axes that were translated. This was a *parallel displacement* of the axes which, at times, offered us advantageous views of relations. There are times too when an **angular rotation** of the axes provides a similar advantage. By way of illustration, let us rotate a set of axes 45°.

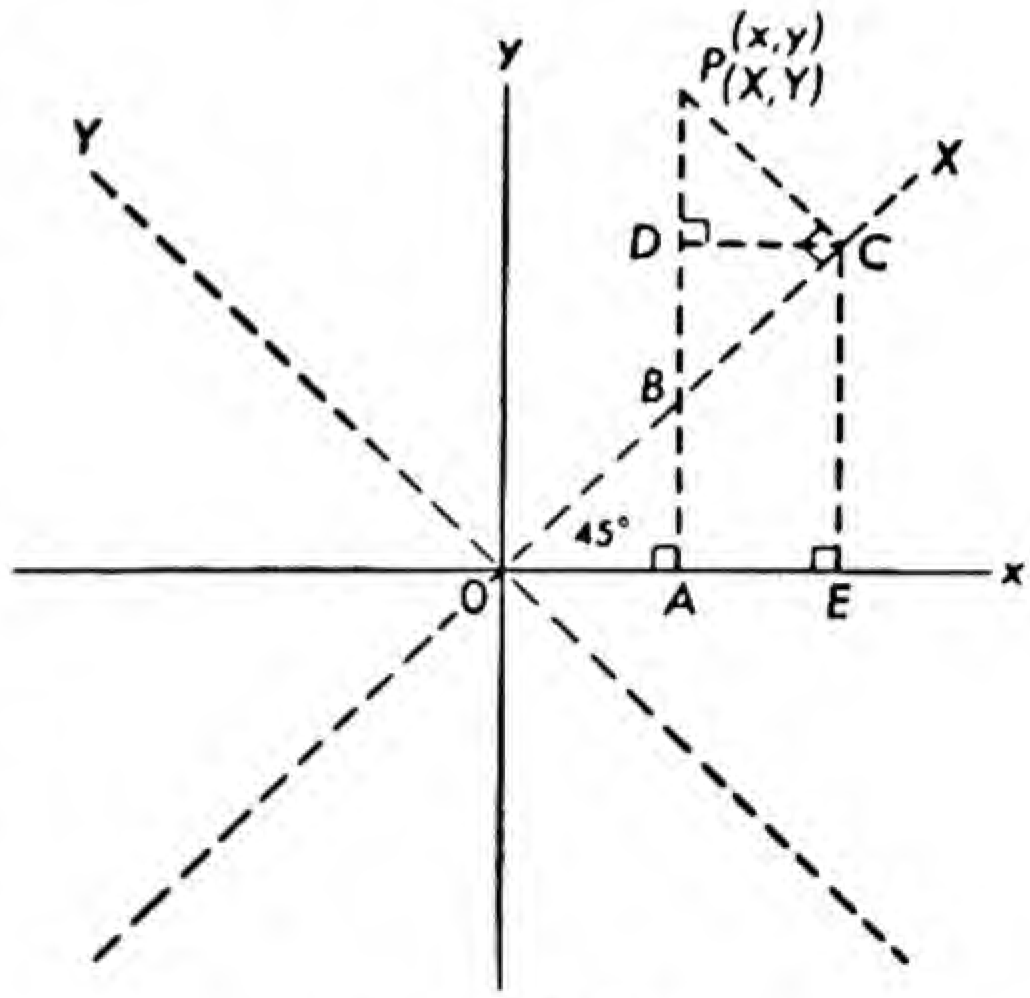


Fig. VII–38

For any point P (Fig. VII–38), we have a choice of two frames of reference, and therefore two sets of coordinates (x, y) and (X, Y). We can rightly anticipate a relationship between these coordinates.

$$OA = x,\ PA = y; \qquad Oc = X,\ PC = Y$$

Since OCE and PDC are isosceles right triangles,

$$DC = \frac{Y}{\sqrt{2}},\ PD = \frac{Y}{\sqrt{2}},\ CE = \frac{X}{\sqrt{2}},\ \text{and } OE = \frac{X}{\sqrt{2}}$$

Now $\qquad x = OA = OE - DC$

So

(1) $\qquad x = \dfrac{X}{\sqrt{2}} - \dfrac{Y}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}(X - Y)$

Also, $\qquad PA = CE + PD$

(2) $\qquad y = \dfrac{X}{\sqrt{2}} + \dfrac{Y}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}(X + Y)$

Equations (1) and (2) are the *equations of rotation* of the axes through an angle of 45°. The origin is the same as is the unit of measure in both frames of reference. The very same diagram and the substitution of θ for 45° would enable one to derive the more general equations:

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

By way of illustration (Fig. VII–39), let us take a simple hyperbola, $y^2 - x^2 = 1$, and refer it to the axes rotated 45°. Of course we make the substitutions, equations (1) and (2).

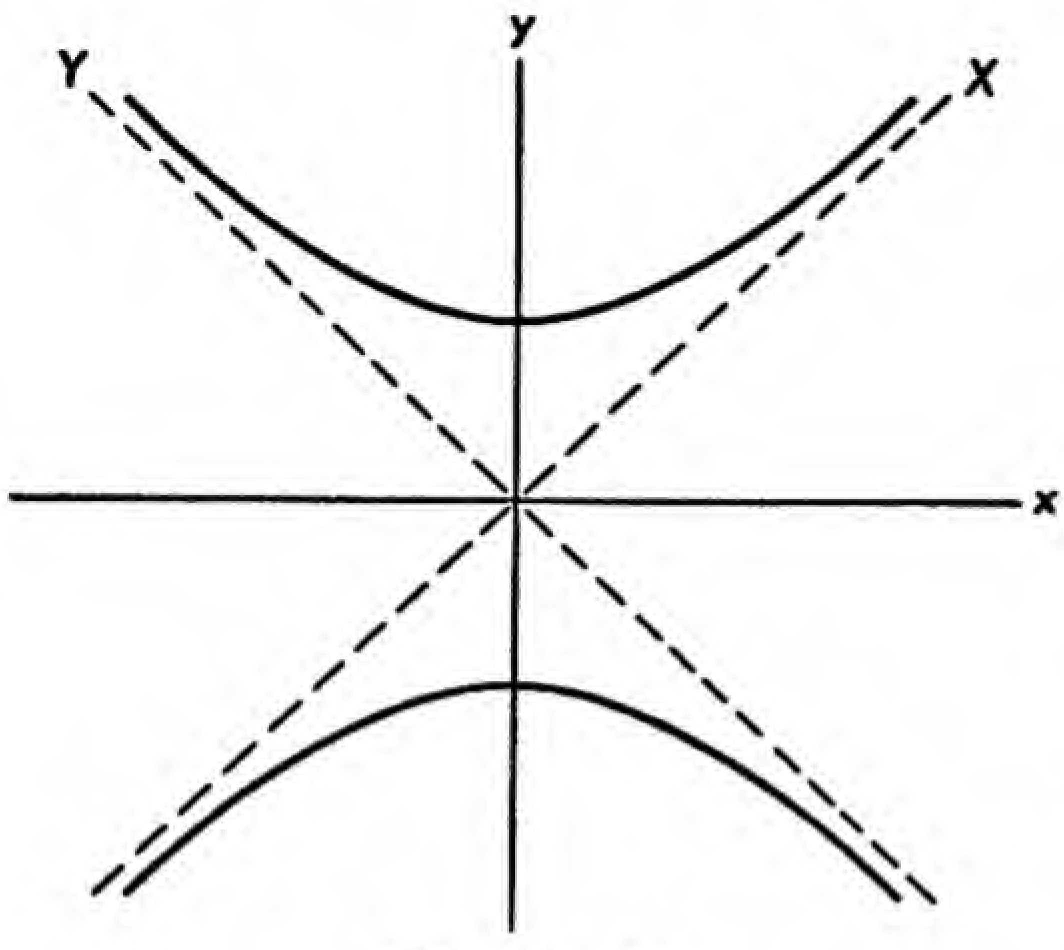


Fig. VII–39

$$\tfrac{1}{2}(X + Y)^2 - \tfrac{1}{2}(X - Y)^2 = 1$$
$$2XY = 1$$
$$XY = \tfrac{1}{2}$$

The occurrence of the XY term is an interesting turn of events. To begin with, because the term is a product of two variables, each of the first degree, the product term is considered to be a second-degree term. The new equation is a second-degree equation, as is the original one. The degree of an equation is not changed by a rotation of axes.

Should any of the other second-degree curves be rotated through any number of degrees, other than a multiple of 90°, xy terms will crop up. The only exception to this is in the case of the circle whose orientation is indistinguishable with respect to rotated axes.

EXERCISES (VII–15)

1. Refer $9x^2 - y^2 = 1$ to axes rotated 45° and write the new equations.
2. Do the same as in exercise 1 with $5x^2 + 6xy + 5y^2 = 8$.
3. Find the common solutions for $x^2 + 3xy = 10$ and $y = 2x - 3$.
4. With the introduction of the XY term in our discussions, we have considered every variety of the second-degree equation. Now, some graphs of second-degree equations may seem to be disappointing, for they resemble first-degree equations. The graphs may consist of straight lines. Without meaning to be snobbish, we call such cases *degenerate second-degree curves.*

These cases arise, in fact, when the second-degree equation is in reality the product of two first-degree factors. Consider $(x - 3y + 1)(x + 2y + 3) = 0$. Every member of the set $\{x, y\}$ defined by this equation must make at least one of the factors 0. The graph then will consist of two lines, each one representing one of the factors set equal to 0. Thus the equation with the parentheses removed, $x^2 - 6y^2 - xy + 4x - 7y + 3 = 0$, only appears to be of the nature of a curve.

Graph each of the following and rewrite the equations without parentheses:

a. $(x - y)(x + y) = 0$　　　b. $(x - y + 1)(x + y + 1) = 0$

5. If the equations of rotation are applied to the general second-degree equation,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

the result may be symbolized by

$$AX^2 + BXY + CY^2 + DX + EY + F = 0$$

wherein, for example, A comes out to be $a \cos^2 \theta + b \sin \theta \cos \theta + c \sin^2 \theta$. This suggests that the new coefficients are likely to be different. The curves will be identical but each will be oriented with respect to its own axes.

However, one quantity will remain the same, no matter the nature of the rotation. Such a quantity is called an **invariant** quantity and is in this case,

$$b^2 - 4ac = B^2 - 4AC$$

For example, we rotated $x^2 - y^2 = 1$ and got $2XY - 1 = 0$. Here we have: $a = 1, b = 0, c = -1$, and $A = 0, B = 2, C = 0$. Substituting these values in the last equation, we get the identity $4 = 4$.

The invariant in this case is called the quadratic *discriminant* or *characteristic*.

We take this opportunity to point out some general facts through some of the special cases we have studied.

a. For
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
we have
$$B^2 - 4AC = 0 - 4\left(\frac{1}{a^2}\right)\left(\frac{1}{b^2}\right) = -\frac{4}{a^2b^2}$$

This indicates that the discriminant is a negative quantity. So, we assume conversely that if $B^2 - 4AC < 0$, the graph of the equation is an ellipse or a degenerate, or a special case of the ellipse, which would include the circle, a point, or a nonexistent graph. Example:

$$2x^2 - 3xy + 2y^2 - x + 7y - 5 = 0; \qquad B^2 - 4AC = -7$$

so the graph is an ellipse or a degenerate case of the ellipse.

b. Now
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
leads to
$$B^2 - 4AC = \frac{4}{a^2b^2}$$

which is always a positive quantity for real values of the coefficients. We assume then, that if $B^2 - 4AC > 0$, the graph is a hyperbola or a degenerate case of the hyperbola, which would be two intersecting lines.

c. In $y^2 = 2px$, $B^2 - 4AC = 0$. We assume that when the discriminant is 0, the graph will be a parabola or the degenerate cases of the straight line or two parallel lines.

6. Classify each of the following as to nature of graph:

a. $3x^2 - 2xy + 5y - 6 = 0$
b. $5xy + 2y^2 - 3x + 5y + 2 = 0$
c. $x^2 + xy + y^2 + x - y - 8 = 0$
d. $2x^2 - 3xy - 2y^2 - x + 2y - 5 = 0$
e. $x^2 + 2xy + y^2 - 3x + 5y = 8$

7. Find the common solutions, where they exist, and sketch the graphs:

a. $x^2 + y^2 = 25$
   $x^2 - y^2 = 7$

b. $2x^2 + 3y^2 = 93$
   $x^2 = y + 4$

c. $x^2 + y^2 = 25$
   $xy = 12$

16. CONIC SECTIONS

Ellipses, including the circle, the parabola, and the hyperbola, are called *conic sections*. This is due to the fact that they, or degenerates of them, are all the only possible sections of a circular conical surface. Such a surface (Fig. VII–40) is obtained by holding a point V on a line QVR in a fixed position directly over or under the center of a circle C and by permitting some other point on the line to trace the circle. The line is called an *element* of the surface.

Consider any point T not on the conical surface. A plane through T and parallel to an element will intersect the surface in a section that will be a parabola, P. Planes through T above this one will give intersections that are circles and ellipses. Planes through T below the first one will give hyperbolas as the intersections.

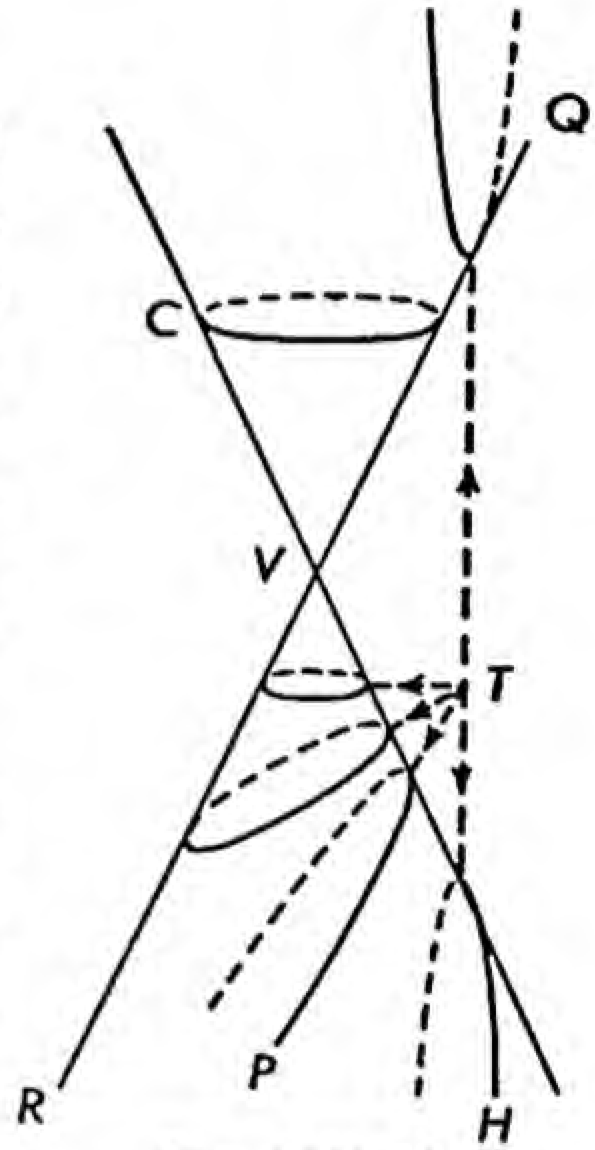


Fig. VII–40

VII–16 REVIEW

1. Sketch the graph of $y = x + \sqrt{x}$ by the method of addition of ordinates. Determine the domain and range.
2. Describe the graph of $4x^2 + 12x - 12y + 3 = 0$.
3. Prove that the abscissa of the vertex of the parabola

   a. $y = x^2 + kx + m$ is $-(k/2)$.
   b. $y = ax^2 + bx + c$ is $-(b/2a)$.

4. Find the points of intersection of the curves whose equations are $x^2 = 6y$ and $x^2 + y^2 = 16$.
5. Find the equation of a set of points, each of which is the same distance from the X-axis as it is from the point $(1, 2)$.
6. An object is fired at an angle of 30° with the horizontal and with an initial speed of 400 feet per second Find the range and the maximum height of the trajectory.
7. If only the abscissa of each point of the set of real numbers that is defined by $x^2 + y^2 = 25$ is doubled, the set of the resulting ordered pairs of numbers will satisfy the equation of an ellipse. Prove.

8. A line segment $a + b$ units in length has one end on the Y-axis and the other on the X-axis. Show that the locus of a point a units from the end of the segment which rests on the X-axis is an ellipse as the segment assumes all possible positions under the conditions given.

9. Sketch $(x - 1)^2 + 4(y + 2)^2 = 16$.

10. Find the equation of an ellipse with the center at the origin and passing through the points $(7, 1)$ and $(2, -5)$.

11. Find the coordinates of the foci, the lengths of the axes, and the coordinates of the vertices:

a. $9x^2 + 25y^2 = 225$

b. $2x^2 + y^2 = 4$

c. $x^2 + 4y^2 - 6x + 8y + 9 = 0$

d. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

12. Find algebraically the set of points in common to $x^2 + 4y^2 = 20$ and $x - 2y + 2 = 0$. Sketch the graphs.

13. Find the product (common elements) of the sets given by $x^2 + y^2 = 25$ and $x^2 - y^2 = 7$.

14. a. Find the center of the hyperbola $x^2 - y^2 - 2x + 4y - 12 = 0$ and translate the axes to this point, getting thereby an equation of the hyperbola without any first-degree terms.

b. Do the same for $x^2 - 4y^2 - 4x + 8y = 4$.

c. In both cases, show both frames of reference and the curve.

15. Graph the equation $xy = 16$.

16. A point moves so that its distance from $(0, 6)$ is always twice its distance from the X-axis. Find the equation.

17. Use the rotation formulas above to rotate the axes 45° for $xy = 4$. Graph the equation and show both sets of axes.

18. Rotate the axes an amount $\theta = \arctan \frac{3}{4}$ for each of the following, and in each case show the graph and the two sets of axes:

a. $4x + 3y = 2$

b. $xy = 1$

19. Rotate the axes 45° for each of the following and then translate the new axes so that the first-degree terms are eliminated:

a. $x^2 - xy + y^2 - 4x - 6y = 10$

b. $3x^2 + 2xy + 3y^2 - 8x + 8y + 20 = 0$

VIII — THREE-DIMENSIONAL GEOMETRY

1. INTRODUCTION

The recent references to the ellipsoid and paraboloid were intimations of the fact that one must begin to face the question or enigma of three dimensions. Of course, as in so many other instances, our ancestors recognized that their own existence was set in a three-dimensional manifold. (The limitations of this view began to become apparent only recently with the advent of the four-dimensional space-time view of the Theory of Relativity.) Concepts of space, surface, and volume were real recognizable entities, with only their quantitative relationships needing discovery and specification.

The occurrence of the ordered pairs $\{x, y\}$ quickened the introduction of rectangular axes and a two-dimensional continuum of points. One can anticipate a three-axis framework if **ordered triples** make an appearance. There is no dearth of practical illustrations wherein a concurrence of three numbers is practically necessary to describe an event. Day, time, and place; latitude, longitude, and altitude are suggestive triples. One can also think of abstract number situations, such as the set of numbers $\{x, y, z\}$ where $x + y + z$ is always 8 for any member of the set. Sample members of this set are (2, 5, 1) and (−3, 4, 7).

In the manner of previous extensions, this new outlook suggests the introduction of another axis perpendicular to the first two to provide for the **Z continuum** of points and numbers. Our drawing (Fig. VIII–1) will have to be in perspective, since the Z-axis, perpendicular to the X- and Y-axes in the plane of this paper, will be perpendicular to this paper. Thus

some visual distortion is necessary. There are arbitrary choices of labeling the axes, one of which is suggested here.

The solid line segments of the axes (Fig. VIII–1) are taken as the positive ends of the axes. The three axes divide the space into eight parts, with the first part separated by the three positive segments and called the *first octant*.

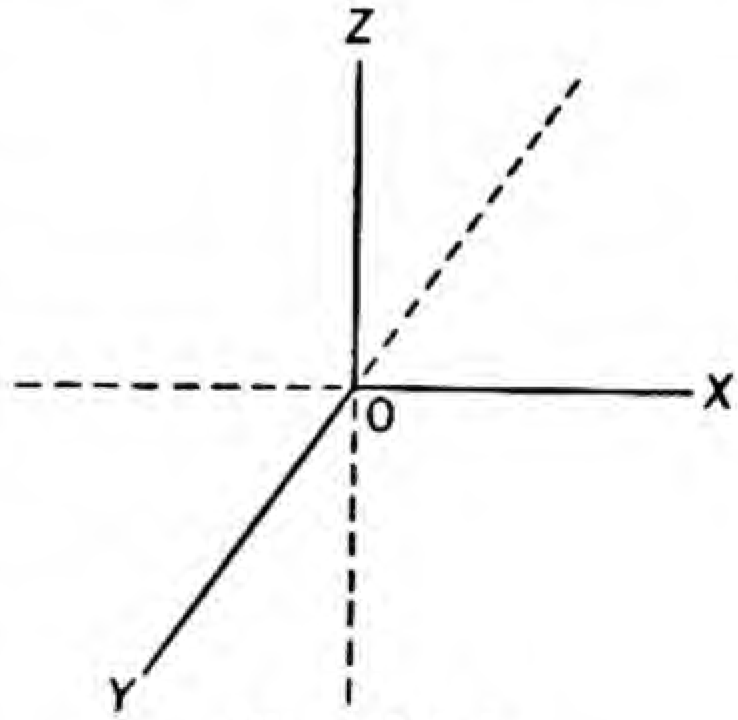


Fig. VIII–1

It is necessary to admit a postulate, actually implicit earlier, which is that *two intersecting lines determine a plane*. The word "plane", like the word "line," is undefined. Thus the three axes form with each other three planes, which we shall designate by the intersecting lines as the XY plane, the XZ plane, and the YZ plane. The designations XY *plane* and XZ *plane* indicate that they have the X-axis in common. This is the line of intersection of the two planes. The three planes have the point O in common, which will be the *origin* or (0, 0, 0) point.

According to the preceding postulate, two distinct planes cannot have more than one line in common, since two or more lines would entail a contradiction. This admits of the possibility that two planes may not have any line in common, that the planes do not intersect. Of course we shall say of such planes that they are *parallel*.

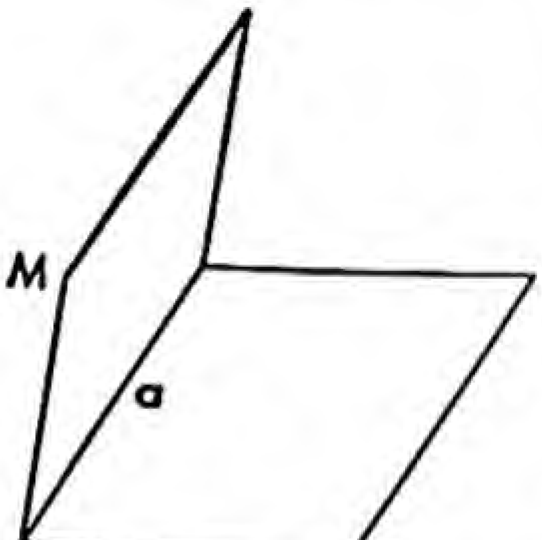


Fig. VIII–2

A plane, being infinite in extent, can be visualized by some finite section which is usually indicated by a parallelogram. Consider the nonparallel *half-planes* M and P (Fig. VIII–2), with their line of intersection a. The figure seems to suggest an *angular* relationship, which is actually called a **dihedral angle.** After the existence of a quantity is recognized, or invented, the next step is to consider a method of measurement.

The two half-planes, M and P, may be placed together, forming no opening between them. The planes are then described as being *coplanar.* Consistency with earlier systems of measurement requires that we assign 0 to this condition. On the other hand, the two half-planes may be so oriented as to lie on a single plane (Fig. VIII–3). This is reminiscent of the straight angle with half-lines. We accept the analogy and refer to the present case as a *straight dihedral angle.*

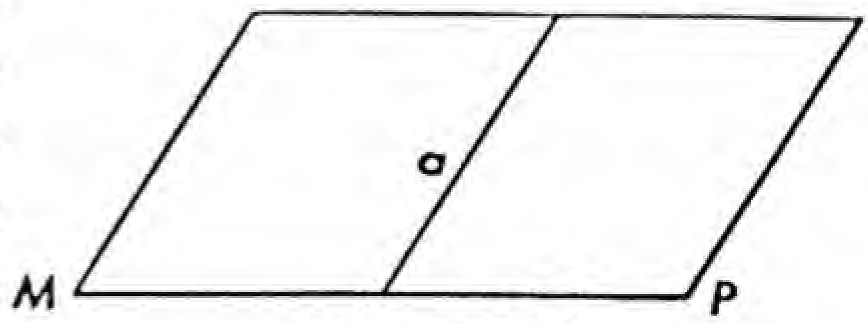


Fig. VIII–3

These instances are enough to recommend the measurement of dihedral angles by means of linear angles. What is needed is a half-line in each of the planes M and P (Fig. VIII–4) meeting on the line of intersection, such that the two lines will form a 0° angle when the half-planes are coplanar, and will form a straight angle when the half-planes form a straight dihedral angle. These restrictions are fully met when, and only when, the two lines are each perpendicular to the line of intersection. *We take the measure of this angle as the measure of the dihedral angle*, and call the linear angle the **plane angle** of the dihedral angle.

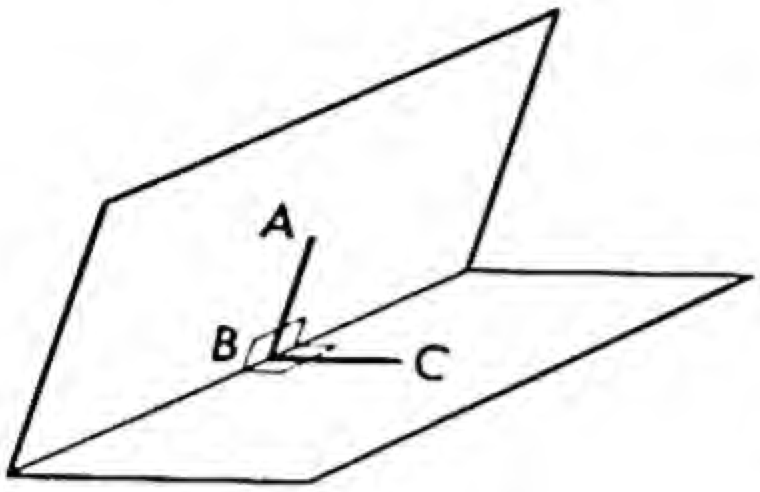


Fig. VIII–4

A plane angle is a constant for any dihedral angle. If the plane angle is drawn at any point other than at B of the line of intersection, the sides of these angles will be respectively parallel to each other. It could be proved that the angles are equal to each other under the circumstances. Indeed the plane angle would be of dubious value were it not constant for a given dihedral angle.

If $AB \perp BC$ (Fig. VIII–4) the plane angle is a right angle and the dihedral angle is a right angle. We say that the planes are perpendicular to each other. This case suggests the possibility of a line and a plane perpendicular to each other. This too turns out to be a useful concept.

We define a line perpendicular to a plane if it is perpendicular to every line in the plane through its foot.

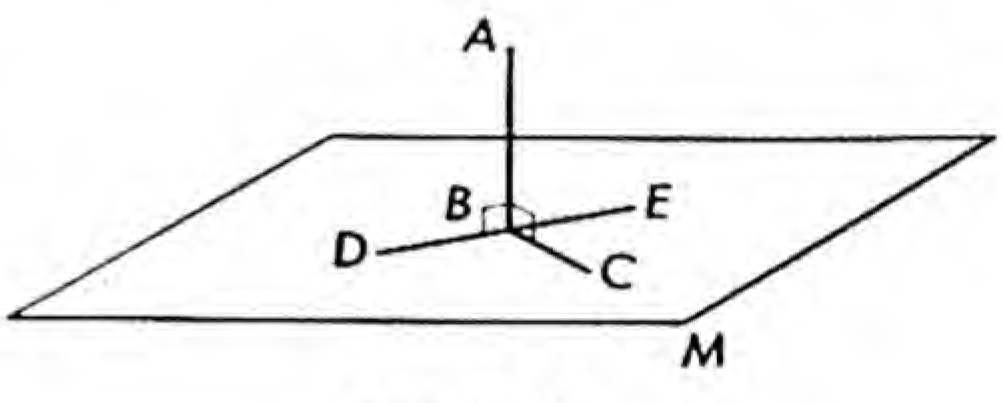


Fig. VIII–5

It is possible to show that *if a line is perpendicular to two lines in the plane through its foot, it is perpendicular to the plane*. The *foot* of a perpendicular is the point of contact between line and plane (see Fig. VIII–5).

Looking back at our three-dimensional coordinate system now, we see that the three axes are mutually perpendicular and any pair of them is a plane angle for two of the reference planes. Consequently the three planes also are perpendicular to each other.

EXERCISES (VIII–1)

1. Draw two angles in a plane with their sides respectively parallel to each other. This may be done generally in two different ways. In one case the angles will be

equal to each other, and in the other the angles will be supplementary. Prove both cases.

2. Prove that vertical dihedral angles are equal to each other.

3. Prove that the intersections of two parallel planes by a third plane are parallel to each other.

4. By comment and sketch, show that two planes that are perpendicular to the same plane are not necessarily perpendicular or parallel to each other.

5. Prove that any edge of a rectangular solid (Fig. VIII–6), such as FB, is perpendicular to two face diagonals, as DB and HF. (All faces of a rectangular solid are rectangles.)

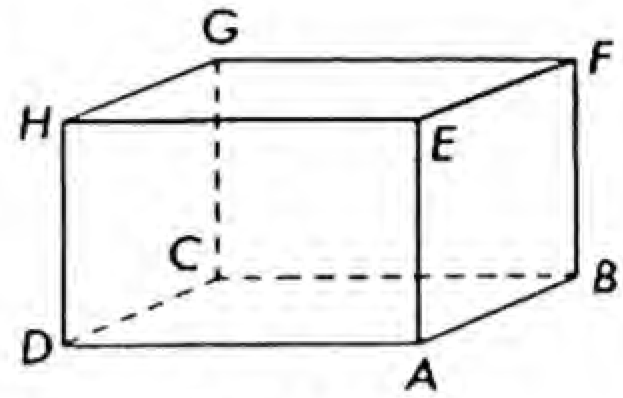


Fig. VIII–6

6. a. Why must the diagonals of a rectangular solid (HB and DF, for instance, Fig. VIII–6) intersect?
   b. Prove that $(FD)^2 = (AD)^2 + (AB)^2 + (FB)^2$.
   c. Prove that any diagonals of a cube are equal to each other.

7. a. Show that the diagonals of a cube are each $e\sqrt{3}$, where e is the length of an edge.
   b. Show that the angles formed by the diagonals of a cube are each given by $2 \text{ arc sin } 1/\sqrt{3}$ and $2 \text{ arc sin } \sqrt{6}/3$.

8. Prove that two planes that are parallel to a third plane are parallel to each other.

9. Prove that the line that is perpendicular to one of two parallel planes is perpendicular to the other.

10. Prove that a plane passing through only one of two parallel lines is parallel to the other.

11. Prove that two planes perpendicular to the same line are parallel to each other.

12. Prove that if each of two intersecting lines is parallel to a given plane, the plane of these lines is parallel to the given plane.

2. PLANES AND POINTS

In two dimensions, equations such as $x = m$ and $y = k$ were represented by lines parallel to the axes. Analogously, in three dimensions, these are planes parallel to two axes simultaneously. Thus, $x = 3$ (Fig. VIII–7a) is a plane parallel to the Y- and Z-axes, respectively, and so is parallel to the YZ-plane.

The two simultaneous equations $x = 3$ and $y = 4$ must describe that which is conjunctively true of the two equations, that is, of the two planes.

Thus $x = 3$ and $y = 4$ (Fig. VIII–7b) describe the line of intersection of the two planes.

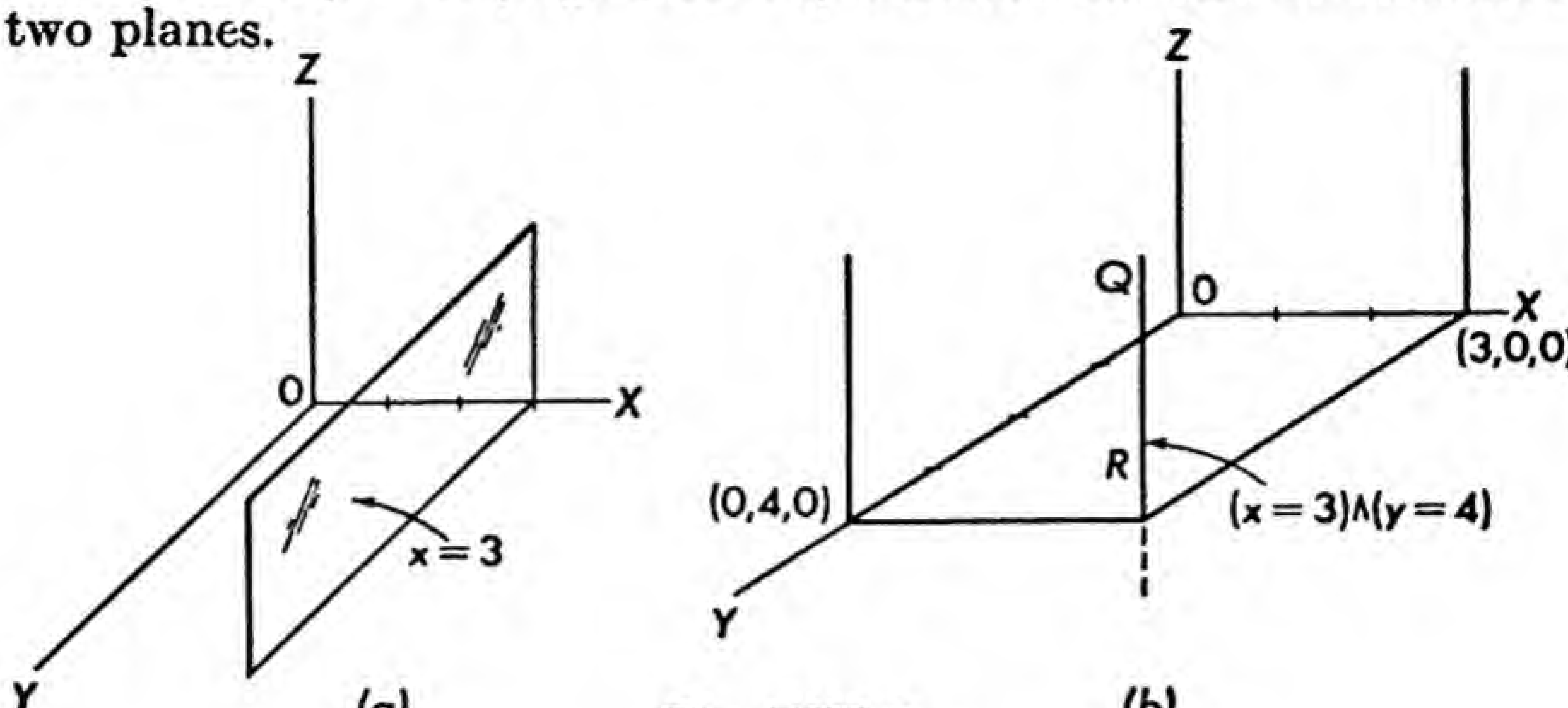


Fig. VIII–7

Every point on the line QR has the value 3 for x and 4 for y. The only thing that varies is the value of z along this line. In fact, z is a free variable, there being no condition imposed on it by the equations of QR. The line QR is parallel to the Z-axis and, like it, perpendicular to the XY-plane.

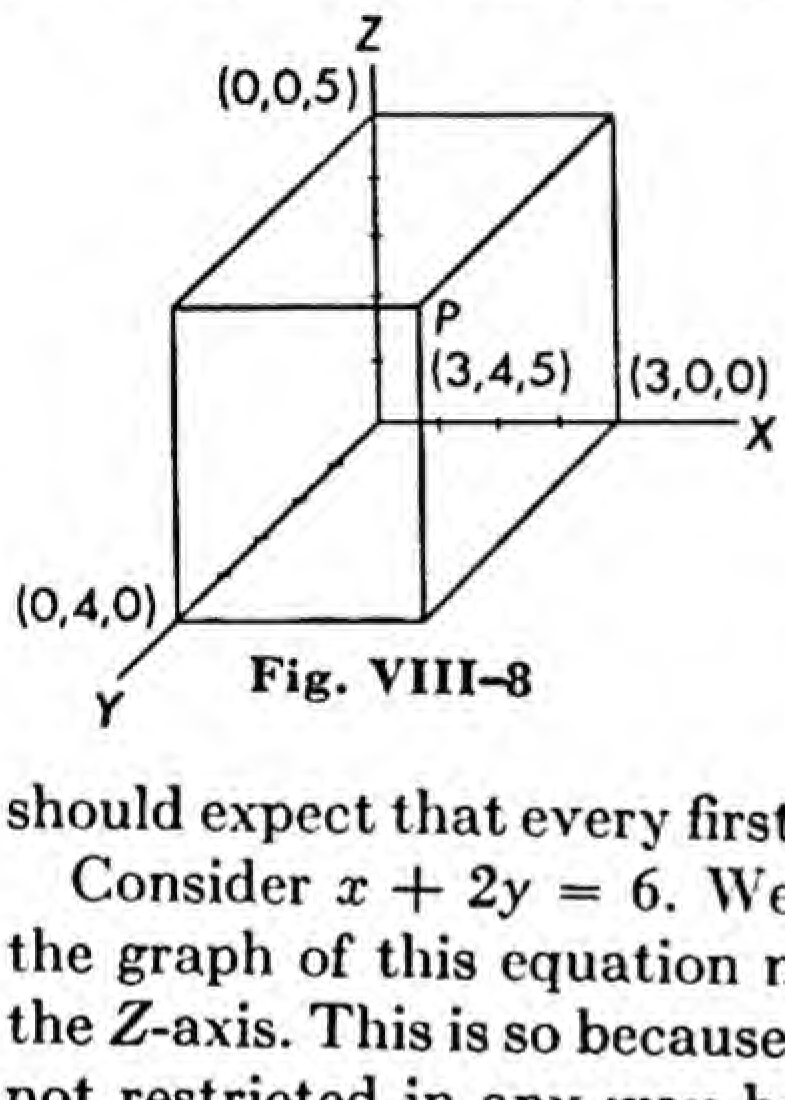


Fig. VIII–8

Finally the three simultaneous equations $x = 3$, $y = 4$, and $z = 5$ describe the conjunction of the three planes, the point P (3, 4, 5) in Fig. VIII–8.

We have noted that first-degree equations such as $x = 3$ represent a plane. If the analogy, or consistent extension from two dimensions holds, where all first-degree equations have the same geometric locus, we should expect that every first-degree equation now should represent a plane.

Consider $x + 2y = 6$. We can safely anticipate the fact that whatever the graph of this equation may be, it must be parallel or coincident with the Z-axis. This is so because z is a free variable, not restricted in any way by the equation.

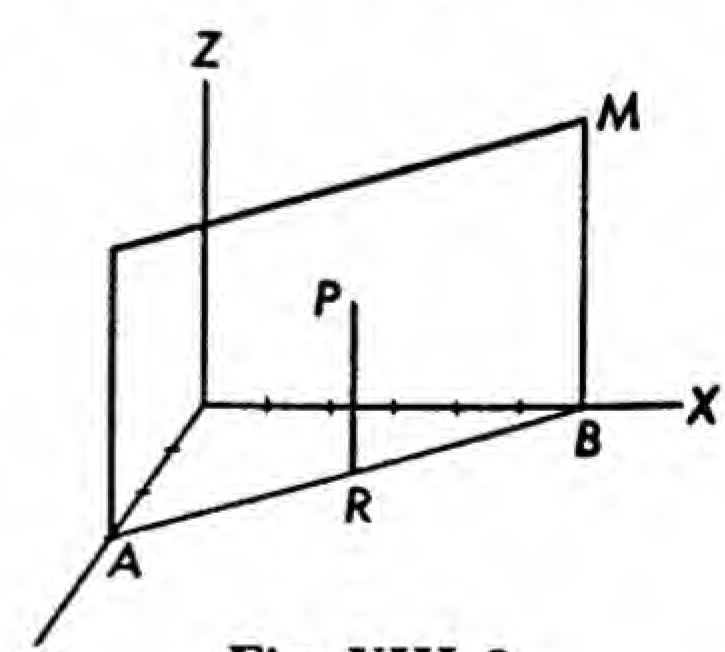


Fig. VIII–9

If AB is the graph (Fig. VIII–9) of $x + 2y = 6$ in the XY-plane, then every point on it satisfies the equation. If R is any point on AB, and PR is taken to be perpendicular to the XY-plane, then every point on PR also satisfies the equation, for the x and y values along PR are identical with those at R. The plane M determined by PR and AB is the locus of all points that satisfy the equation $x + 2y = 6$. As anticipated earlier, this plane is parallel to the Z-axis, since it passes through PR which is parallel to that axis. Again, only the first octant has been

sketched. AB is called the *trace* of plane M in the XY-plane. Similarly, we have plane S representing $y + z = 5$ and plane T representing $z - 2x = 4$ in Fig. VIII–10.

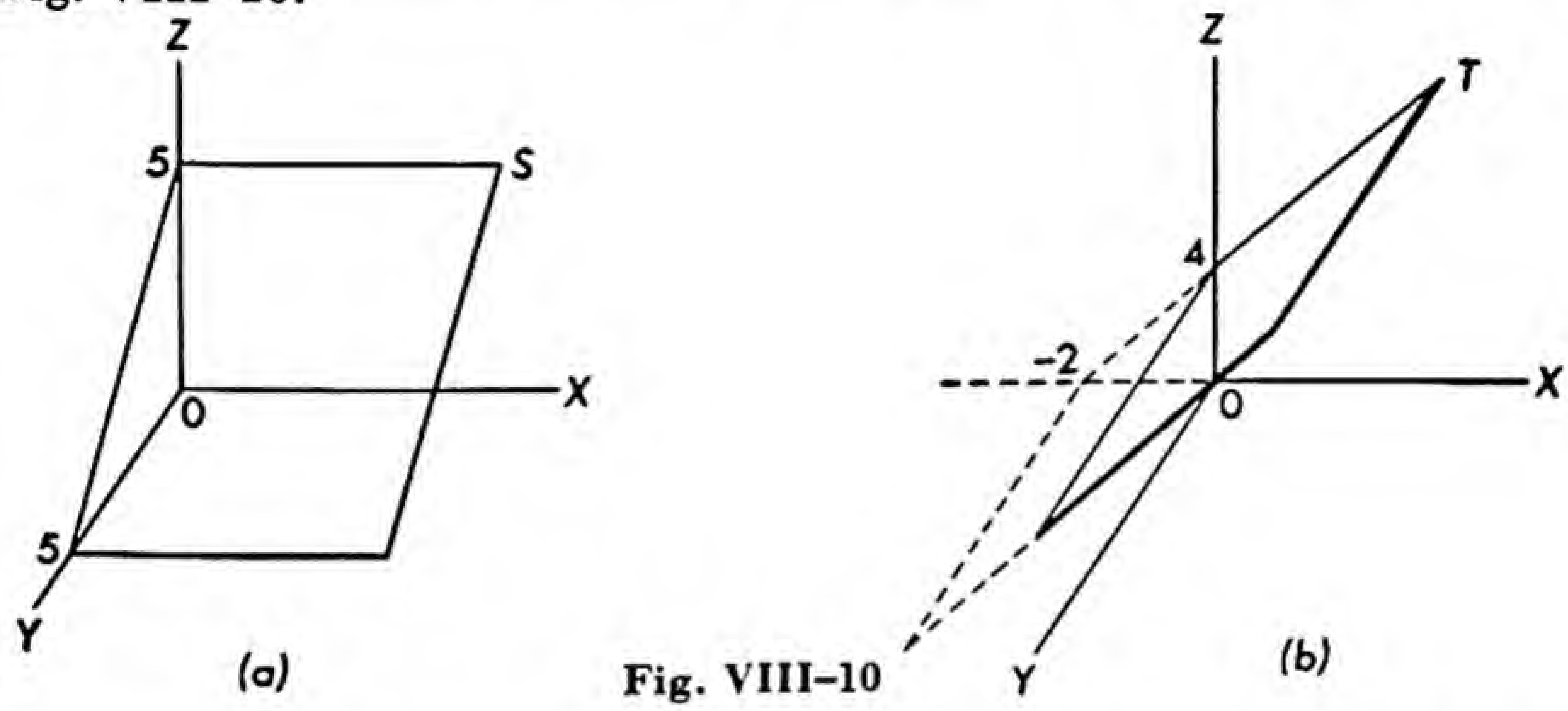


Fig. VIII–10

In the light of the foregoing facts, we may anticipate that the graph of an equation in the first degree which contains all three variables will not be parallel to any of the axes. The graph of the equation

$$2x + y + 3z = 6$$

can be sketched (Fig. VIII–11) from various viewpoints. First consider finding the intercepts on all three axes. Along the X-axis, both y and z are 0; so, by substitution of these values, we get the x-interecept equals 3. Likewise we find that the y-intercept is 6, since $x = 0$ and $z = 0$ along the Y-axis. Similarly, the z-intercept is 2.

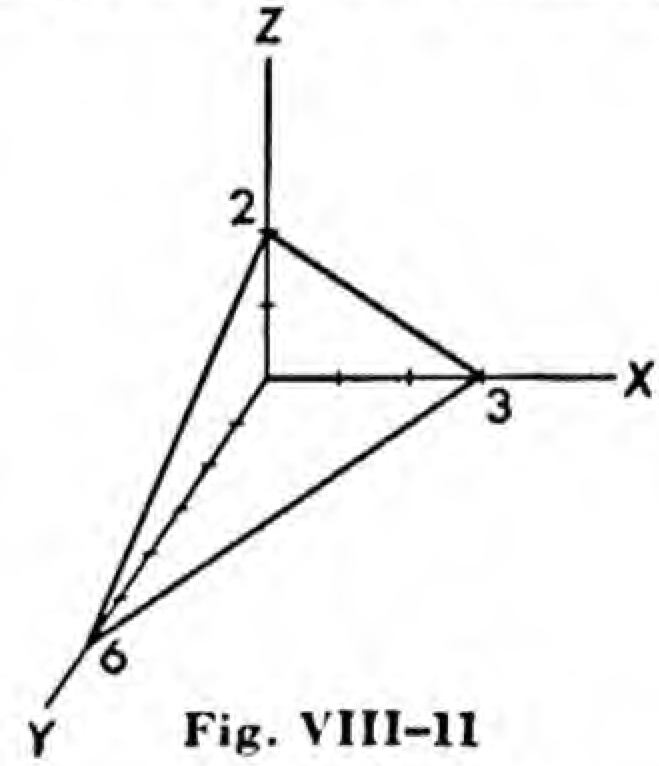


Fig. VIII–11

We can get the equations of each trace merely by letting one of the variables be 0. Thus, in the XY-plane, $z = 0$ everywhere. By setting $z = 0$ in the original equation, we get the trace $2x + y = 6$, which, of course, intercepts the X-axis at 3 and the Y-axis at 6, as we discovered earlier.

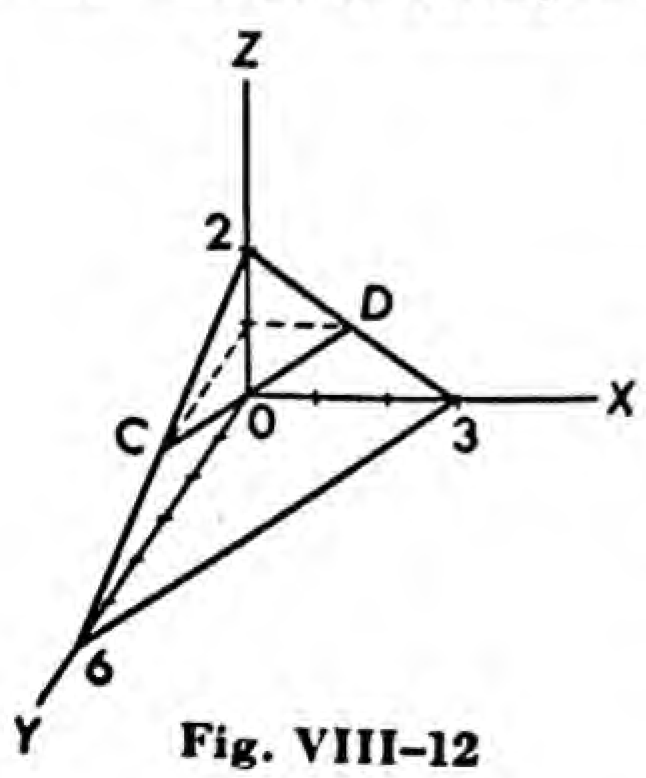


Fig. VIII–12

The triangular portion that was drawn in Fig. VIII–11 for the graph of the equation is only the portion of the plane in the first octant that is bounded by the three traces in the three reference planes. The traces in the reference planes are not the only ones possible. Thus the trace of $2x + y + 3z = 6$ in the plane $z = 1$ is the line CD (Fig. VIII–12), whose equation is $2x + y = 3$. However, this equation by itself does not give the equation of the line because there is no indication that it is in the plane $z = 1$. Rather, the equation of the line can be represented

as the intersection of the two planes $z = 1$ and $2x + y + 3z = 6$; that is, by the conjunction $(z = 1) \wedge (2x + y + 3z = 6)$.

EXERCISES (VIII–2)

1. Sketch each of the following:

a. $x = +3$ b. $y = 2$ c. $z = 4$ d. $x = -3$
e. $y = -5$ f. $z = -4$ g. $(x = 2) \wedge (z = 4)$
h. $(x = 3) \wedge (y = 4)$ i. $(y = 5) \wedge (z = 3)$ j. $(2, 2, 2)$
k. $(5, 4, 6)$ l. $(3, 1, 4)$ m. $(3, 0, 0)$ n. $(0, 3, 0)$
o. $(0, 0, 3)$

2. Sketch each of the following:

a. $x + y = 8$
b. $3x + 2y = 12$
c. $z + y = 6$
d. $3x + 4y = 6$
e. $2x + 3z = 6$
f. $x - y = 6$
g. $3y - x = 9$

3. Sketch the following:

a. $x + y + z = 6$
b. $2x + y + 3z = 9$
c. $3x + 2y + z = 12$
d. $z - 2x - y = 4$

4. Write the equations of the traces in the reference planes of (*a*) and (*b*) in exercise 3.

5. Sketch the lines that are determined by:

a. $(x = 3) \wedge (z = 2)$
b. $(x + y = 4) \wedge (z = 3)$
c. $(x + z = 5) \wedge (y = 3)$
d. $(x + 2y = 8) \wedge (y + 3z = 9)$
e. $(3x + z = 6) \wedge (5x + 2y = 10)$
f. $(2x + 3y + 4z = 12) \wedge (4x + y + 4z = 8)$

6. Give the coordinates of two points that lie on each of the lines in exercise 5.

7. Solve the simultaneous equations:

$$x + y + z = 7,\ 2x + y + 3z = 17,\ x - y + 2z = 9$$

8. The equations in exercise 7 represent three planes and their solutions, simultaneously conceived, consist of finding the coordinates of points that satisfy all three equations. As pointed out earlier, this may vary from no point to one point to an infinity of points.

Let us look at this more generally. The equation of any plane can be represented by $ax + by + cz = d$. By using subscripts for the coefficients, we can represent three equations in general terms as we have done on other occasions. By a further step in the direction of abstraction, we can represent three such equations more compactly by

$$a_ix + b_iy + c_iz = d_i \qquad \text{for } i = 1, 2, 3$$

Now if these equations are solved by determinants, as we have seen, the three variables will all have the same determinant in the denominator, which will be:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If D is not equal to zero, the values of x, y, and z will be unique and finite, and the planes determined by the three equations meet in a point. However, if D takes the nonpermissible value of zero, the other possibilities arise.

State some of the special conditions that will make a determinant zero. Refer these cases to graphs of the equations generally, and describe the circumstances when three equations will not have a unique solution.

9. If $a_ix + b_iy + c_iz = d_i$ for $i = 1, 2$ represents any two planes, what does $a_1x + b_1y + c_1z - d_1 + k(a_2x + b_2y + c_2z - d_2) = 0$ represent? Explain.

3. DISTANCE IN 3-D

Whatever measurements of lines may have been made so far in three dimensions, they have been along the axes. As in two dimensions, we find it desirable and necessary to investigate the matter of distance in any direction.

Consider the points $P(x_2, y_2, z_2)$ and $Q(x_1, y_1, z_1)$. The length PQ can be determined in a number of ways. We could start by dropping the perpen-

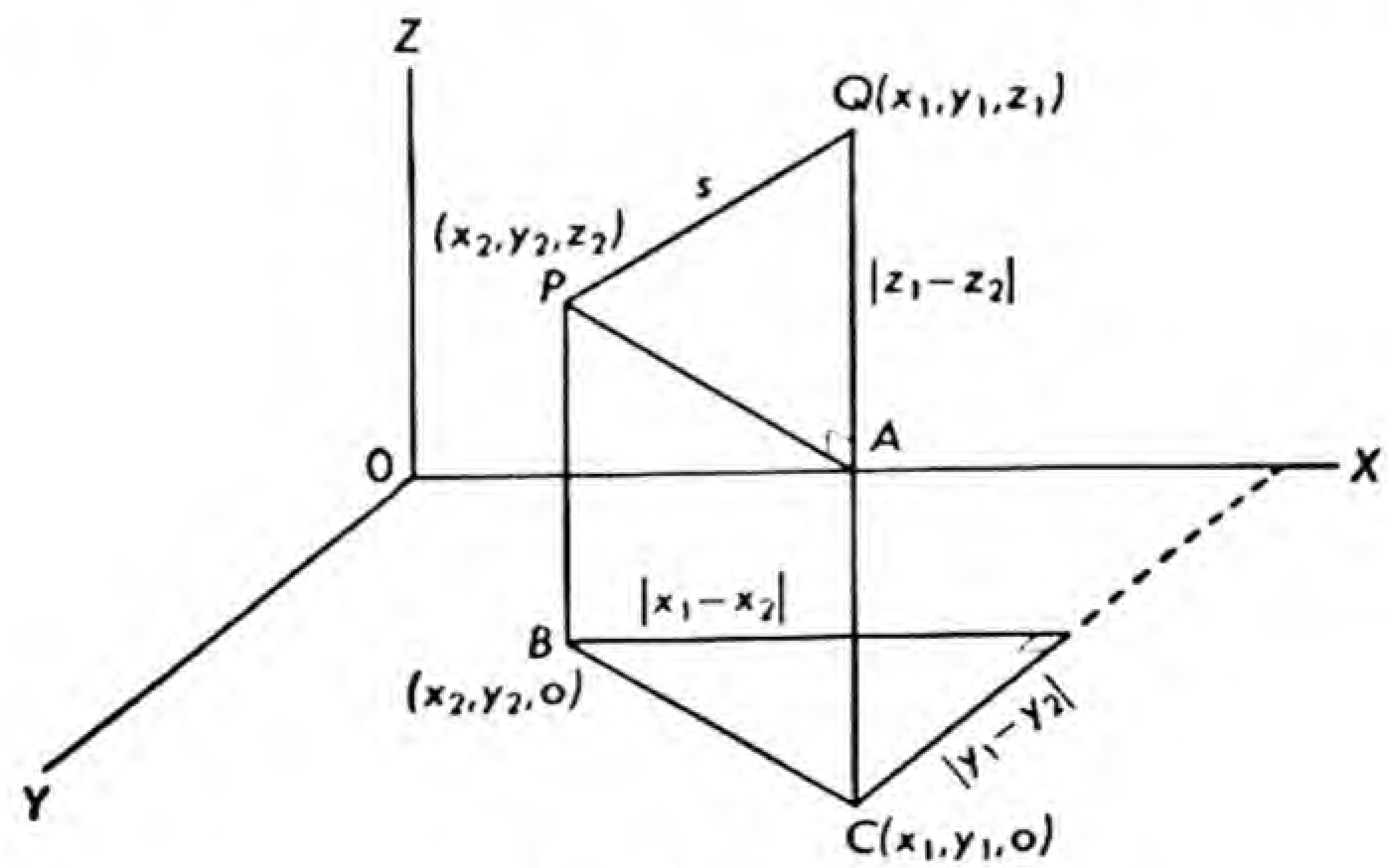


Fig. VIII-13

dicular QC to the XY plane (Fig. VIII-13). PA can be drawn perpendicular to QC. In right triangle PQA, $QA = |z_1 - z_2|$, so that

$$s^2 = (PA)^2 + (z_1 - z_2)^2$$

If PB is drawn perpendicular to the XY plane and BC is drawn, we have $PA = BC$ in the rectangle $PBCA$. Then

$$s^2 = (BC)^2 + (z_1 - z_2)^2$$

Since BC is in the XY plane, we can apply our known distance formula there. The coordinates of B and C are (x_2, y_2) and (x_1, y_1), respectively, in the XY-plane, which is, of course, the plane $z = 0$. Now we have

$$(BC)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

So, by substitution, we have

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

or

$$s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

where $\Delta x = |x_1 - x_2|$, etc.

Also,

$$s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

which is seen to be but an extension of the corresponding formulas of two dimensions.

The distance between the points $(-2, 3, 1)$ and $(5, -2, 4)$ is

$$s = \sqrt{7^2 + (-5)^2 + 3^2} = \sqrt{83}$$

EXERCISES (VIII–3)

1. Find the distance between the indicated pairs of points:

a (3, 5, 7), (1, 4, 2)
b. (−3, 1, 4), (2, 5, −4)
c. (8, 0, 0), (0, 4, 0)
d. (0, 0, 0), (a, b, c)

2. Find the equation of the locus of a point that is five units from the origin.
3. Find the equation of the locus of a point that is equidistant from (2, 0, 3) and (5, 4, 0).
4. a. Show that the points (8, 4, 6), (6, 7, 0) and (3, 1, −2) are the vertices of a right triangle.
   b. Show also that (1, 5, 5), (3, 4 2), and (−2, −3, 1) form a right triangle.
5. Derive the equation of the locus of a point whose distance from the Y-axis is equal to its distance from (1, 2, −2).
6. Find the equation of the locus of a point that is twice as far from the XY plane as it is from (0, 0, 4).

4. QUADRICS

Enough experience has been accumulated to state the fact that a *surface* described by an equation is everywhere parallel or coincident with an axis if that variable is missing from the defining equation. Thus $x^2 + y^2 = 16$ is a surface parallel to the Z-axis. Since the XY trace is a circle with the center

at the origin and with radius equal to 4, the surface must be a circular cylinder parallel to the Z-axis (Fig. VIII–14). Any trace in the plane $z = k$, where k is any real number, will leave the original equation unchanged, and so, every such trace is an identical circle. While the surface is interminable, physical necessity requires termination in our drawings, although sometimes marks are made on the drawings to indicate that the end of the surface has not been reached.

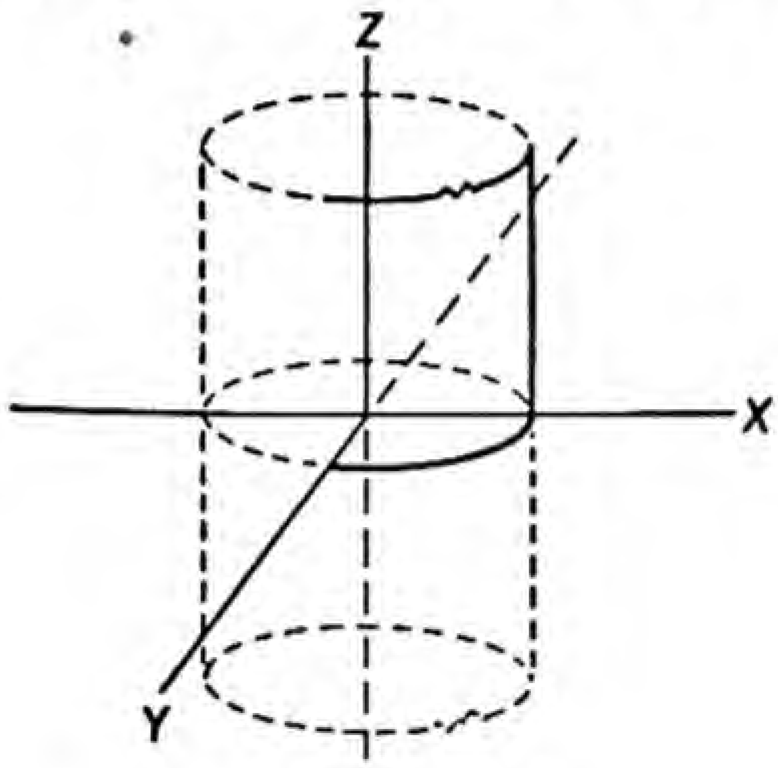


Fig. VIII–14

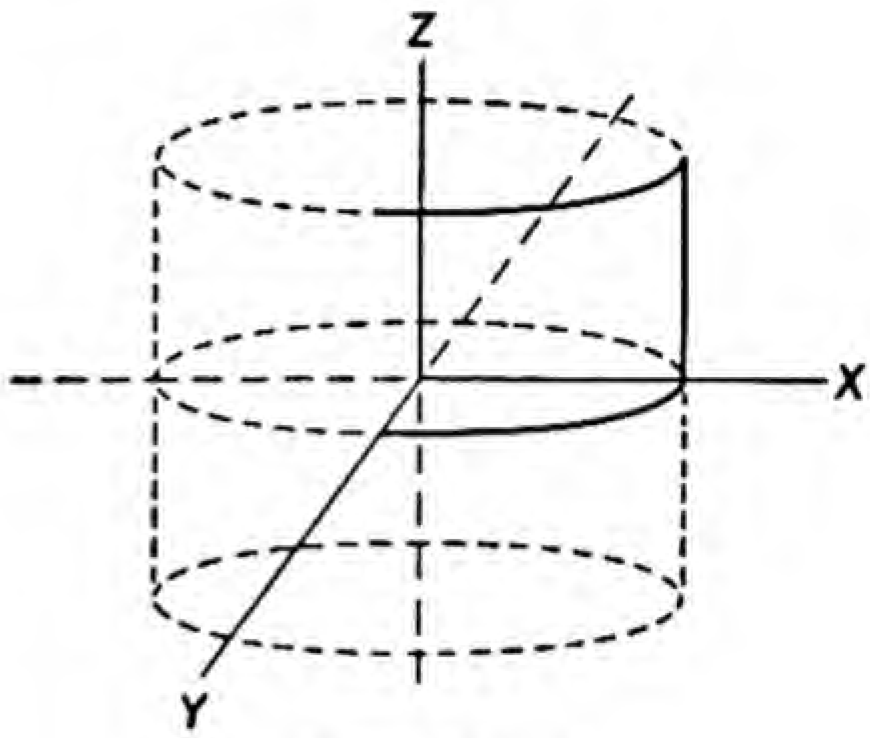


Fig. VIII–15

Analogously $4x^2 + 9y^2 = 36$ is an elliptical cylinder (Fig. VIII–15), $x^2 = 8y$ is a parabolic cylinder (Fig. VIII–16), and $x^2 - y^2 = 12$ is an hyperbolic cylinder (Fig. VIII–17).

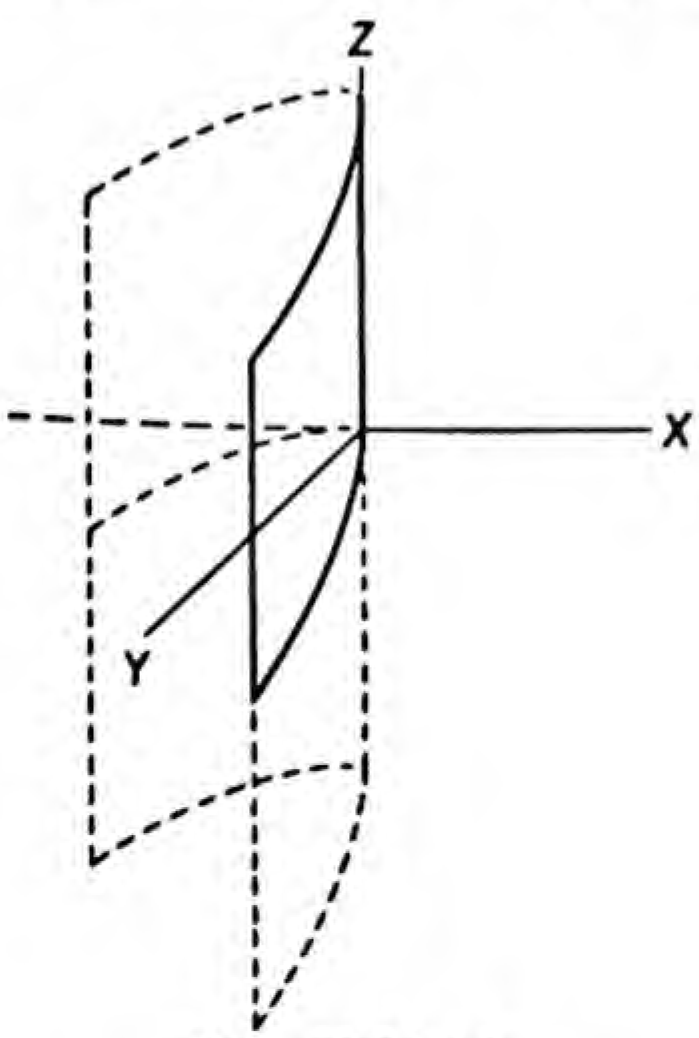


Fig. VIII–16

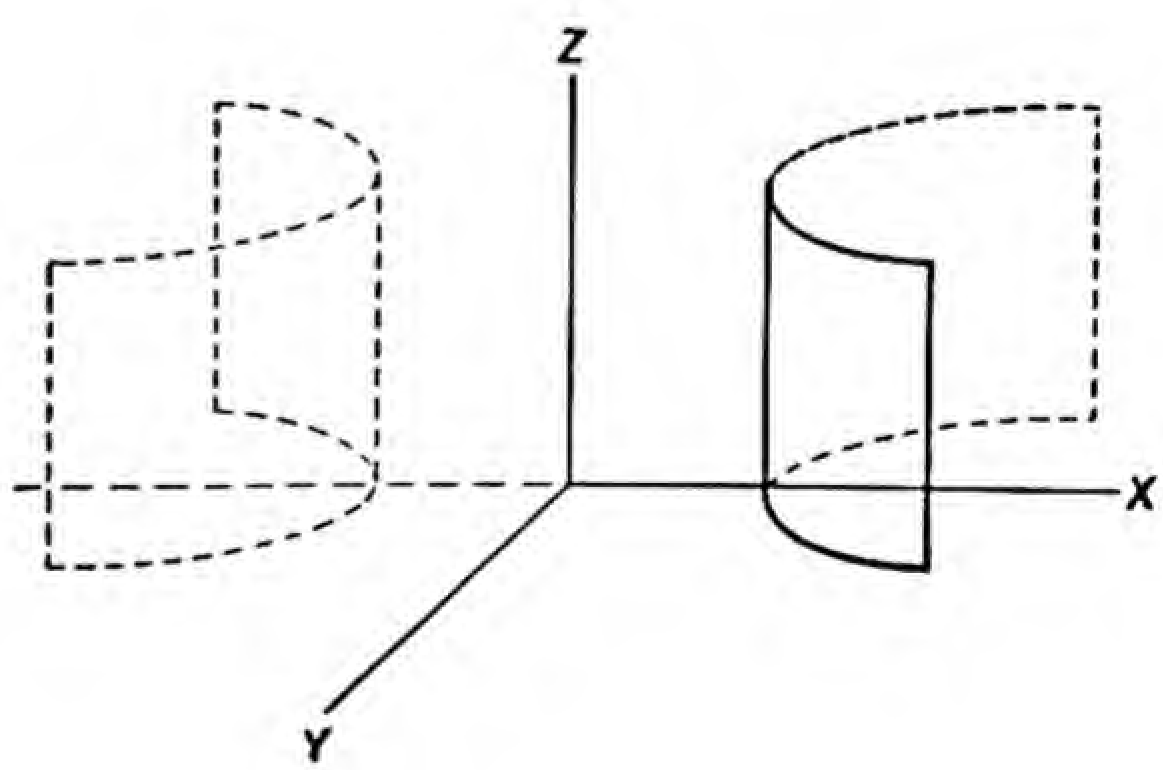


Fig. VIII–17

A special case of the last surface comes from the degenerate hyperbola $x^2 - y^2 = 0$, which in the XY plane represents two intersecting lines. In

three dimensions this becomes two intersecting planes intersecting in, rather than parallel to, the Z-axis (Fig. VIII–18).

We turn our attention now to second-degree equations in which all the variables are present. We start with

$$x^2 + y^2 + z^2 = 25$$

Again various approaches are possible. We begin with the intercepts. The values of two of the variables are zero on any of the axes. For example, any point on the X-axis may be represented by $(x, 0, 0)$. Substituting this in the equation, we discover that there are two intercepts on the X-axis, $+5$ and -5. By substituting $(0, y, 0)$, we learn that there are two y-intercepts, ± 5. Similarly, Z has the intercepts of ± 5, too (Fig. VIII–19).

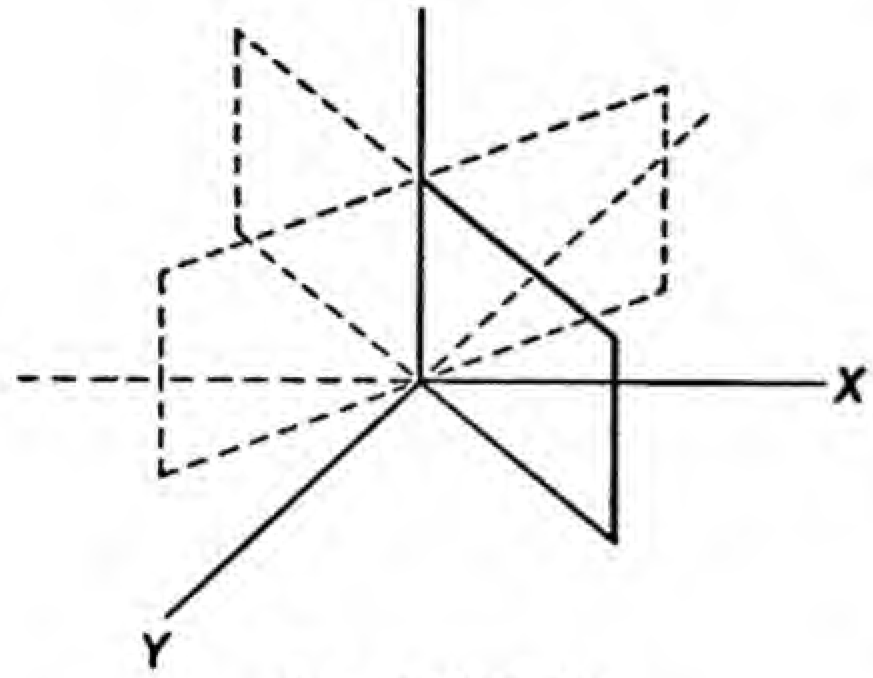


Fig. VIII–18

We can go further. By substituting $z = 0$ in the equation, we find that the equation of intersection with the XY plane is the equation of a circle, $x^2 + y^2 = 25$, whose center is at the origin and whose radius is 5. The substitution of $y = 0$ yields the equation of intersection, $x^2 + z^2 = 25$, which is a circle of the same nature and orientation but in the XZ plane. Finally the yz trace is the same sort of circle. Surely, more than enough information has already been collected to be certain of the fact that the original equation represents a sphere whose center is at the origin and whose radius is 5 (see Fig. VIII–19).

The conclusion can be further supported by a locus view of the situation. We take the sphere as being the set of points, each of which is a fixed distance from a fixed point. If the fixed point is the origin, the fixed distance is five units, and if P is the variable point (x, y, z), then according to the distance formula, the equation turns out to be exactly the equation we have been analyzing.

We can generalize the last case. The center of the sphere may be taken at (h, k, m) and the fixed distance as r. Either by translation of the axes or by the distance formula, we get

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

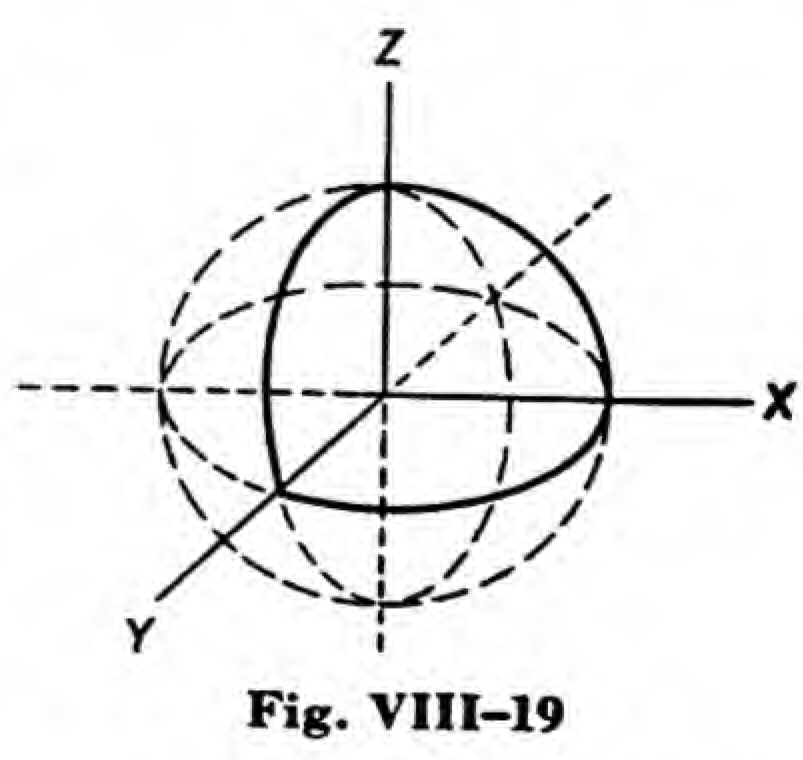


Fig. VIII–19

Should we take the center at the origin, we would have the special case

$$x^2 + y^2 + z^2 = r^2$$

Consider the intersection of the sphere with the plane $z = c$. The equation, by substitution, is $x^2 + y^2 = r^2 - c^2$. If r^2 is larger than c^2, the intersection is a circle. If $r^2 = c^2$, we have the degenerate case of the point circle. And if r^2 is less than c^2, making the difference of the two negative, we have no intersection at all in real space. In similar fashion it can be shown that all real intersections of the sphere and any plane are circles.

The definition of the sphere is apparently only a one-dimensional extension of the circle definition. One may easily suspect that similar extensions are possible for the other two-dimensional conic sections we met earlier. Indeed, when illustrating certain of their properties, we did refer to a paraboloid and an ellipsoid.

The parabola, it will be recalled, represents a set of points, each of which is equidistant from a fixed point and a fixed line. All that we need do for a three-dimensional analogue is to replace the fixed line with a fixed plane (see Fig. VIII–20). So, we shall seek the set of points (each of which is equidistant from a fixed point), $(0, 0, p)$, and from a fixed plane, $z = -p$.

$$(PF)^2 = x^2 + y^2 + (z - p)^2$$
$$PQ = |z + p|$$

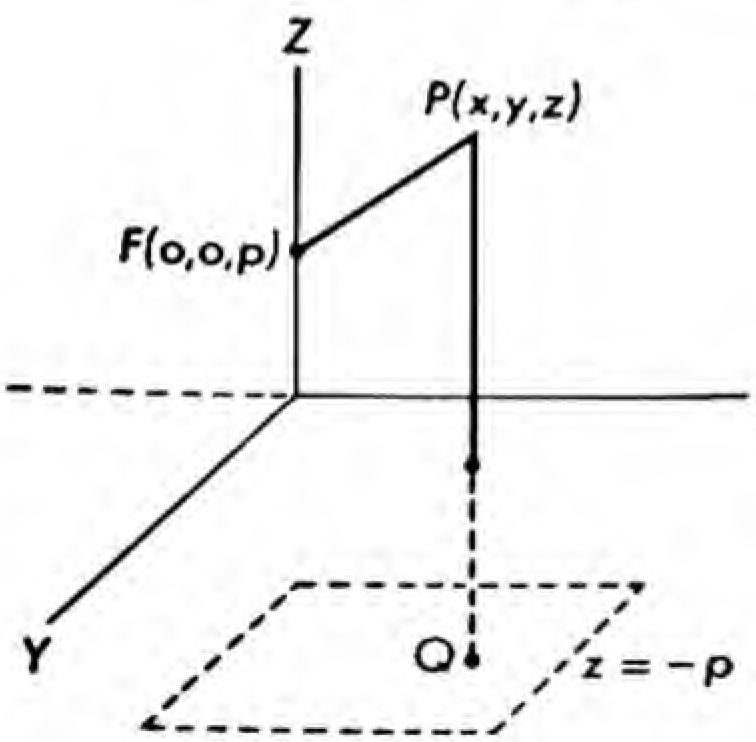


Fig. VIII–20

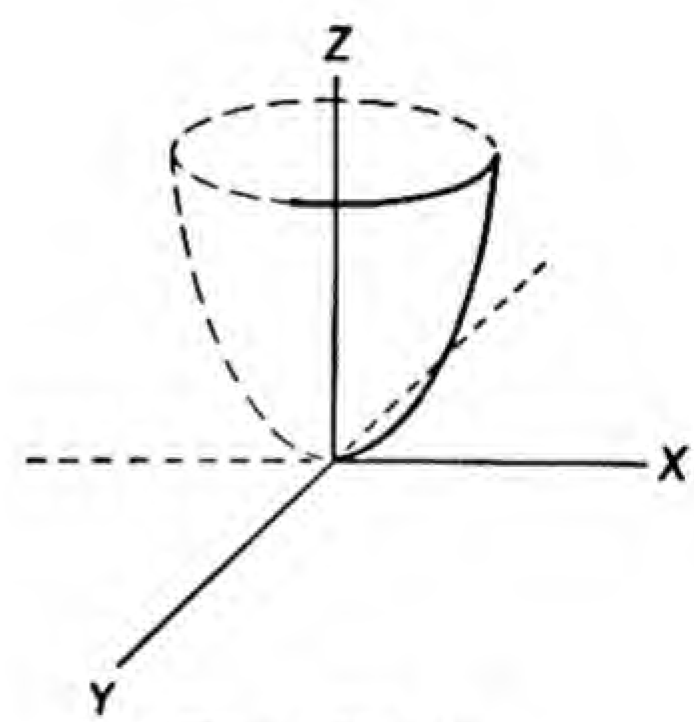


Fig. VIII–21

Taking $(PF)^2 = (PQ)^2$, which is in accordance with the preceding definition or stipulation, we get

$$x^2 + y^2 + (z - p)^2 = (z + p)^2$$

and

$$x^2 + y^2 = 4pz$$

is the equation of the paraboloid (Fig. VIII–21).

We note that by setting $y = 0$, we get the xz trace, $x^2 = 4pz$, which is a parabola with the vertex at the origin and, with z the first-degree term, the principal axis is along the Z-axis. By letting $x = 0$, we get the yz trace, which is a parabola in the YZ plane similarly oriented. If we now assign $z = 0$, we get $x^2 + y^2 = 0$, a point circle; that is, the vertex

of the paraboloid. However, if we assign to z increasingly larger values, we get circles of increasing radii. For negative values of z, there are no real intersections.

The name that is assigned to this and the other quadric surfaces is derived from the traces in the reference planes. Since two of the traces in the last case are parabolas, the name **paraboloid** is assigned. The other trace, in a plane parallel to the XY plane, is a circle. Because of this we say that we have a **circular paraboloid** or a **paraboloid of revolution.** The latter is due to the fact that the entire surface is obtainable by revolving either of the first parabolas about its axis.

It is well to illustrate this further. Considering

$$y^2 + z^2 = 8x$$

we have

$$\begin{array}{lll} xz \text{ trace: } z^2 = 8x & & \text{a parabola about } x \\ xy \text{ trace: } y^2 = 8x & & \text{a parabola about } x \\ yz \text{ trace: } y^2 + z^2 = 0 & & \text{a point circle} \end{array}$$

A parallel yz trace (for $x = 2$, for example) is $x^2 + y^2 = 16$, which is a circle. This, then, is a paraboloid of revolution about the X-axis (Fig. VIII–22).

There is a variation of the paraboloid that we could not have anticipated earlier. Suppose that the numerical values of the coefficients of the second-degree terms are unequal, as in

$$x^2 + 4y^2 = 8z$$

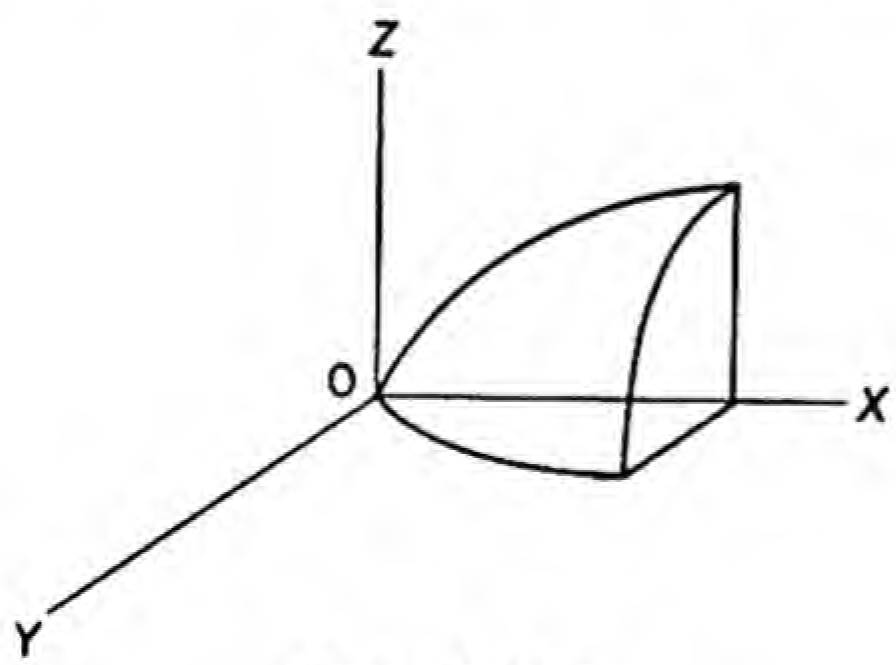


Fig. VIII–22

Two traces are still parabolas, but the trace parallel to the XY plane is an ellipse. This, by our convention, is an **elliptical paraboloid.** Because of the elliptical cross-section, this surface is not obtainable by a revolution of any cross-section. The xz parabola and the yz parabola have different focal points, and therefore the curves are not congruent to each other. Neither one can revolve into the other. If a model of this elliptical paraboloid were to be made, it would have to be molded three-dimensionally. Of course our perspective sketch of this surface would not differ from the previous one since ellipses and circles look the same in perspective.

The ellipse, we found, was a set of points the sum of the distances of each of which, from two fixed points, was a constant. The same definition can

be taken in three dimensions for the **ellipsoid.** We take, analogously to two dimensions, $(-c, 0, 0)$ and $(c, 0, 0)$ as the fixed points, $2a$ as the fixed distance, with $a > c$, and $P(x, y, z)$ as the representative point. Then,

$$\sqrt{(x + c)^2 + y^2 + z^2} + \sqrt{(x - c)^2 + y^2 + z^2} = 2a$$

$$\sqrt{(x + c)^2 + y^2 + z^2} = 2a - \sqrt{(x - c)^2 + y^2 + z^2}$$

$$(x + c)^2 + y^2 + z^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2 + z^2} + (x - c)^2 + y^2 + z^2$$

$$4a\sqrt{(x - c)^2 + y^2 + z^2} = 4a^2 - 4cx$$

$$a\sqrt{(x + c)^2 + y^2 + z^2} = a^2 - cx$$

$$a^2[(x - c)^2 + y^2 + z^2] = a^4 - 2a^2cx + c^2x^2$$

$$a^2x^2 + a^2c^2 + a^2y^2 + a^2z^2 = a^4 + c^2x^2$$

$$(a^2 - c^2)x^2 + a^2y^2 + a^2z^2 = a^2(a^2 - c^2)$$

For the same good reasons as in two dimensions, we introduce a substitute for the binomial coefficient. We let $b^2 = a^2 - c^2$. Here, too, b will turn out to be the length of the semiminor axis of the ellipsoid (Fig. VIII–23).

$$b^2x^2 + a^2y^2 + a^2z^2 = a^2b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

The xy and xz traces are ellipses, with a and b the semimajor and semiminor axes, respectively. The yz trace is a circle with radius b (see Fig. VIII–24). Consequently we have the equation for a *circular ellipsoid,* or an *ellipsoid of revolution.*

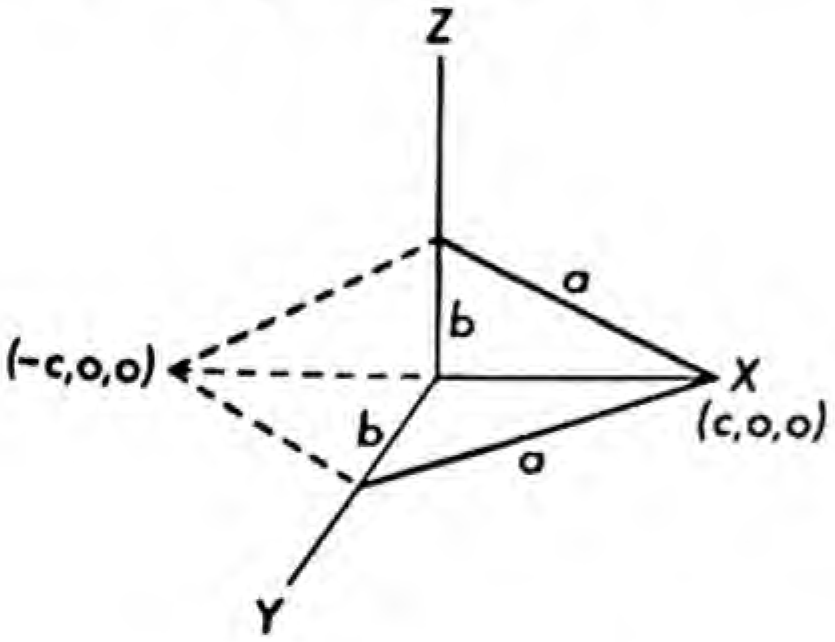


Fig. VIII–23

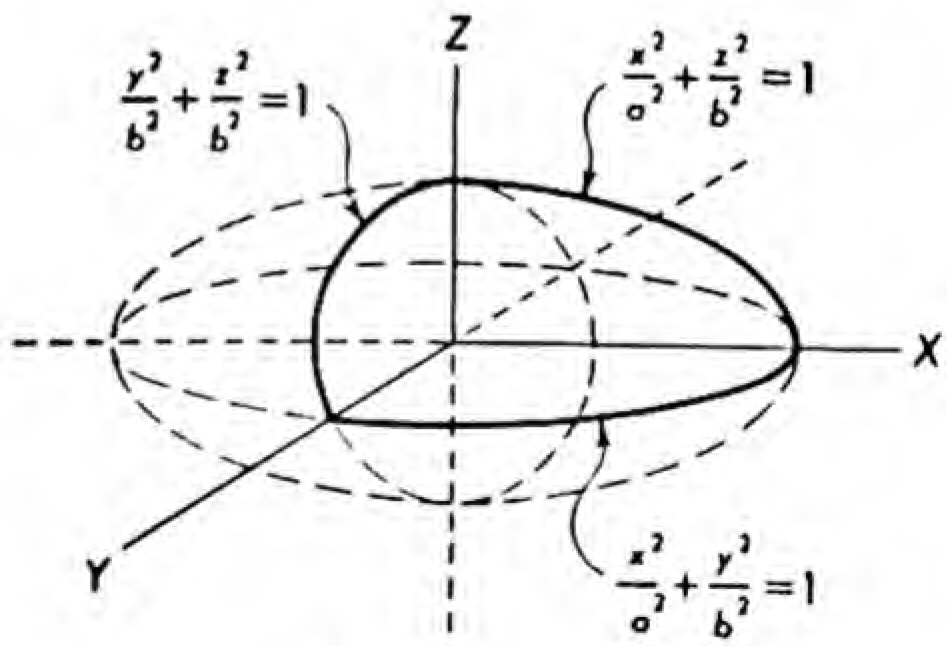


Fig. VIII–24

As with the paraboloid, a change in coefficient would change this to an *elliptical ellipsoid* or, just to avoid redundancy, an *ellipsoid.* This comes about by changing the yz trace from a circle to an ellipse by the simple

device of making the y and z coefficients numerically unequal. This is illustrated by

$$\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{4}=1 \qquad \text{which is} \qquad 9x^2+16y^2+36z^2=144$$

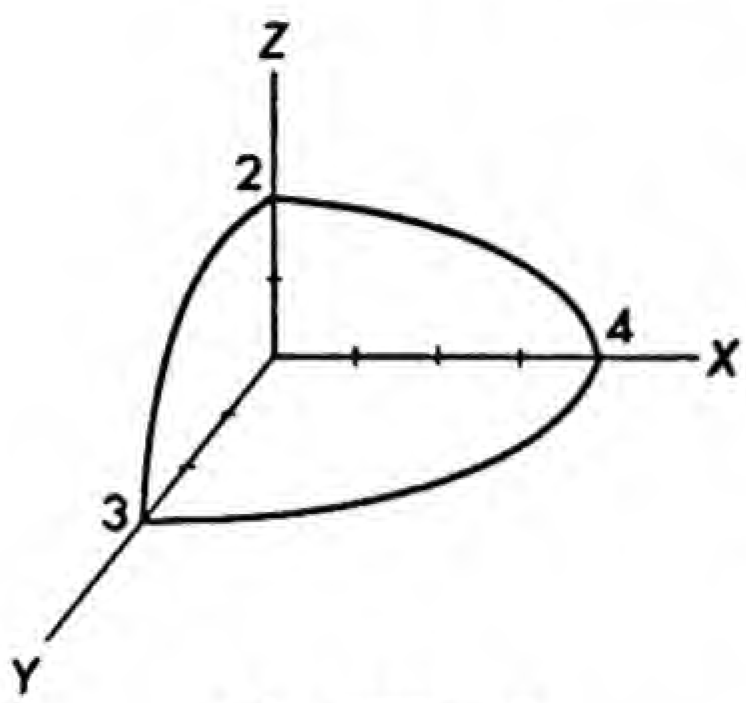


Fig. VIII–25

All three reference traces are ellipses, and for quick sketching, the intercepts are found to be 4, 3, and 2 on the positive portions of the X-, Y-, and Z-axes, respectively (See Fig. VIII–25).

The essential difference between the locus description of the hyperbola and the ellipse consisted in using the difference rather than the sum of the distances from the two fixed points. This difference is maintained for the definition of the **hyperboloid,** and the various constants are taken as for the ellipsoid excepting that of $c > a$. We have, with the change in sign only,

$$\sqrt{(x+c)^2+y^2+z^2}-\sqrt{(x-c)^2+y^2+z^2}=2a$$

The expansions and simplifications are almost identical as for the ellipsoid. The binomial coefficients will turn out to be opposite in sign, as $c^2 - a^2$, but b^2 will still be substituted for them. We get

$$b^2x^2 - a^2y^2 - a^2z^2 = a^2b^2$$

or

$$\frac{x^2}{a}-\frac{y^2}{b^2}-\frac{z^2}{b}=1$$

The xy trace, $(x^2/a^2) - (y^2/b^2) = 1$, and the xz trace, $(x^2/a^2) - (z^2/b^2) = 1$, are hyperbolas; so, the surface is an *hyperboloid.* The yz trace, $(y^2/b^2) + (z^2/b^2) = -1$, is a circle with an imaginary radius. However, for $|x| > a$, the radius will be real, and the traces are real circles. This, then, is an *hyperboloid of revolution* (Fig. VIII–26).

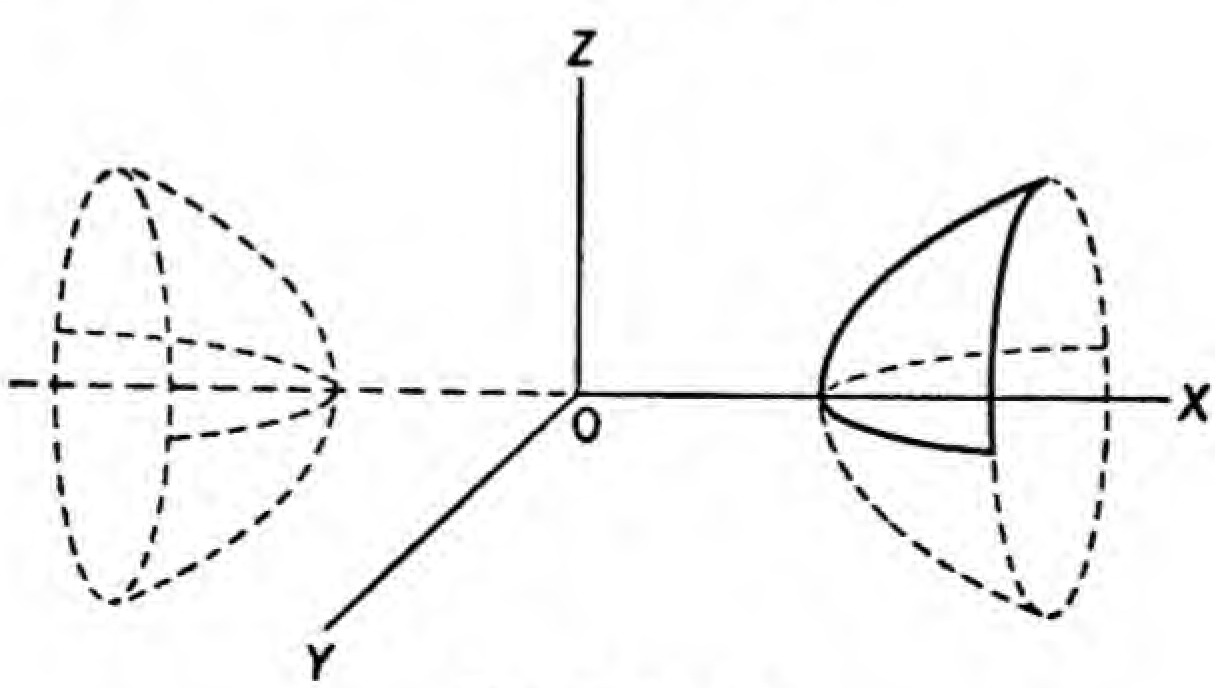


Fig. VIII–26

For reasons that will develop momentarily, this is called, more descriptively, a *circular hyperboloid of two sheets.* If, in the last equation, one of the negative signs is changed to positive, two traces are still hyperbolas. The surface is a hyperboloid, but we have instead a *single sheet hyperboloid.* This may be illustrated by

$$x^2 + y^2 - 4z^2 = 16$$

The hyperbola cross-sections are $x^2 - 4z^2 = 16$ and $y^2 - 4z^2 = 16$. The third reference plane intersection is the circle, $x^2 + y^2 = 16$ (Fig. VIII–27).

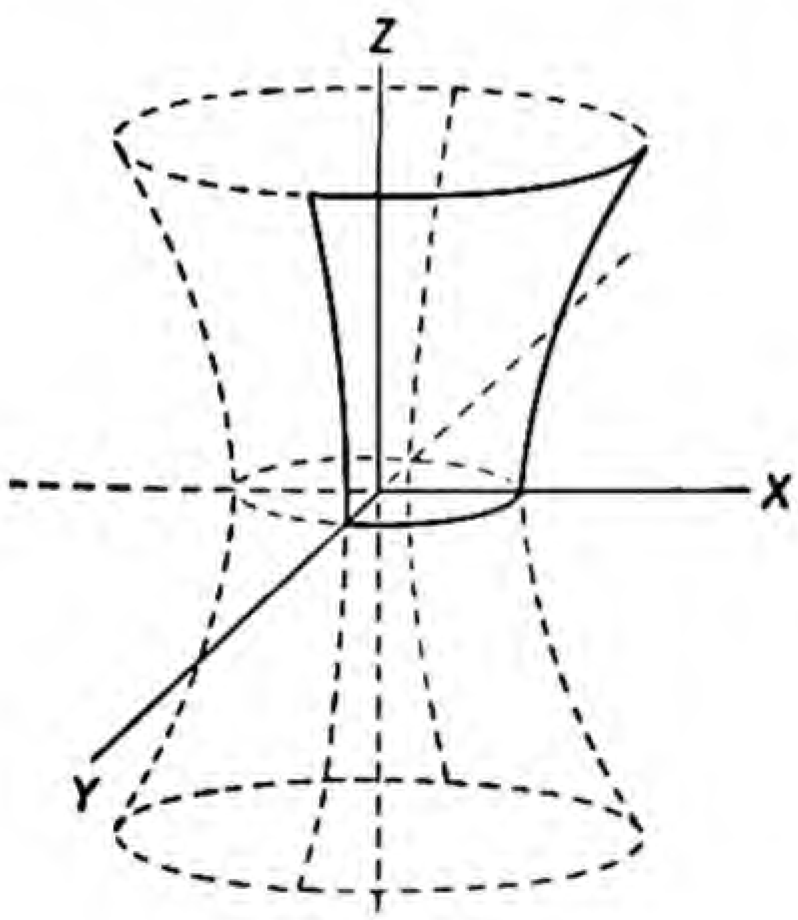


Fig. VIII–27

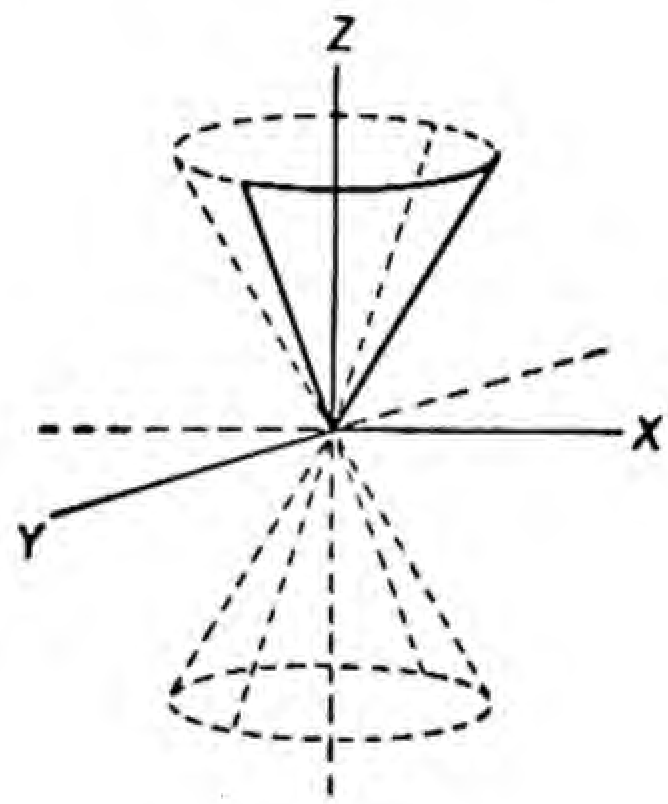


Fig. VIII–28

If the xy trace is constricted until it becomes a point circle, $x^2 + y^2 = 0$; that is, if the whole equation becomes $x^2 + y^2 - 4z^2 = 0$, the hyperboloid degenerates into a conical surface (Fig. VIII–28). The yz trace in the process has also degenerated into two intersecting lines. The same is true of the xz trace.

There is another possibility of considerable interest. If we take the earlier paraboloid, $x^2 + y^2 = 8z$, and introduce a sign change, we get

$$x^2 - y^2 = 8z$$

Two traces are still parabolas: $x^2 = 8z$ and $y^2 = -8z$. This is still a paraboloid, but because of the change in sign, one of the parabolas faces downward on the Z-axis, while the other faces upward. In the original paraboloid they both faced in the same direction. For positive values of z, the cross-sections parallel to the XY plane will be hyperbolas parallel to the XY plane and facing away from the Y-axis, as in $x^2 - y^2 = 16$, for example.

For negative values of z, the hyperbolas are turned 90° and face away from the X-axis, as in $y^2 - x^2 = 16$ when $z = -2$. At $z = 0$ (that is, in the

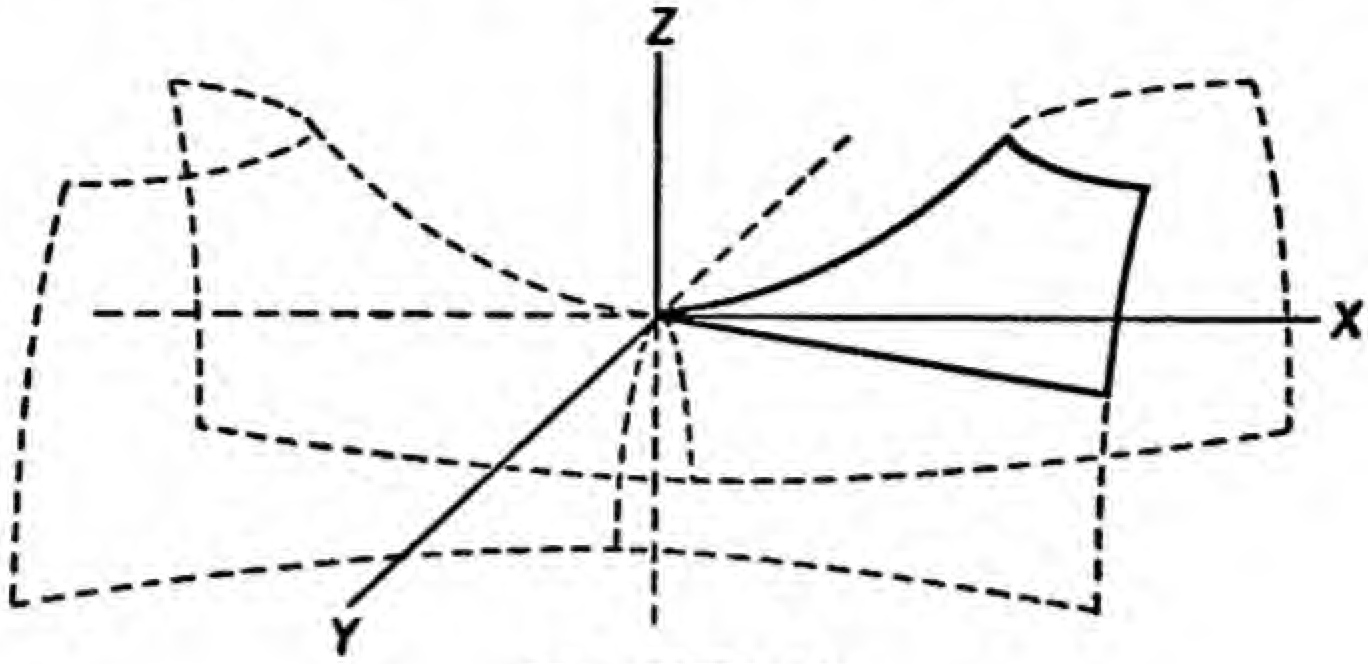


Fig. VIII–29

XY plane itself), we have a degenerate hyperbola, $x^2 - y^2 = 0$, which consists of intersecting lines. Well, then, we have something of a complicated surface that is called a **hyperbolic paraboloid** (Fig. VIII–29). We may think of this as a saddle-shaped surface, whose curvature is *negative* throughout.

EXERCISES (VIII–4)

1. Sketch each of the following:

a. $x^2 + z^2 = 9$
b. $y^2 + 4z^2 = 36$
c. $xy = 12$
d. $x^2 - y^2 = 4$
e. $y^2 - x^2 = 12$
f. $y^2 + z^2 = 16$
g. $2yz = 9$
h. $4x^2 + 9z^2 = 36$
i. $x^2 - z^2 = 0$

2. Find the locus of a point that is four units distance from $(-3, 1, 5)$.
3. Find the equation of the sphere that is tangent to the XY plane and has its center at $(3, 5, 4)$.
4. Find the equation of the locus of a point:
   a. That is equidistant from $x = -3$ and $(3, 0, 0)$.
   b. Whose distance from the Z-axis is equal to its distance from the XY plane.
   c. Whose distance from the X-axis equals its distance from $(3, 1, 1)$.
   d. The sum of whose distances from $(0, 4, 0)$ and $(0, -4, 0)$ is 12.
   e. Whose distance from $(0, 0, 3)$ is equal to twice the distance from the XY plane.
   f. That is equidistant from $x = 3$ and $(6, 0, 0)$.
5. The distance from a point to a plane is given by a formula quite analogous to the two-dimensional counterpart

$$s = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where (x_1, y_1, z_1) is the point and $ax + by + cz + d = 0$ is the plane. Find the distance from $(3, 5, -2)$ to $2x + 3y + z = 6$.

6. Find the equation of the locus of a point:
   a. The sum of whose distances from (4, 0, 0) and (−4, 0, 0) is 10.
   b. The sum of whose distances from (0, 0, 4) and (0, 0, −4) is 12.
   c. The difference of whose distances from (4, 0, 0) and (−4, 0, 0) is 6.
   d. The difference of whose distances from (0, 5, 0) and (0, −5, 0) is 8.
7. Name and sketch each of the following:

a. $x^2 + y^2 + z^2 = 36$	g. $x^2 + y^2 - z^2 = 4$
b. $x^2 + y^2 - 4z = 0$	h. $x^2 - 2y^2 - z^2 = 1$
c. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$	i. $x^2 + 4y^2 + z^2 = 16$
	j. $4x^2 + y^2 + 9z^2 = 36$
d. $x^2 + y^2 = 4z^2$	k. $x^2 + y^2 - 9z^2 = 16$
e. $4x^2 + y^2 - 4z = 0$	l. $4x^2 - y^2 - z^2 = 4$
f. $z^2 - 2y^2 + 4x = 0$	m. $x^2 - y^2 = 4z$

8. There is another very useful approach to the graphics of equations of the type $z = f(x, y)$ whereby only the XY plane is utilized for discrete values of z. Consider, for example, $z = x^2 + y^2$. For various positive values of z, we get a family of concentric circles with the origin as the center in the XY plane (Fig. VIII–30). These illustrative members of the family are called *level curves* of the equation.

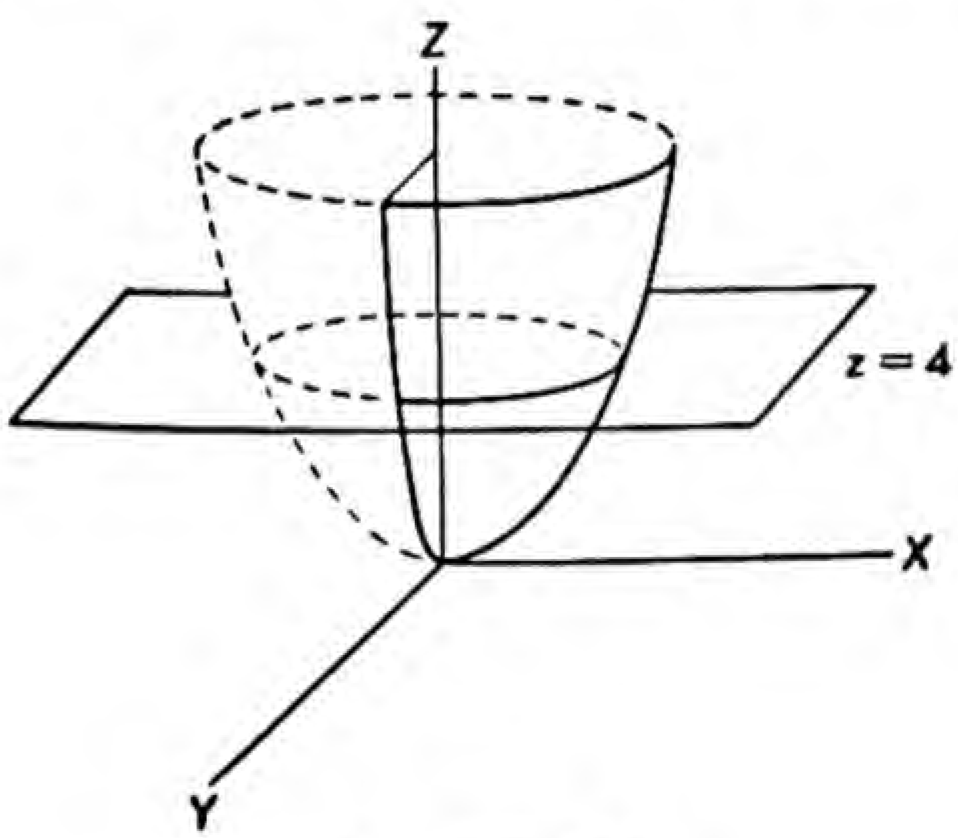


Fig. VIII–30

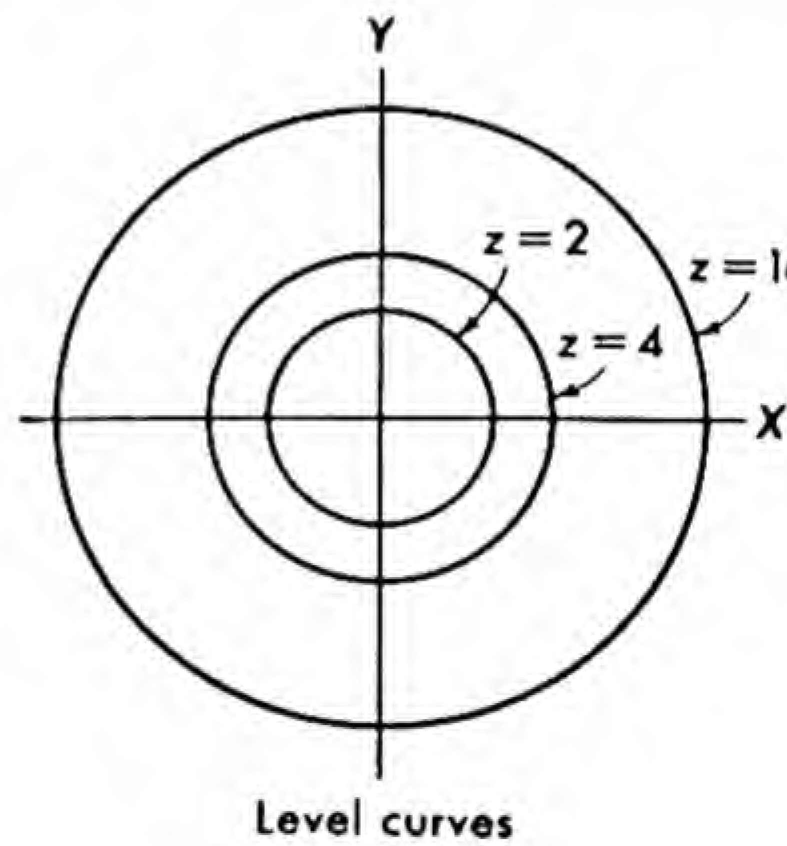


Fig. VIII–31

The level curves correspond to the curves of intersection, the traces of the surface with an appropriate plane. The level curve for $z = 4$, for example, is but the projection of the trace of $z = x^2 + y^2$ in $z = 4$ (Fig. VIII–31) in the XY plane. The trace, the intersection, or the cross-section, is also called the *contour curve*. This is a situation where we seem to be unusually prolific of names. Sketch a number of level curves of:

a. $z = x^2 - y^2$	d. $z = \dfrac{y^2}{x}$
b. $x^2 + y^2 + z^2 = 25$	e. $z = \dfrac{x^2 + y^2 - 1}{2y}$
c. $z = 2x^2 + 4y^2$	

5. FOURTH DIMENSION

We have met first- and second-degree equations in both two and three dimensions. The degree of correspondence between various elements is, as we have found, quite striking. The correspondence can be carried forward to four or even higher dimensions. In the process, however, we must sacrifice our capability of visualization. Our limits in this direction have been reached.

From the facts that $ax + by + c = 0$ is a line in two dimensions, and $ax + by + cz + d = 0$ is a plane in three dimensions, extension suggests that $ax + by + cz + dw + e = 0$ is a **hyperplane** (for lack of a better word) in four dimensions.

The circle and sphere equations, $x^2 + y^2 = r^2$ and $x^2 + y^2 + z^2 = r^2$, suggest $x^2 + y^2 + z^2 + w^2 = r^2$ for a *hypersphere*. Each of the four major traces, obtained by setting in turn one of the variables equal to 0, is a three-dimensional sphere. The equation $x^2 - y^2 - z^2 - w^2 = k$ could consist of hyperboloids in three traces and a sphere in the fourth. We call this a *spherical hyper-hyperboloid.*

The distance formula in a four-dimensional Euclidean framework is easily imagined as $s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta w)^2$.

VIII-5 REVIEW

1. Consider any two half-lines, a and b, meeting in the edge of a dihedral angle. Show by sketches that:
   a. a and b can form a 0° angle when the dihedral angle is 0°, and yet not a straight angle when the dihedral angle is a straight angle.
   b. a and b can form a straight angle when the dihedral angle is a straight angle, but not a 0° angle when the dihedral angle is 0°.
   c. a and b can be perpendicular to each other when the planes are perpendicular, but can be neither 0° nor 180° when the dihedral angle assumes these angles, respectively.

2. The 8-inch base of an isosceles triangle rests on a plane. The vertex of the triangle is 5 inches above the plane, and the arms of the triangle are each 12 inches. Find the distance from the foot of the perpendicular, dropped from the vertex to the plane, to the base of the triangle.

3. The projection of a curve or line onto a plane is the curve or line formed by the feet of the perpendiculars dropped from the curve or line to the plane.

The angle that a line makes with the plane is the acute angle formed by it and its projection on the plane.

Find the angles that the diagonal of a rectangular solid, $3 \times 4 \times 5$ inches, makes with the faces of the solid.

4. According to the determinant D in exercises VIII–2, two planes are parallel if the coefficients of the variables are proportional; that is, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

It can also be shown that planes are perpendicular to each other if the sum of the respective products of the coefficients is zero. That is, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Show that by dropping the z terms and their coefficients, we get the conditions for parallelism and perpendicularity of lines in two dimensions.

5. Show that the plane $3x - 2y + z = 6$ is parallel to $6x - 4y + 2z = 8$ and perpendicular to $x + 2y + z = 12$.

6. Find the equation of a plane through (2, 2, 4) and parallel to $3x + y + z = 6$. Sketch the given and resulting planes.

7. Find the equation of a plane through (5, 2, 3) and (1, 8, 2) and parallel to the Z-axis.

8. Prove that the equation of a plane may be written as $(x/a) + (y/b) + (z/c) = 1$, where a, b, and c are the x-, y-, and z-intercepts, respectievly.

9. A plane has equal intercepts on the axes and passes through the point (5, 4, 6). Find the equation of the plane.

10. Determine the nature of the intersection of each of the following sets of three planes:

a. $2x - 4y + 6z = 7$; $3x - 6y + 4z = 12$; $4x - 8y + 12z = 1$
b. $4x + 3y + 8z = 24$; $5x + 2y + 10z = 10$; $10x + 4y + 20z = 15$
c. $2x + 3y - z = 11$; $5y + 2z = 3$; $x + y + z = 2$

11. Show that the points $(3, 4, -1)$, $(6, 3, 5)$ and $(5, 2, 5)$ are the vertices of a right triangle.

12. Find the distances of $P(a, b, c)$ from each of the axes.

13. Use the distance formula of a point from a plane to write the special formula for the distance of a plane from the origin.

14. Draw the trace of $x^2 + z^2 = 4$ in the plane $y = 3$.

15. Sketch $z = 9 - x^2 - y^2$

16. a. Locate the center and determine the radius of the sphere

$$x^2 + y^2 + z^2 - 6x - 8y - 12z + 60 = 0.$$

b. By a procedure entirely analogous to that used in two dimensions, translate the axes so that the first-degree terms are eliminated.

c. Sketch the sphere and show both sets of axes.

17. Sketch:

a. $xy = 12$	d. $x^2 - 3y^2 - 3z^2 = 15$
b. $x^2 + 4y^2 - 4z = 0$	e. $x^2 + 3y^2 - 3z^2 = 15$
c. $y^2 + z^2 = x^2$	f. $x^2 + y^2 + z = 4$

IX —

PARAMETRIC AND POLAR EQUATIONS

1. THE THIRD PARTY—THE PARAMETER

There are times in the description of a locus of points when it is very difficult or almost impossible to determine an explicit or an implicit equation between the coordinates of a variable point $P(x, y)$. Yet it may be comparatively simple to relate each of the coordinates of P to a third variable that is intrinsically involved in the description of the locus. It may turn out that this third party, this intermediary, will act as a convenient or necessary link. Thus, instead of describing the function through equations such as $y = F(x)$ or $g(x, y) = 0$, we may be able to find something like $x = h(\theta)$ and $y = k(\theta)$, where θ is a third variable that can be related as an independent variable to each of x and y separately. This new variable is called a **parameter.**

Suppose that we know that a set of points is related to some time change; that the abscissa of any point of the set is directly proportional to the cube of the time, and that the ordinate is directly proportional to the square of the time. So, for any $P(x, y)$ and any proportionality constants k and m, we write, using the letter t to represent the time variable,

$$x = kt^3 \qquad \text{and} \qquad y = mt^2$$

Suppose further that the description of events includes the additional data that at time $t = 1$, the value of x is precisely 1 and the value of y exactly 2. By substituting these values, we determine the values of the constants as $k = 1$ and $m = 2$. So, we have

$$x = t^3 \qquad \text{and} \qquad y = 2t^2$$

The set of ordered numbers $\{x, y\}$ in which we are concerned cannot be directly determined one from the other as in the past but rather through the intermediate letter t. The assignment of permissible values to the parameter will yield members of the set $\{x, y\}$; t does the paring. (See Fig. IX–1). Equations so conceived are known as **parametric equations.**

$$x = t^3 \quad ; \quad y = 2t^2$$

t	0	1	2	3	4
x	0	1	8	27	64
y	0	2	8	18	32

One may have anticipated that an explicit or an implicit equation is possible in this case. It may be possible for some readers to see through the description of the locus situation and dispense with the parameter. However, it is surely possible to see that we may dispense with the parameter now. The variable t can be eliminated from the scene by solving one of the equations for t and substituting in the other. Thus, from $x = t^3$, we can get $t = x^{1/3}$ which, when substituted in the y-equation, yields

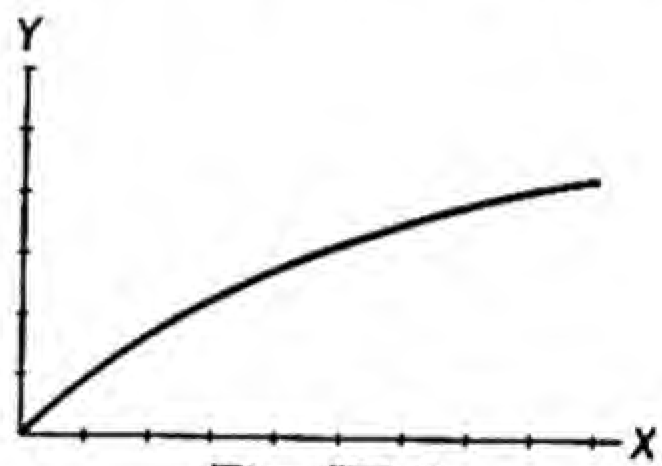


Fig. IX–1

$$y = 2(x^{1/3})^2$$

and

$$y = 2x^{2/3}$$

Unfortunately the elimination of the parameter is not often easy or even possible. In such cases, parametric equations may be almost the only avenue to a mathematical description of a set of points. The following discussion is illustrative of this.

The conditions imposed on $P(x, y)$ is descriptive of the conditions imposed on the entire set. P (Fig. IX–2) always lies on the extension of the two-unit radius of the circle. The amount of extension is to be controlled by the fact that PB will always be equal to $\overset{\frown}{AB}$. And, for uniqueness, $\overset{\frown}{AB}$ will be permitted to vary only counterclockwise.

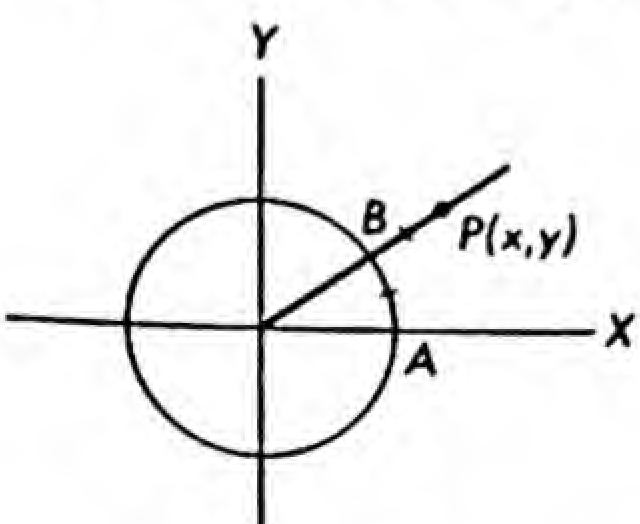


Fig. IX–2

The position of P, and so the values of its coordinates, is dependent on the arc length AB, which in turn is directly dependent on its central angle. It seems, then, that the utilization of the central angle as a

parameter is the key to the analysis (Fig. IX–3):

$$\widehat{AB} = 2\omega \qquad \text{(from } s = r\omega\text{, in radians)}$$

Therefore $OP = 2 + 2\omega \qquad (\omega = \text{omega})$

$$OM = x = (2 + 2\omega)\cos\omega \qquad (OM/OP = \cos\omega)$$
$$PM = y = (2 + 2\omega)\sin\omega \qquad (PM/OP = \sin\omega)$$

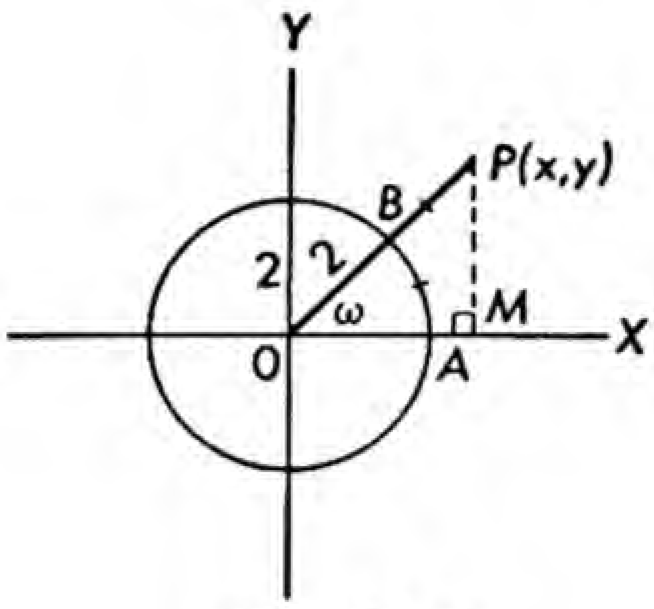


Fig. IX–3

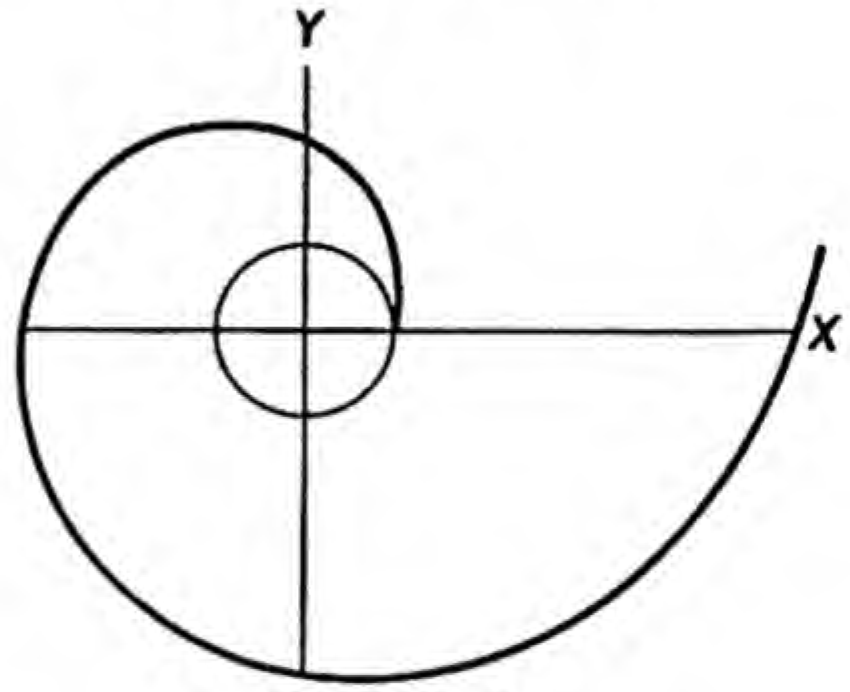


Fig. IX–4

Thus we have the parametric equations, which may be rewritten slightly as

$$x = 2(1 + \omega)\cos\omega$$
$$y = 2(1 + \omega)\sin\omega$$

The graph (Fig. IX–4) can be obtained as before through permissible values of ω, and it will spiral out from the X-axis. However, the reader can see that the elimination of ω from the equations is no simple matter.

Another illustration can be presented which will serve to introduce a really fascinating curve. Consider a circle of radius r rolling along a line. One can fairly easily picture and sketch the locus of a point on the circle.

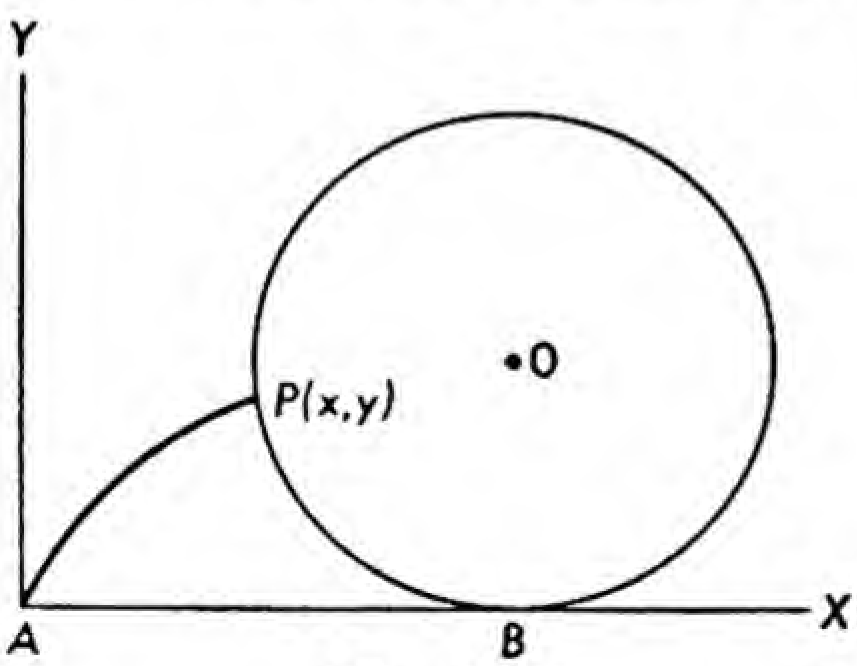


Fig. IX–5

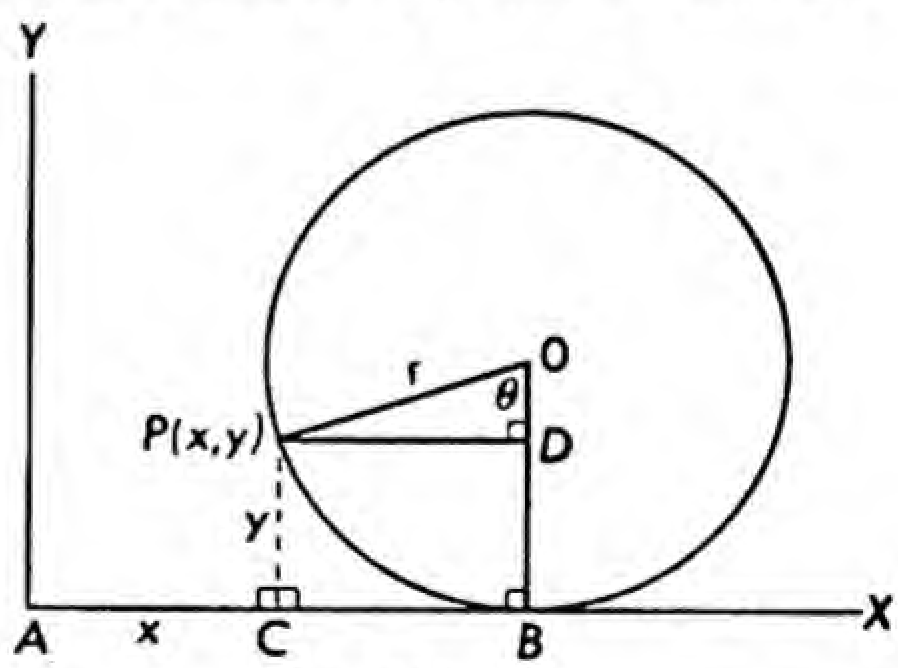


Fig. IX–6

When the circle has reached the position B (Fig. IX–5), the variable point $P(x, y)$ has already moved some distance from its starting position at A. What conditions prevail? The most important observation is that $AB = \widehat{PB}$; the circumference has *unrolled* that distance. Direct approach

to a mutual relationship between x and y is out of the question. We must search for a parameter that is clearly linked to x and to y. The equality $AB = \widehat{PB}$ contains a clue because the arc PB, as any arc, is intimately tied up with its central angle (Fig. IX–6). The size of the arc will be directly proportional to the size of the central angle. The central angle, in turn, varies with the position of the circle at B with respect to the starting position at A. Thus the central angle seems to be an appropriate parameter.

$$\begin{aligned} \widehat{PB} &= r\theta \\ AC = x &= AB - PD \\ x &= \widehat{PB} - PD \\ x &= r\theta - r\sin\theta \\ PC = y &= OB - OD \\ y &= r - r\cos\theta \end{aligned} \qquad \begin{aligned} x &= r(\theta - \sin\theta) \\ y &= r(1 - \cos\theta) \end{aligned}$$

The curve is called a **cycloid** (Fig. IX–7). If the curve is inverted and resting on the ground (Fig. IX–8), it can be proved that a frictionless object will slide down from P to G in the least possible time. Along any other path between P and G, the particle will take a longer time. This property is described by the special name *brachistochrone*. A corollary of this fact is that any two particles situated along the cycloid would take the same time to reach G even though one particle might be 0.001 inch from G and the other 1 mile up. This property is recognized by still another name, *tautochrone*. The latter property has been of particular value in the construction of pendulums for precision clocks. The length of the path of a pendulum bob would be inconsequential if it moved in a cycloidal path.

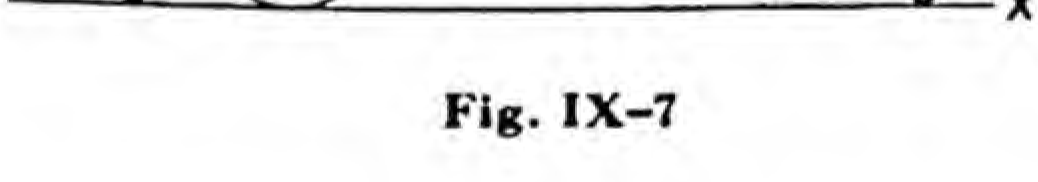


Fig. IX–7

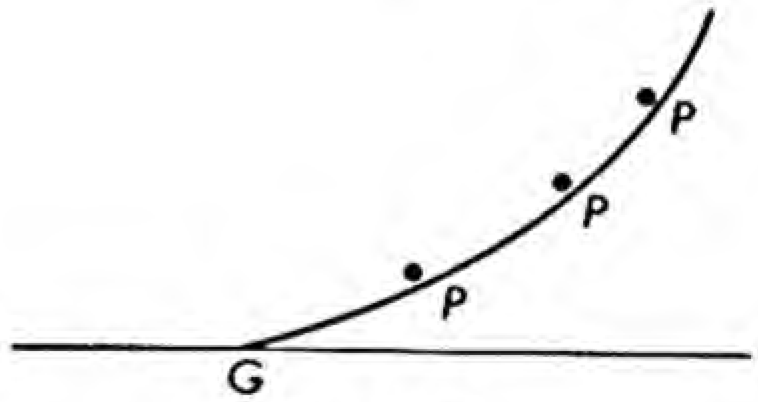


Fig. IX–8

EXERCISES (IX–1)

1. Sketch the graphs of each of the following:

a. $x = t,\ y = \dfrac{12}{t}$

b. $x = t^2,\ y = \frac{1}{4}t^3$

c. $x = t - 1,\ y = t(t + 2)$

d. $x = 3\sin\theta,\ y = 2\cos\theta$

e. $x = \dfrac{2t^2}{t^2 + 1},\ y = \dfrac{2t^3}{1 + t^2}$

f. $x = \sin^3\theta,\ y = \cos^3\theta$

2. If a projectile is fired at an angle θ with the horizontal and with an initial speed of v_0, its position at any time t is given by the parametric equations

$$x = v_0 t \cos \theta, \; y = v_0 t \sin \theta - \tfrac{1}{2}gt^2$$

Sketch the trajectory if $v_0 = 100$ ft/sec, $\theta = \text{arc sin } \frac{4}{5}$, and $g = 32$ ft/sec^2.

3. Eliminate the parameters in each of the equations in exercise 1.

4. Care must be exercised in graphing equations derived from parametric equations. The region of definition may not be the same. Consider, for example, $y = \cos 2\theta$, $x = \sin \theta$. By the very nature of our trigonometric functions we know that $|y| \leq 1$ and $|x| \leq 1$. Now eliminate the parameter and graph the equation.

5. Show that the parametric equations of a circle are $x = r \cos \theta$, $y = r \sin \theta$, (see Fig. IX–9).

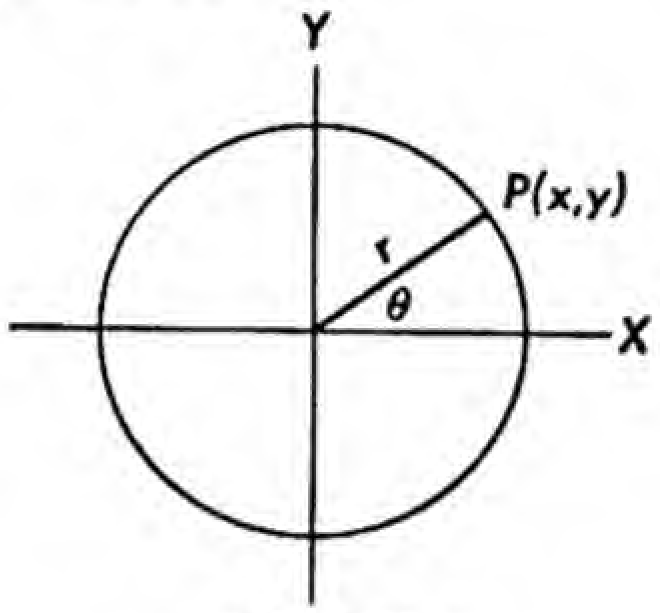


Fig. IX–9

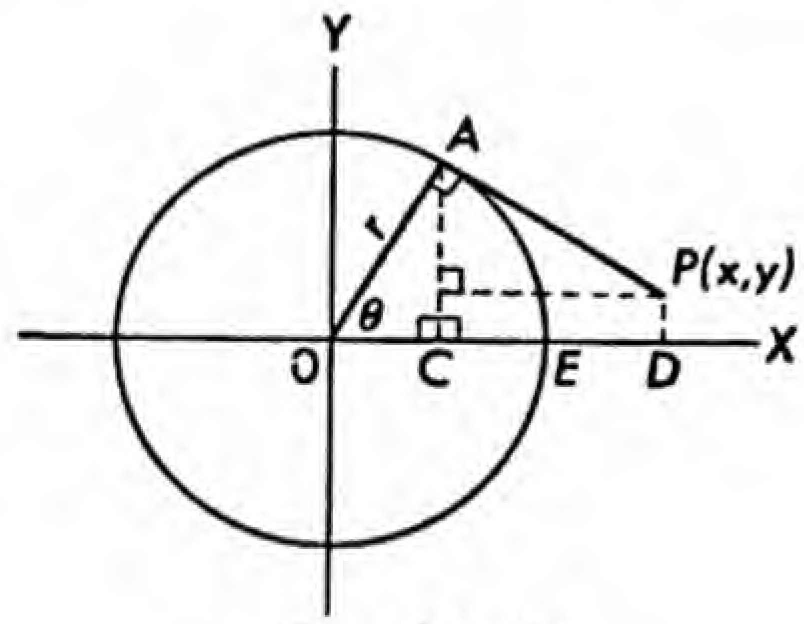


Fig. IX–10

6. A string that has been wound around a fixed circle is unwound and kept taut at all times in the plane of the circle. Find the equation of the locus of the end of the string. Take the end of the string, P, as having been initially at E (Fig. IX–10). (Note that $PA = \widehat{AE}$ and that $\angle PAC = \theta$). The locus is called an *involute of the circle.*

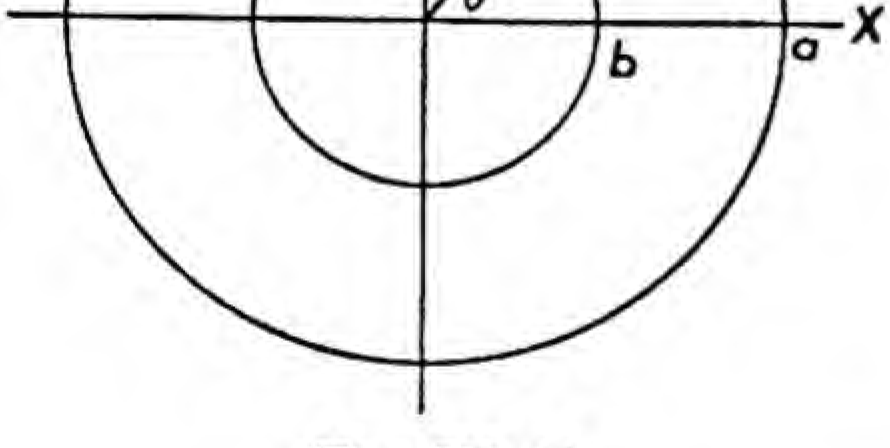


Fig. IX–11

7. Two concentric circles have radii a and b, as indicated in Fig. IX–11. A point P is located as shown.

a. Find the parametric equations of P.

b. Show that the equation of the locus is that of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

c. Show how to draw an ellipse by this means.

8. Given a circle with radius a and center at the origin and the line $y = 2a$. Any line from the origin, making $\angle\theta$ with the positive end of the X-axis, intersects the circle at A and the line at B. Through A and B, parallels are drawn to the X- and Y-axes, respectively. These meet at $P(x, y)$. Find the locus equation of P.

9. Prove that $x = h + a \sin t$ and $y = k + b \cos t$ for $0 \le t \le 2\pi$ are the equations of an ellipse.

10. Describe the graphs of $x = 3[t]$, $y = [t] + 2$.

11. The parametric equations $x = u$, $y = u^2$, and $z = u^3$ define a *twisted* (in contrast to a plane) curve in space. Show that it cuts the plane $x - 2y - z = -2$ in three distinct points.

2. A NEW TECHNIQUE—POLAR COORDINATES

We have utilized and extended the rectangular coordinate system in various ways. We give some attention now to another system. The **polar coordinate system** is an important method. In this an angle θ (Fig. IX–12) becomes a variable and one of the coordinates of the system. In keeping with earlier conventions, θ will be measured about a point O, called *pole* or *origin*, counterclockwise from a line called the **polar axis.** The other side of the angle is the **radius vector,** or just the *radius.*

Since the radius will sweep through the entire two-dimensional continuum, any point in the plane will be identified uniquely by its distance from the pole, as measured along one of these radii and by the angle that the radius makes with the polar axis (See Fig. IX–13). The ordered number pairs will be shown as (r, θ) where r is the length of the radius vector and θ is the **argument**, or *vector angle*. As with other coordinates, these too can be negative. The angle can be negative, as in trigonometry, through a clockwise rotation from the polar axis. The radius vector may be negative if it is extended past the pole. This is all in consonance with our basic concept of a negative quantity. Figure IX–13 shows several examples.

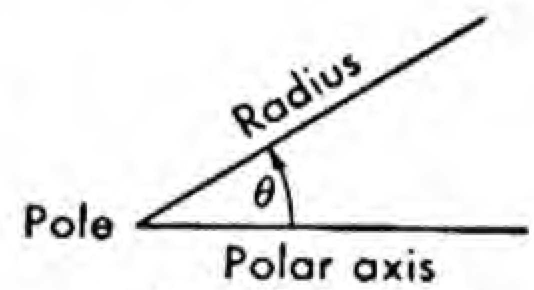


Fig. IX–12

Fig. IX–13

The reader will surely notice a singular difference in this coordinate system as compared with others, in that it does not have a *biunique correspondence.* To every pair of coordinates there is but one and only one point. But, conversely, to every point in the plane there may be an infinity of pairs of coordinates.

Consider the point P (Fig. IX–14) and the radius vector passing through it. This position is obtainable not only by means of the angle θ but also by $\theta + 360°$, $\theta + 720°$, and so forth. As far as the angle goes, for integral values of n any $\theta + 2n\pi$ would do, even including negative values for n. The possibilities are even further increased if we recognize the fact that OP itself may be negative and that OP is an extension of a radius vector in the third quadrant. Thus, to any point there is not a unique set of coordinates.

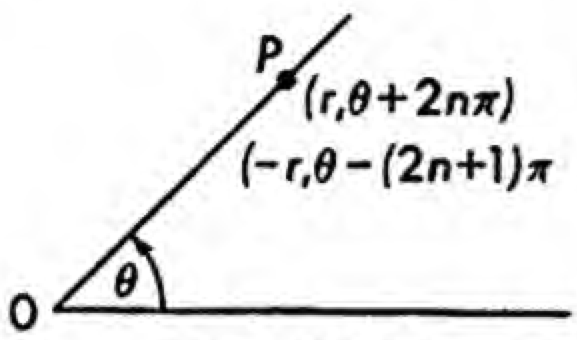


Fig. IX–14

While this does impose limitations on the system, the limitations are more in the nature of cautions for use than impediments. The merit of the system is still enormous.

To begin with, we could try as an application of this system a problem that was solved earlier by means of parametric equations. The variable point P (Fig. IX–15) was such that $PB = \widehat{AB}$.

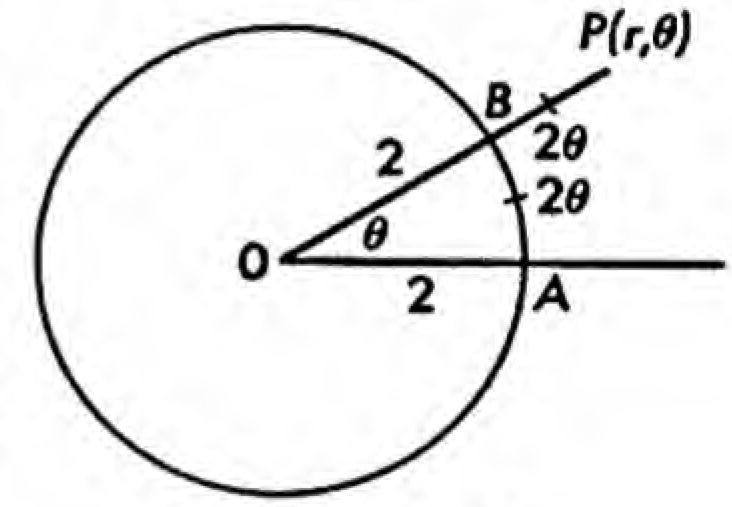


Fig. IX–15

For a polar coordinate approach, we characterize the coordinates of P as (r, θ), and we begin a search for an equation relating the variables r and θ.

Now, if θ is in radians, then $\widehat{AB} = PB = 2\theta$. Further, $OP = r = OB + PB$. So, we have immediately the polar equation

$$r = 2 + 2\theta$$

The graph of this equation will yield the identical curve that we obtained for the parametric equations. The intrinsic relationship is the same in both cases. The difference is, as it were, that the same relationship is being described in different symbolisms. Actually it will be possible shortly to demonstrate the identity of the two descriptions by converting both sets of equations to the same equation in rectangular coordinates.

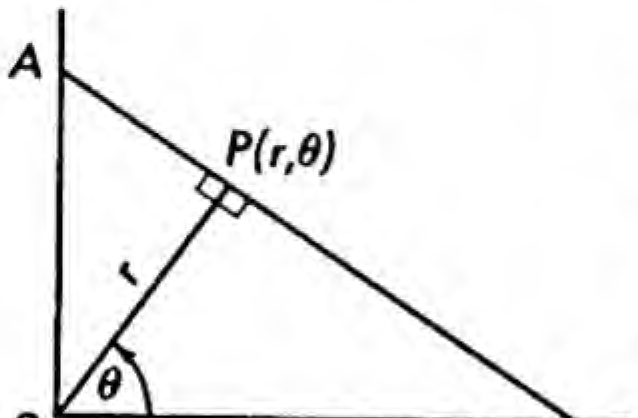


Fig. IX–16

Let us consider another case, shown in Fig. IX–16. Suppose that the line AB, six units long, always forms a right triangle with AO and OB and that A is moved continually upward on OA. Suppose further that OP is always kept perpendicular to AB. What is the locus of P?

From the right triangles shown, we find PB and PA and then impose on them the condition that their sum is 6.

$$PB = r \tan \theta$$
$$PA = r \cot \theta \qquad (A = \theta \text{ via complements})$$

$$r \tan \theta + r \cot \theta = 6$$

$$r (\tan \theta + \cot \theta) = 6$$

$$r = \frac{6}{\tan \theta + \cot \theta}$$

By means of various substitutions and simplifications, the last equation may be written as

$$r = 3 \sin 2\theta$$

For greater generality, we disregard the practical limitations of the preceding problem and consider the graph of the equation without the first quadrant limitation of the line AB. To begin with, $\sin 2\theta$ has a period of 180°, and so there is no need to seek values in our table beyond this, since no new ones can arise. As in other graphic techniques, we can only choose a finite number of points that will be suggestive of the graph. We choose the ones indicated, which are actually more than are needed for a simple sketch (Fig. IX–17).

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
r	0	1.5	2.6	3	2.6	1.5	0	−1.5	−2.6	−3	−2.6	−1.5	0

The first point is (0, 0) and the seventh is (0, $\pi/2$). We started at the pole and returned to it at 90°. This gives us the loop in the first quadrant. The next seven points provides us with a similar loop. However, since all the r values are negative or zero, the loop turns up in the fourth quadrant because the radii are all extended backward. Continuing with values for θ, from 180° to 225°, the values of 2θ will range from 360° to 450°. The values of the sine will be those of the first quadrant. Consequently we shall have the beginnings of another loop in the third quadrant, one that is congruent to the other corresponding portions. Continuing in this manner (see Fig.

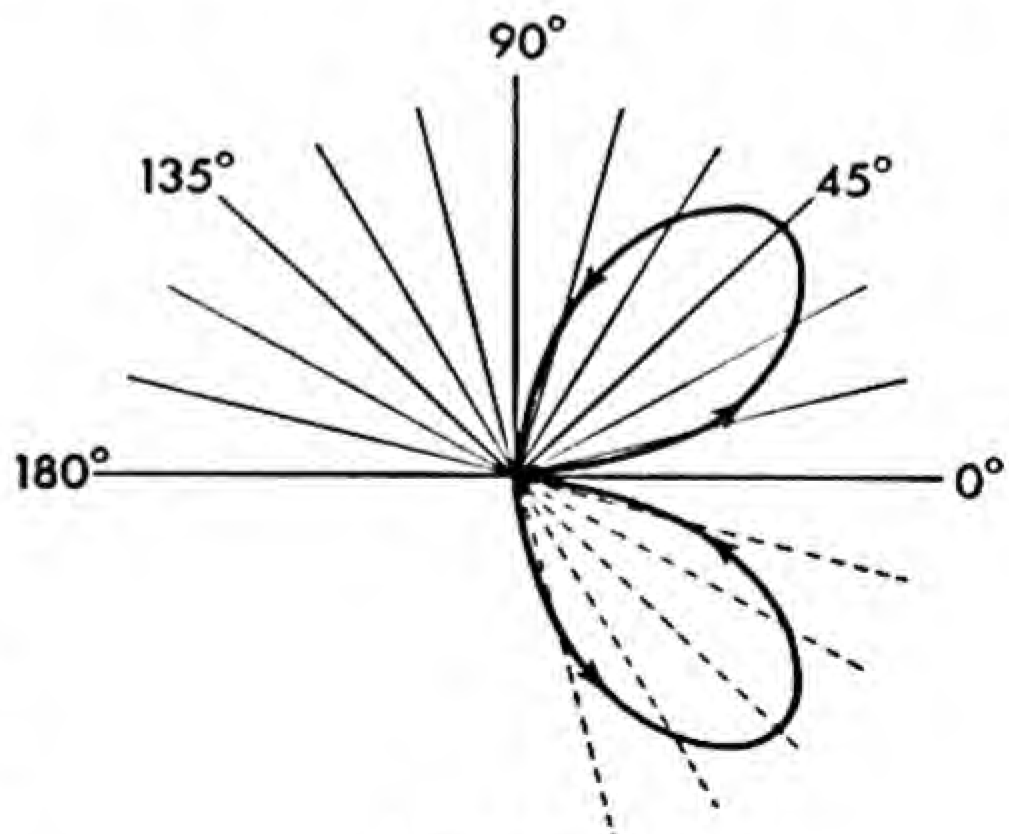


Fig. IX–17

IX–18), we get a "four-leaf rose" with the petal in the second quadrant turning up last. This is a continuous curve, with the arrows on the graph indicating the progression of points (Fig. IX–18).

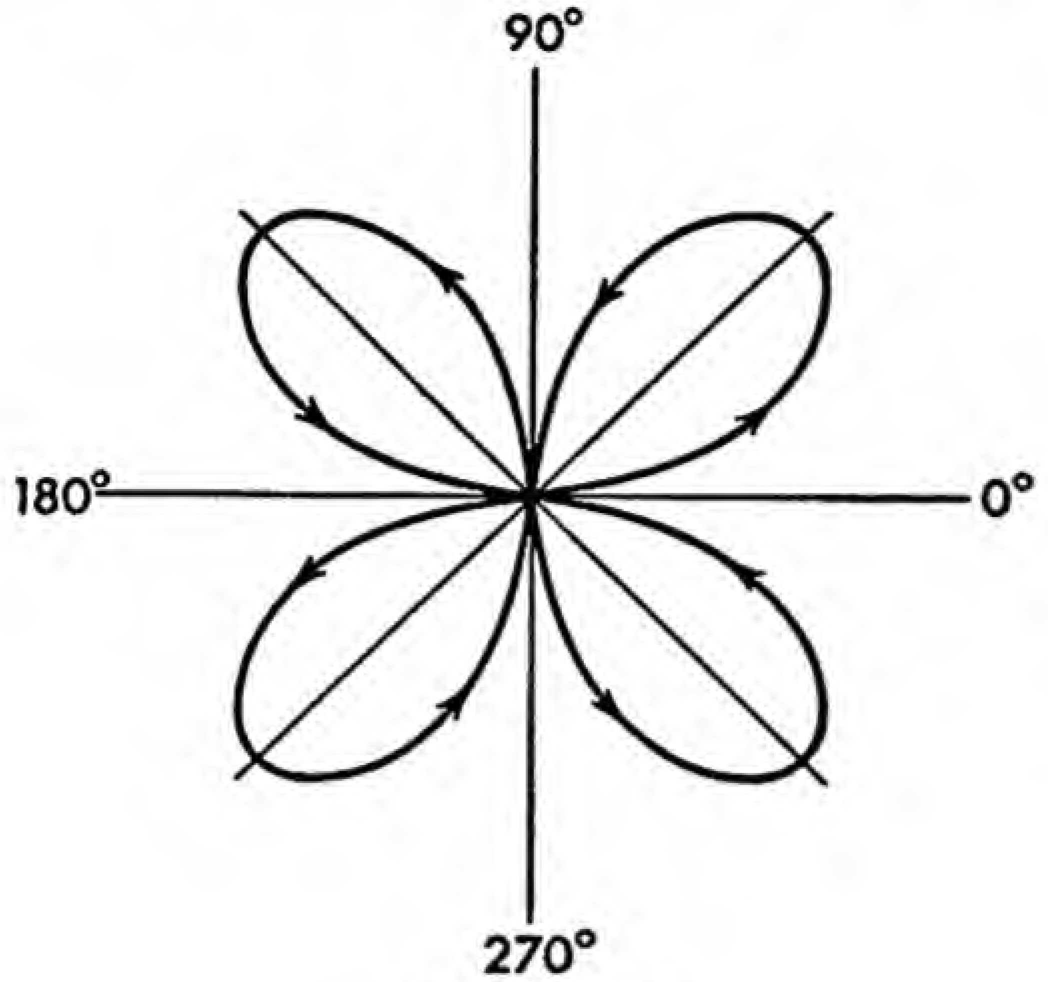


Fig. IX–18

The sketching is facilitated by the employment of special *polar coordinate* graph paper on which the angle measures are indicated to a considerable degree of accuracy and the units on the radii are clearly indicated.

We have had occasion before to find equations in different systems for the same locus. We may look to the development of *conversion formulas* now.

If P is a point in a two-dimensional continuum (Fig. IX–19), its coordinates may be (x, y) if viewed from rectangular axes or (r, θ) in the polar coordinate system. In either case we have in mind the same point P. To determine relationships between these variables, we need only display them explicitly as we have done in the diagram (Fig. IX–19). The right triangle indicates a host of possible equations, of which the following are, perhaps, the most useful:

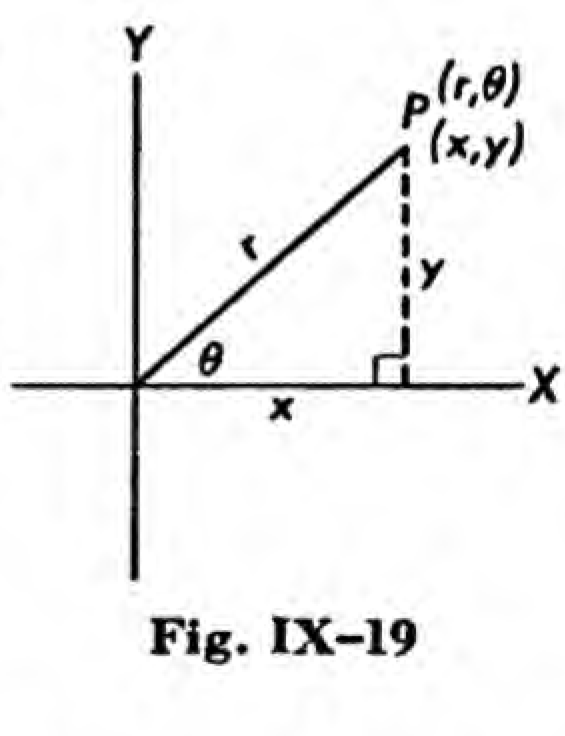


Fig. IX–19

$$x = r\cos\theta \qquad \frac{y}{x} = \tan\theta$$

$$y = r\sin\theta \qquad \theta = \arc\tan\frac{y}{x}$$

$$x^2 + y^2 = r^2$$

Let us employ some of these formulas immediately by converting the equation of the four-leaf rose to rectangular coordinate form.

$$\begin{aligned} r &= 3\sin 2\theta \\ r &= 6\sin\theta\cos\theta \qquad \text{(sub. for } \sin 2\theta) \\ r &= 6\left(\frac{y}{r}\right)\left(\frac{x}{r}\right) \\ r^3 &= 6xy \\ (x^2 + y^2)^{3/2} &= 6xy \end{aligned}$$

The reader may find some difficulty in reaching this equation directly from the initial locus. Certainly, it is easier to graph the equation in the new system.

Earlier we examined a locus set up from two viewpoints. By polar coordinates (this section) we got $r = 2(1 + \theta)$, and by parametric equations (last section) $x = 2(1 + \omega) \cos \omega$, $y = 2(1 + \omega) \sin \omega$. We can show the equivalence of the results. Suppose that we start by squaring the last two equations for reasons that will become immediately apparent.

$$\begin{aligned} x^2 &= 4(1+\omega)^2 \cos^2 \omega \\ y^2 &= 4(1+\omega)^2 \sin^2 \omega \\ \hline x^2 + y^2 &= 4(1+\omega)^2(\cos^2 \omega + \sin^2 \omega) \\ r^2 &= 4(1+\omega)^2 \\ r &= 2(1+\omega) \end{aligned}$$

Since ω in the original parametric set-up is the same as θ in polar coordinates, the last equation is equivalent to the preceding polar form. Should one desire, the last equation can be stated in rectangular form:

$$\sqrt{x^2 + y^2} = 2\left(1 + \arctan \frac{y}{x}\right)$$

EXERCISES (IX–2)

1. Show that

$$\frac{6}{\tan \theta + \cot \theta} = 3 \sin 2\theta$$

2. Sketch the following points in polar coordinates:
   a. $(3, 45°)$, $(2, 120°)$, $(3, 0°)$, $(1, 270°)$
   b. $(-2, 60°)$, $(-3, 90°)$, $(-1, 315°)$, $(-1, 360°)$
   c. $(2, -50°)$, $(-3, -160°)$, $(-4, -270°)$, $(5, -180°)$
   d. $\left(3, \frac{\pi}{4}\right)$, $\left(-2, -\frac{3\pi}{4}\right)$, $\left(11, \frac{2\pi}{3}\right)$

3. For each of the following coordinates, give two other sets of coordinates that would identify the same point:

   a. $(2, 40°)$ b. $(-3, 150°)$ c. $\left(-2, \frac{\pi}{5}\right)$ d. (k, α)

4. $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are two distinct points in polar coordinates.
   a. Show that the distances between them is given by

$$s^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$$

   b. Show that the area of $\triangle OP_1P_2$ is given by

$$K = \tfrac{1}{2}r_1r_2 \sin(\theta_1 - \theta_2)$$

5. The accompanying diagram (Fig. IX–20) recalls the definition of the parabola. This time the parabola is being referred to polar coordinates, where the

pole is taken at the focal point of the parabola and the polar axis coincides with the axis of the parabola. Show that

$$r = \frac{2p}{1 - \cos\theta}$$

6. In polar coordinates, the general equation of a conic is

$$r = \frac{2ep}{1 - e\cos\theta}$$

where e = eccentricity = OP/PQ.

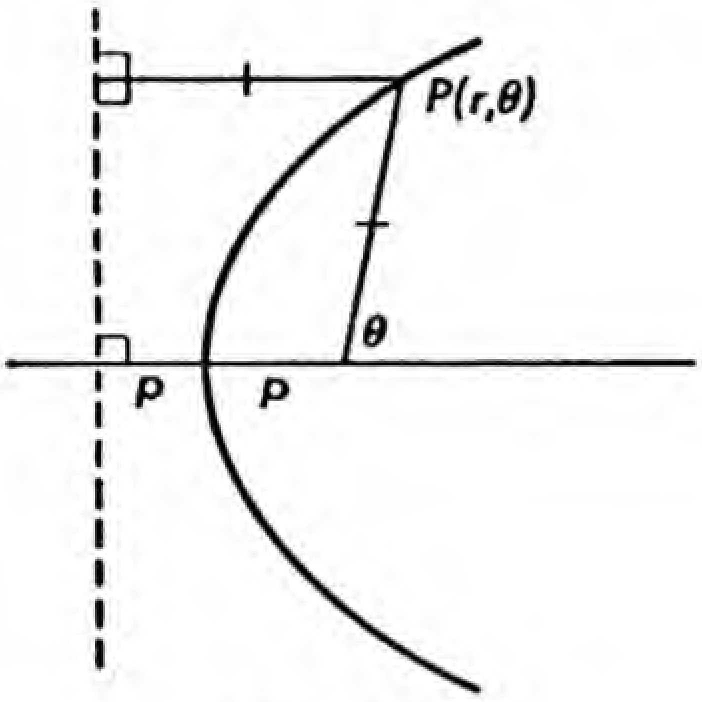


Fig. IX–20

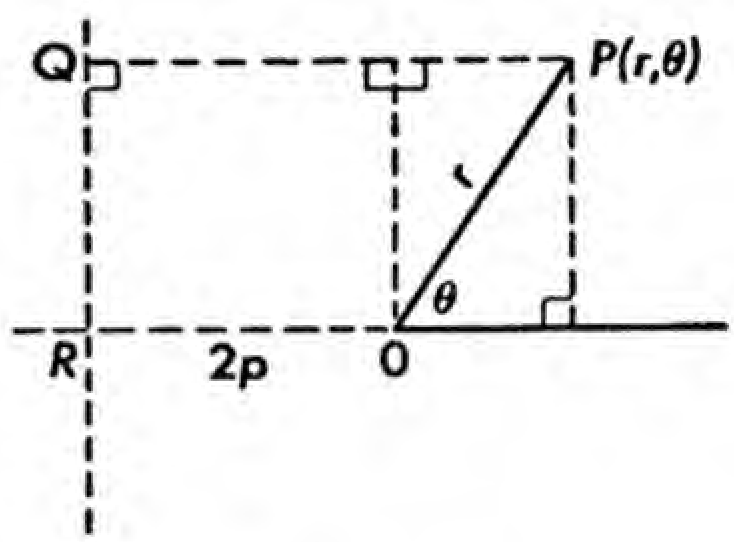


Fig. IX–21

Develop this equation. The rectangles shown in the diagram (Fig. IX–21) ought to be helpful. The locus is a parabola when $e = 1$ as in exercise 5. When e is less than 1, the locus is an ellipse, and when e is greater than 1, the locus is a hyperbola.

7. a. A point $P(r, \theta)$ moves so that the product of its distances from A and B (Fig. IX–22) is always a^2. Show that the equation of the locus is the *lemniscate:*

$$r^2 = 2a^2(2\cos^2\theta - 1)$$

or

$$r^2 = 2a^2\cos 2\theta$$

b. Sketch the curve.

8. A point moves so that its radius vector is proportional to its vectorial angle (Spiral of Archimedes).

a. Write the equation.

b. Sketch the curve.

9. A point moves so that its radius vector is inversely proportional to the vectorial angle.

a. Write the equation.

b. Sketch the curve.

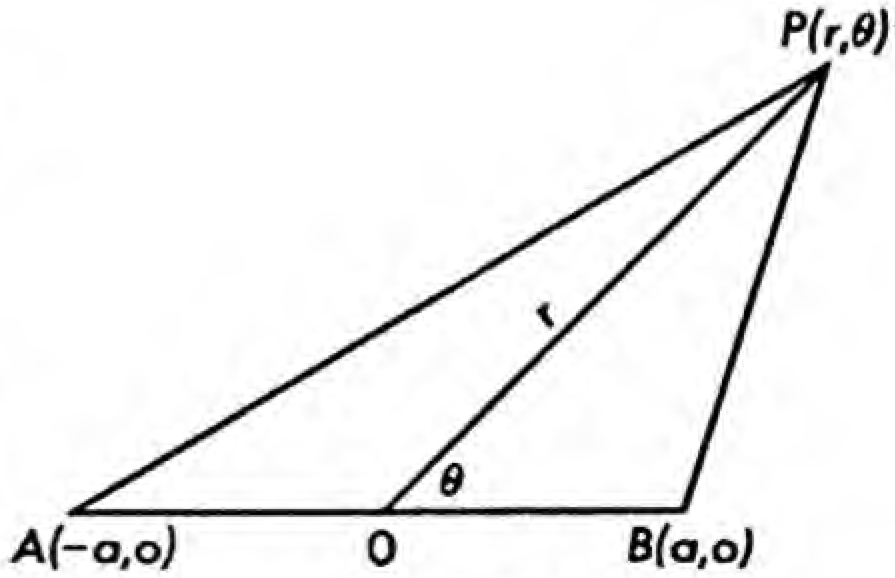


Fig. IX–22

10. Any chord OB is drawn in a circle of diameter $2a$ and is extended through B to $P(r, \theta)$ so that $PB = 2a$ (Fig. IX–23).

a. Find the equation of the locus (Carioid).

b. Sketch the locus.

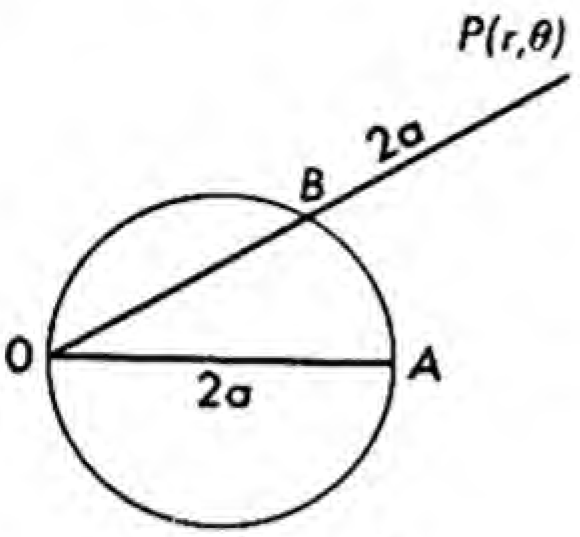


Fig. IX–23

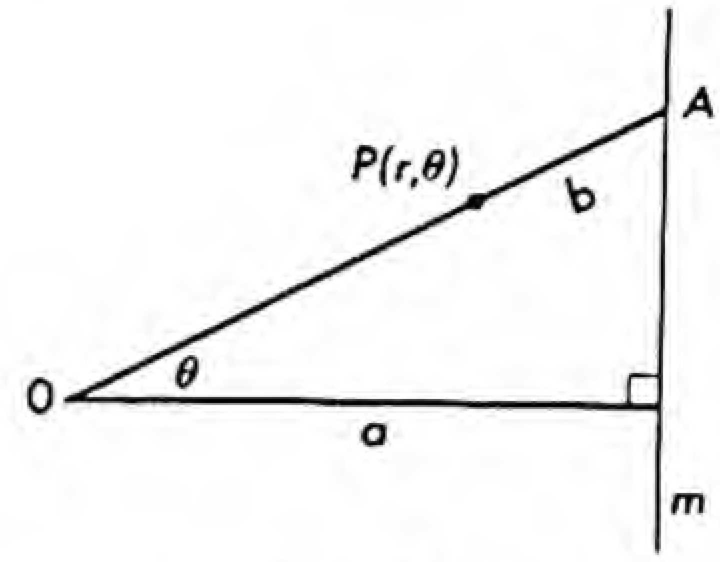


Fig. IX–24

11. $P(r, \theta)$ is on AO, b units from A (Fig. IX–24). Line m is a units from O Find the equation of the locus of P and sketch the curve.

12. Sketch each of the following:

a. $r = 2 - 2 \cos \theta$

b. $r = 2 + 2 \cos \theta$

c. $r = 3 \cos 2\theta$

d. $r^2 = 3 \cos 2\theta$

e. $r = 5 \sin 3\theta$

f. $r = 5 \cos 3\theta$

g. $r = 3 + 6 \sin \theta$ (limacon)

h. $r = 6 + 3 \sin \theta$

i. $r = 4 + 6 \cos \theta$

j. $\theta = \dfrac{\pi}{4}$

k. $r = 5$

13. Transform to polar equations:

a. $x^2 + y^2 - 6x = 0$

b. $(x^2 + y^2)^2 = x^2 - y^2$

c. $x = y$

d. $\sqrt{x^2 + y^2} = \arctan \dfrac{y}{x}$

e. $y^2(8 - x) = x^3$

14. Write the following in rectangular coordinate form:

a. $r = 3 \cos \theta$

b. $r = a \sin \theta$

c. $r = a \sin 2\theta$

d. $r^2 = 4 \cos 2\theta$

e. $r = \dfrac{6}{1 - 2 \cos \theta}$

f. $r = 4 + 4 \sin \theta$

g. $r = 3 \sin \theta + \cos \theta$

h. $r^2 = 6 \cos 2\theta$

3. NEW SPACE COORDINATES

As one may easily suspect, the two-dimensional variants in coordinate systems have their counterparts in three dimensions. There is the kind called **cylindrical coordinates,** where we start with polar coordinates in

the XY plane and tack on a z-coordinate perpendicular to that plane for the third dimension. Thus (Fig. IX–25) $r = 4$, $\theta = 40°$, and $z = 5$ is written as (4, 40°, 5).

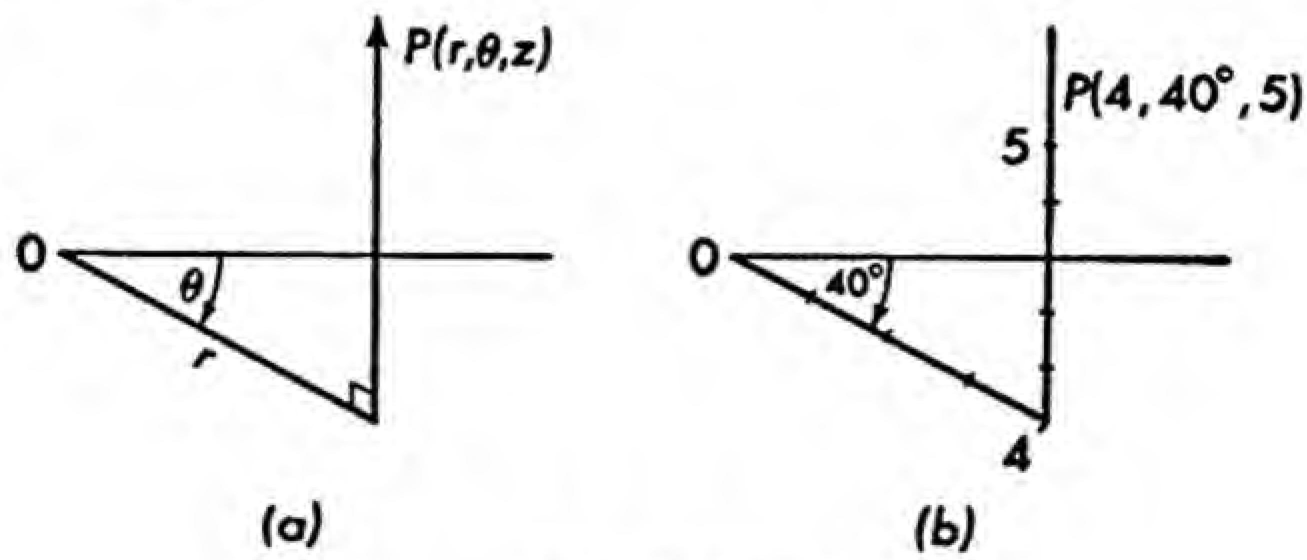


Fig. IX–25

This system is advantageous in describing surfaces that are symmetrical about some line which is taken parallel or coincident with the z direction. The equation of the cylinder, (Fig. IX–26) is simply $r = 5$. This describes every point on the surface for any values of θ and z.

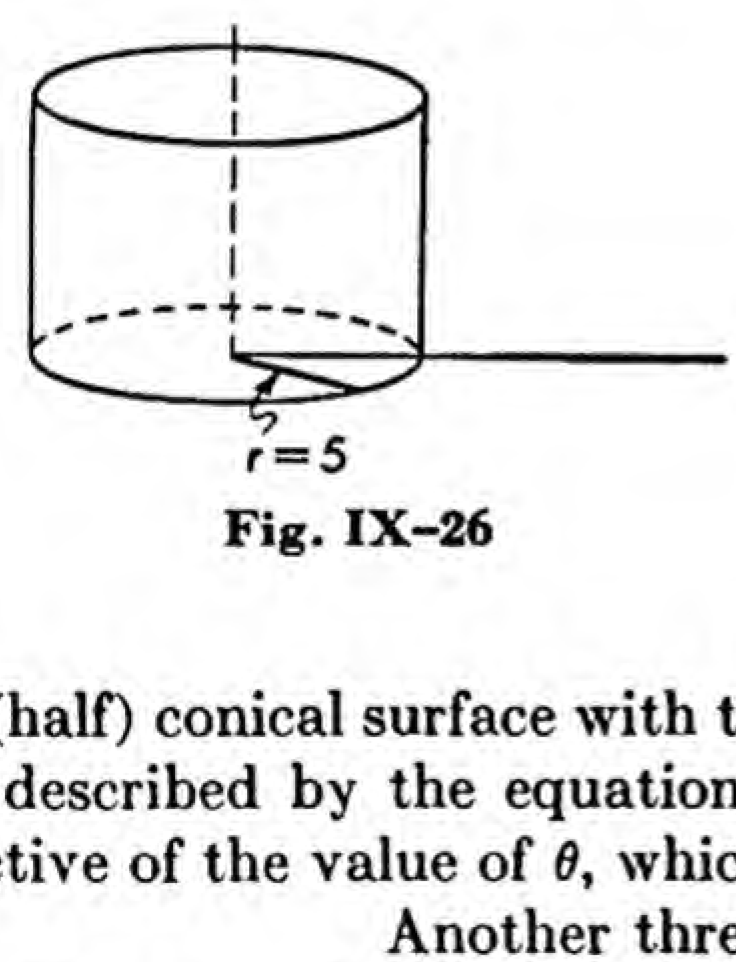


Fig. IX–26

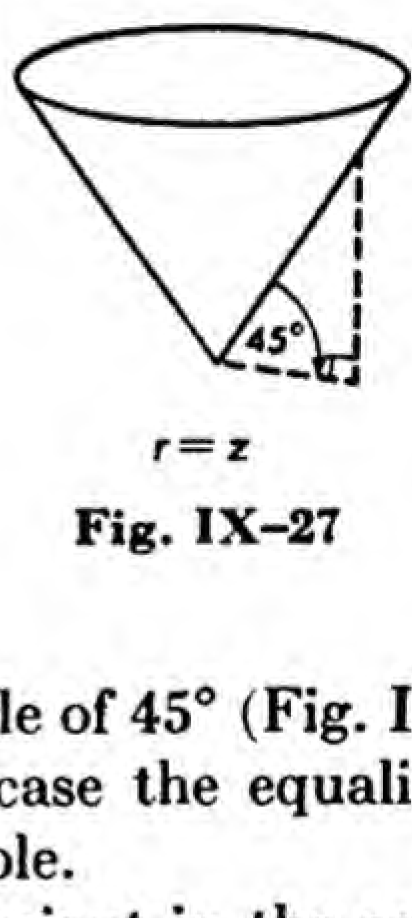


Fig. IX–27

The (half) conical surface with the indicated angle of 45° (Fig. IX–27) is simply described by the equation $r = z$. In this case the equality holds irrespective of the value of θ, which is a free variable.

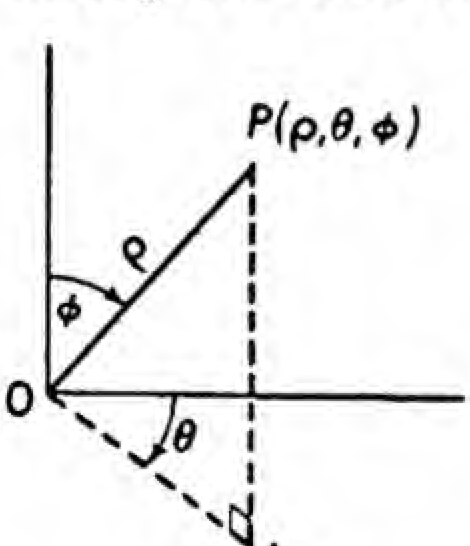


Fig. IX–28

Another three-dimensional variant in the coordinate system is the **spherical coordinate system.** The θ coordinate of the polar system is essentially the same as before excepting that OA (Fig. IX–28) is the projection of the radius vector $OP = \rho$. The third coordinate is ϕ (phi), the angle OP makes with the Z-axis. This system is particularly useful when dealing with surfaces that are symmetric about some point. For example, the sphere with radius 5 and center at O has as its equation $\rho = 5$, since $(5, \theta, \phi)$ satisfies the equation and represents the coordinates of any point on the sphere.

EXERCISES (IX-3)

1. Show that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ represent conversion formulas between cylindrical and rectangular coordinates. Indicate two other conversion formulas that might be used.

2. Show that $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ holds between spherical and rectangular coordinates.

3. Describe the locus of points $P(r, \theta, z)$ whose cylindrical coordinates satisfy the following:

a. $(r = 3) \wedge (z = 4)$

b. $(r = 3) \wedge (z = \theta)$

c. $(z = 3) \wedge \left(\theta = \frac{\pi}{4}\right)$

4. Describe the loci of the following:

a. $(\rho = 3) \wedge \left(\theta = \frac{\pi}{3}\right)$

b. $(\rho = 3) \wedge \left(\phi = \frac{\pi}{3}\right)$

c. $\left(\theta = \frac{\pi}{4}\right) \wedge \left(\phi = \frac{\pi}{3}\right)$

5. Transform each of the following from the given coordinate system (Cartesian, cylindrical, or spherical) to each of the other systems:

a. $x^2 + y^2 = 2pz$

b. $x^2 + y^2 + 9z^2 = 16$

c. $z^2 - 4r^2 = 16$

d. $r = -4\sin\theta$

e. $\rho\sin^2\phi = 4\cos\phi$

f. $\rho^2\cos 2\phi = 9$

IX-3 REVIEW

1. What legend must accompany the equation in rectangular coordinates when the parameter is eliminated from $x = 3[t]$, $y = [t] + 2$?

2. a. Graph $x = |t|$, $y = t^2$.

   b. Define the same equation, explicitly y, in terms of x.

3. Eliminate the parameter and identify the curve $x = a\sec t$, $y = b\tan t$.

4. Sketch the graph of $x = \theta$, $y = 1 - \cos\theta$.

5. Graph $x = \cos^3\theta$, $y = \sin^3\theta$, for $0 \le \theta \le 2\pi$ in intervals of 30° (Hypocycloid).

6. Determine an equation in x and y for each of the following:

a. $y = 3t$, $x = 5t + 2$

b. $x = 4t$, $y = 2t^2$

c. $x = 4\sin\theta$, $y = 3\sin\theta$

d. $x = \frac{1}{t}$, $y = \frac{1}{t^2}$

e. $x = 3\cot\theta$, $y = 2\csc\theta$

7. Find the distance between $(3, \frac{7}{6}\pi)$ and $(3, \frac{1}{6}\pi)$.

8. Write in rectangular form the polar equation $r = a$.

9. Plot each of the following:

a. $r = \cos^2\theta$

b. $r = 2\csc 2\theta$

c. $r = a\sin\frac{1}{2}\theta$

d. $r = 2 - \sin\frac{3}{2}\theta$

10. Find the polar equation for $2y^2 = x^3$.
11. Find the rectangular equivalent for the polar equation $r^2 \cos 2\theta = 2$.
12. If we perform a rotation of axes by an angle α, then any $P(r, \theta)$ becomes $P'(r', \theta')$ where $r = r'$ and $\theta = \theta' + \alpha$.

   a. Show that by a rotation of $\alpha = \left(\frac{\pi}{2}\right)$, the equation

$$r = a(1 + \sin \theta) \text{ becomes } r = a(1 + \cos \theta')$$

   b. Transform the equation $r = a(1 + \sin \theta)$ by a rotation of $\alpha = \frac{3}{2}\pi$.

13. Rotate the polar axis as indicated and find transformed equations:

   a. $r \sin \theta = 4;\ \alpha = \dfrac{3}{2}\pi$

   b $r \cos \theta = 2;\ \alpha = -\dfrac{1}{2}\pi$

   c. $r = 4 \sin 2\theta;\ \alpha = \dfrac{\pi}{4}$

14. Represent the rectangular coordinates (1, 2, 2) in:

   a. Spherical coordinates

   b. Cylindrical coordinates

15. If $\rho = 4$, $\theta = \pi/4$, and $\phi = \pi/3$, represent this spherical coordinate point in rectangular coordinates.

16. Write the following in spherical coordinates:

   a. $x^2 + y^2 = 16$

   b. $x^2 + y^2 = 2z$

X

THE DERIVATIVE

1. A MAJOR BREAKTHROUGH

We have met a great many functions and they have served us in many ways. Yet we have made only a small dent in the analysis of a function.

Consider the points P, Q, and R (Fig. X–1) in the graphic representation of some function f. We assume that the points P, Q, and R lie within an interval over which the function is defined. Suppose that we progress among the elements of the function in a positive sense with respect to the independent variable.

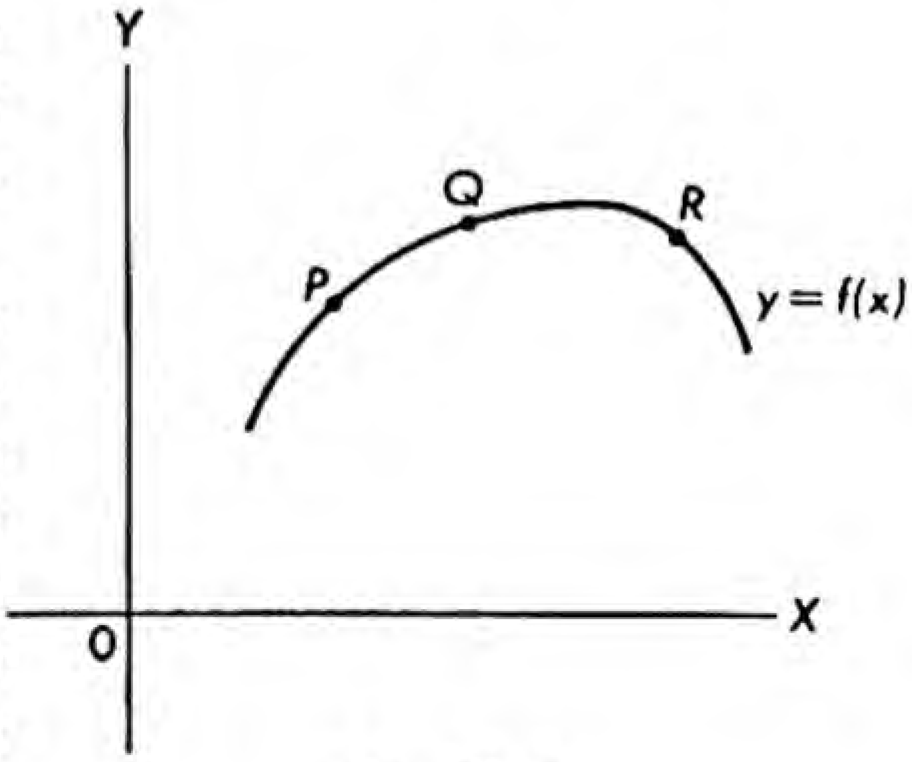


Fig. X–1

Intuitively it seems that P is at the beginning of an interval where the function values are increasing, and R initiates an interval where the function values are decreasing. On the other hand, Q appears to introduce an interval of slower change in the function values. There seems to reside in these observations some quantitative aspects which may be subject to precise mathematical definition. Indeed we are again on the verge of one of the basic concepts of the calculus. The calculus is a landmark in the development of mathematics for which modern technology and science as well as mathematics are indebted to Newton and Leibniz.

As soon as we begin to contemplate this matter of change more closely,

we recognize that as we progress from P the curve may oscillate considerably, as suggested by the wavy construction in Fig. X–2.

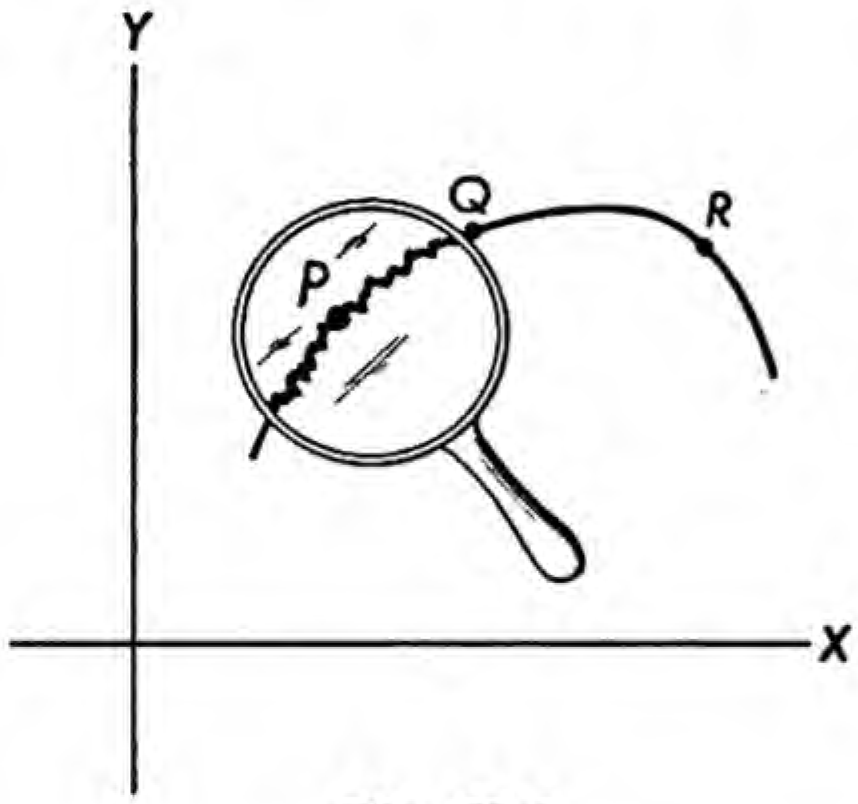


Fig. X–2

Well, this suggests that to get a meaningful measure of the change of a function, we can attempt to reduce the interval of attention in order to eliminate some of the oscillation. Unfortunately this is unavailing. In the interval about P, no matter now small, there is in the continuum an infinite number of members of the function. Well, this is just the kind of situation that is best studied by means of the limit concept.

Let us pursue these thoughts in a concrete instance, with $f:\{x, x^2\}$ at $P(3, 9)$ in Fig. X–3.

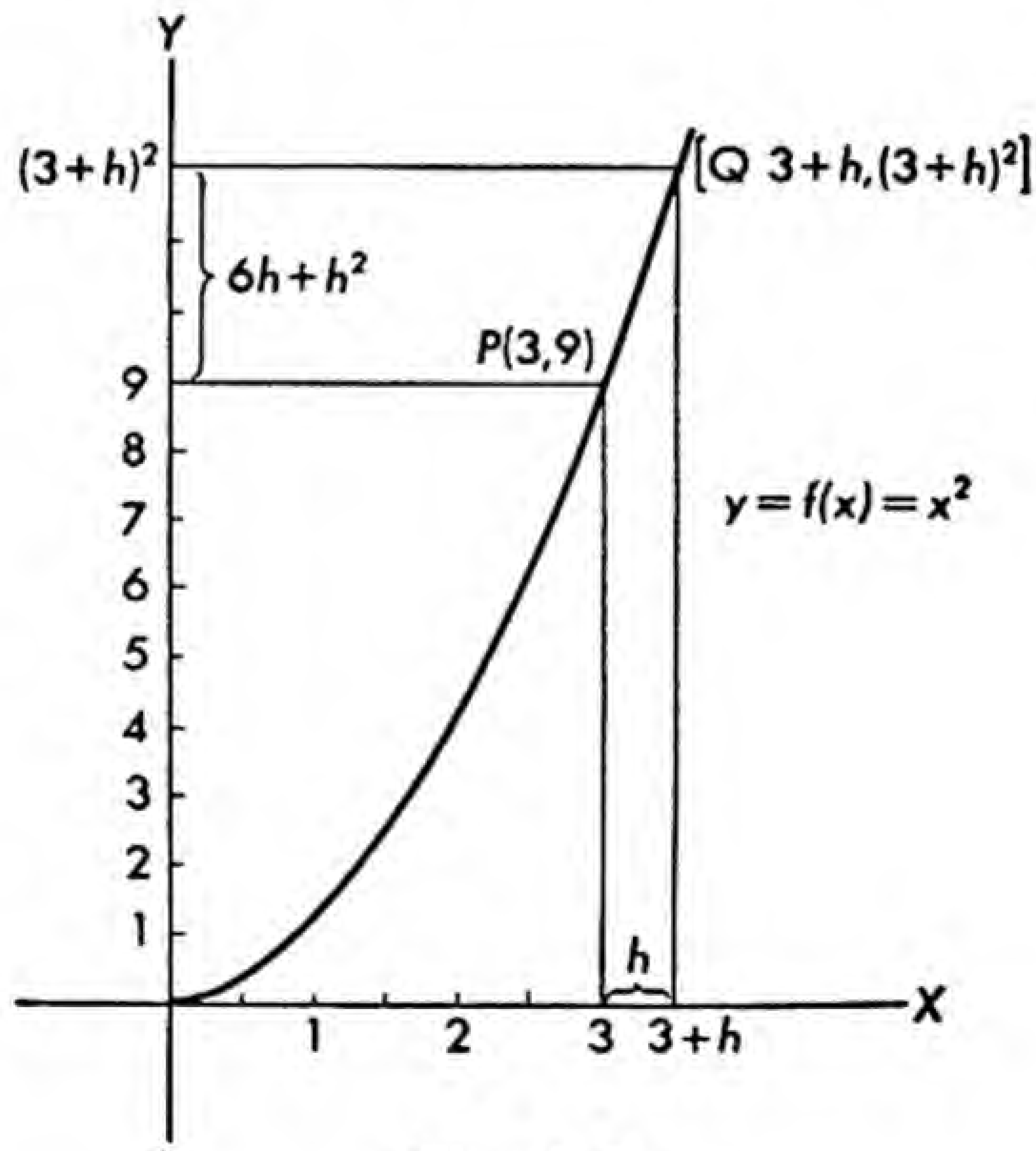


Fig. X–3

We take a nearby point, $Q[3 + h, (3 + h)^2]$. If we let Δy represent the *difference in y-values* in the interval and Δx the *difference in x-values*, we have

$$\Delta y = (3 + h)^2 - 9 = 6h + h^2$$
$$\Delta x = h$$

If we permit $h \to 0$, as indicated in our plan, both limit differences disappear, for $\lim \Delta y = 0$ and $\lim \Delta x = 0$. However, our experience with limits suggests the possibility that the limit of the ratio of the differences may not be 0, infinite, or indeterminate. Thus, in fact,

$$\lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim \frac{6h + h^2}{h} = \lim (6 + h) = 6$$

As with other comparisons (the speed of travel, for instance), the fraction may be read as a *rate of change*. In the case of speed, the rate is usually taken with respect to 1 hour, 1 minute, or 1 second. Now, and this is most significant, we are considering a rate of change at a single point. *At* P the rate of change in y with respect to x is 6.

It would be well to try this again with $f:\{x, x^2\}$ at a different element of the set, $P(2, 4)$. Again, we take a nearby point, $Q[2 + h, (2 + h)^2]$. Then,

$$\Delta y = (2 + h)^2 - 4 = 4h + h^2$$

and

$$\Delta x = h$$

We take the limit of the quotient of these two quantities.

$$\lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim \frac{4h + h^2}{h} = \lim (4 + h) = 4$$

At $P(2, 4)$ the rate of change in y with respect to x is 4.

The rate of change at a point is often referred to as the *instantaneous rate of change*, which is suggestive of the intent although the time factor is not necessarily in the picture.

It takes little effort to generalize the previous study. Let P be a variable point $P(x, y)$ for the function f defined by the equation $y = x^2$. The coordinates of the nearby point Q are $Q(x, x + h)$. Then,

$$\Delta y = (x + h)^2 - x^2 = 2xh + h^2$$

and

$$\Delta x = h$$

$$\lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim \frac{2xh + h^2}{h} = \lim (2x + h) = 2x$$

Thus, for the element (x, y), which represents any element of the set $\{x, x^2\}$, the rate of change of the function at any point in the interval of definition is $2x$, twice the value of the independent variable. The rate of change is 2 for $x = 1$, 10 for $x = 5$, -8 for $x = -4$, and 0 for $x = 0$.

As may be expected, one would want a succinct symbol for this limiting value, when it exists. Actually we are plagued with a plethora of symbols:

$$D_x y,\ \frac{dy}{dx},\ y',\ f'(x),\ Df(x),\ \frac{df(x)}{dx}$$

any of which represents the limit

$$\lim_{h \to 0} \frac{\Delta v}{\Delta x}$$

for any member (x, y) of the set $\{x, y\}$, which is some function f whose defining equation is $y = f(x)$. These are all read summarily by the phrase **the derivative of y with respect to x.**

EXERCISES (X–1)

1. Show that the rate of change of y with respect to x at $x = a$ for $y = mx + b$ is m.
2. Show that for $y = \frac{2}{3}x + 1$, the ratio of the y change to x change at $x = 1$ is $\frac{2}{3}$.
3. Consider the function defined by the equation $y = 1/x$. Show that $dy/dx = -(1/x^2)$ for $x \neq 0$.
4. Find the value of D_xy for each of following:
   a. $y = 3x^2 - 2x$
   b. $y = 5x + 2$
   c. $y = x + \frac{1}{x}$, $x \neq 0$
   d. $y = (x + 1)^2$
5. a. Show that the derivative of $y = 1/x^2$ is $-2/x^3$ for $x \neq 0$.
   b. Find dy/dx when $x = 2$.
6. Find the derivative of the function given by $\{x, x^3\}$.
7. Take the function defined by the equation $y = mx + b + (1/x)$. This is a sum of the function values in exercises 1 and 3. In the light of our knowledge of continuous functions and limits, why would $dy/dx = m - 1/x^2$?

2. ANOTHER VIEW

A geometric interpretation of the derivative adds an additional dimension to a very important and subtle concept. The ratio of intervals that we have been getting is equivalent to finding some ratio such as QR/PR in Fig. X–4. We should recognize this ratio as nothing more than the slope of the secant PQ.

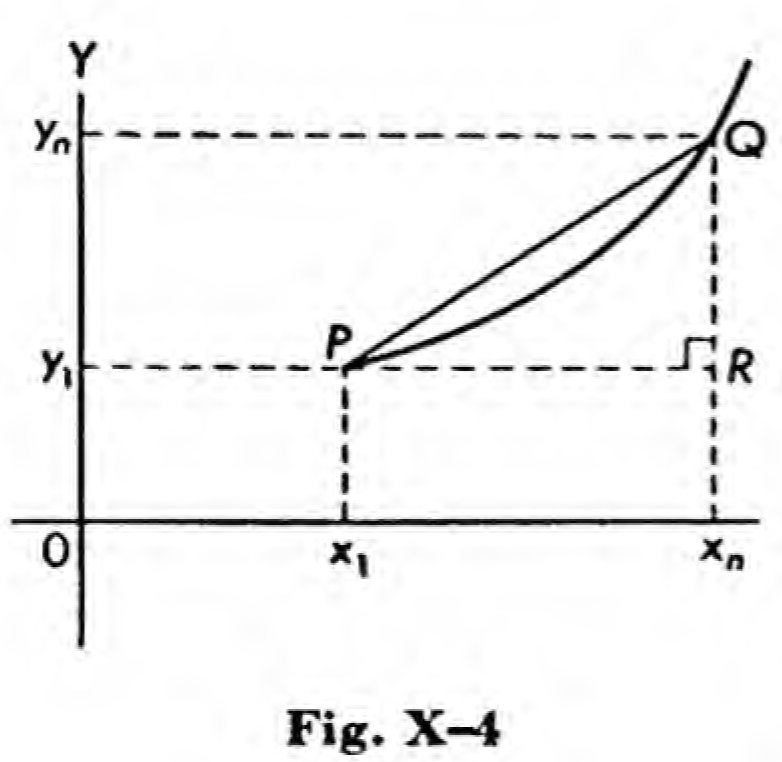


Fig. X–4

As the interval along the arc PQ gets smaller, Q approaches P along the curve. If the secant approaches a limiting position, we define that as the *tangent to the curve* at P. The slope of the secant becomes, in the limit, the slope of the tangent. Thus, *the derivative, where it exists, is the slope of the tangent to the curve.*

Take any function (f) in general, whose ordered pairs are determined by some explicit equation $y = f(x)$. We take two members of the set $\{x, y\}$. One is (x, y) and the other is a nearby pair determined by some interval h about x. The second pair may then be represented by $[x + h, f(x + h)]$. The x-interval between these points is just h, and the corresponding y-interval is $f(x + h) - f(x)$ in Fig. X–5. The derivative is defined as the limit of the ratio of these intervals, with the y-interval appearing in the numerator.

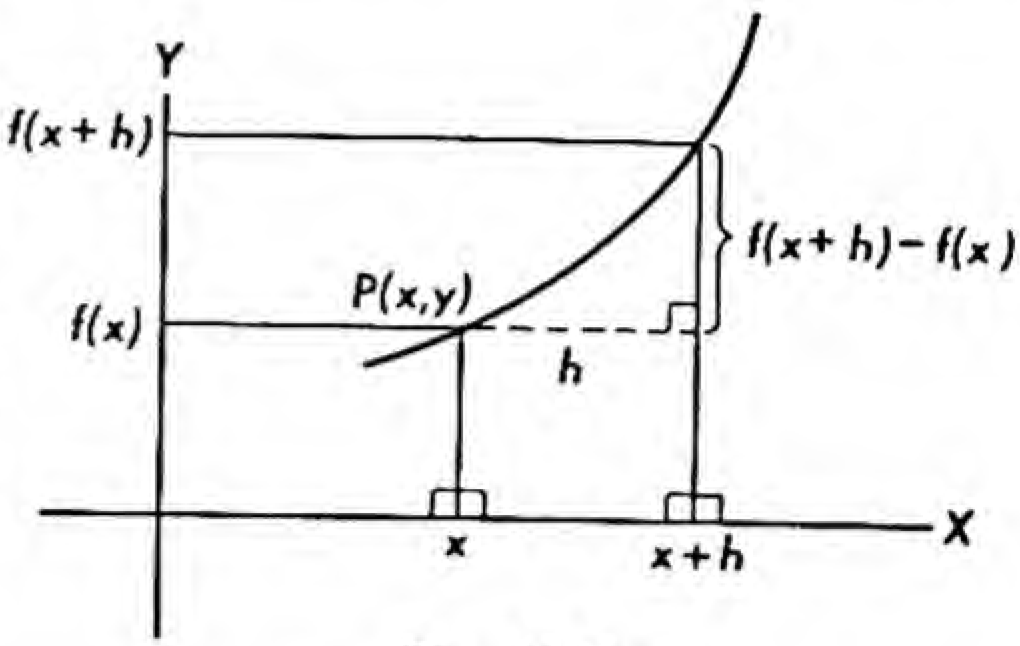


Fig. X–5

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

If the limit exists, we say that the function has a derivative at the point where the abscissa is x; and, also, that the function is **differentiable** at x. The symbol $\boldsymbol{dy/dx}$ for the limit is a single entity, and contrary to appearances, is neither a quotient nor a product of factors.

It is very important to note that the definition of the derivative mandates the conditions necessary for continuity.

Suppose that for some $y = f(x)$ the derivative exists at $x = x'$. This means that dy/dx has some finite value at $x = x'$. Now,

$$\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x$$

and so

$$\lim_{h\to 0} \Delta y = \lim \frac{\Delta y}{\Delta x} \cdot \lim \Delta x = \frac{dy}{dx} \cdot 0 = 0$$

We take note of the fact that $\Delta y = f(x' + h) - f(x')$, and so, by the previous line, we have shown that the limit of this difference is zero. That is,

$$\lim_{h\to 0} [f(x' + h) - f(x')] = 0$$

which is the same as

$$\lim_{h\to 0} f(x' + h) = \lim f(x') = f(x')$$

which proves that $f(x)$ is continuous at x' by our definition of continuity.

Thus, differentiability implies continuity. The converse is not true. It has been known for quite some time that curves can be continuous at isolated points but have no derivatives there. We shall meet instances of

this. Riemann and Weierstrass both constructed curves that are everywhere continuous and nowhere differentiable.

From the foregoing discussion, we see that the derivative of x^2 is $2x$. We try to find the derivative of x^3 and start with the equation $y = f(x) = x^3$.

$$f(x + h) = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim \frac{3x^2h + 3xh^2 + h^3}{h} = \lim (3x^2 + 3xh + h^2)$$

$$\frac{dy}{dx} = \lim (3x^2) + \lim (3xh) + \lim h^2$$

Of the three limits, the quantity in the first, $3x^2$, is independent of h, and so is unaffected by $h \to 0$. The other limits approach 0. Thus, for $y = x^3$,

$$\frac{dy}{dx} = 3x^2$$

Once again the result is dependent on the value of x. For $x = 2$, the derivative is 12 and so forth. The value of 12 also represents the slope of the tangent to the curve at the point where $x = 2$.

From a broader view we have a newly *derived* function $dy/dx = 3x^2$, which is dependent on x. We may call the function f', whose ordered pairs are indicated by $\{x, dy/dx\}$, where dy/dx is defined by the equation $dy/dx = 3x^2$. This, of course, may be expressed in any of the other ways mentioned earlier, as in $f':\{x, y'\}$, where $y' = 3x^2$.

It may have been noticed that the last case and an earlier one are suggestive of a general rule. This can be investigated and determined all at once by starting with the general equation $y = x^n$ for positive integral values of n.

$$f(x + h) = (x + h)^n = x^n + nx^{n-1}h + \frac{n(n - 1)}{2}x^{n-2}h^2 + \cdots + h^n$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n - 1)}{2}x^{n-2}h + \cdots + h^{n-1} \right]$$

As $h \to 0$, the limit of each term (except the first) in the parentheses approaches 0. We have, then,

$$\frac{dy}{dx} = nx^{n-1}$$

Consequently the derivative of x^7 is $7x^6$, and the derivative of x^{20} is $20x^{19}$.

It is important for us to consider the derivative of a constant. If $y = f(x) = c$, then $f(x + h) = c$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{c - c}{h} = \lim \frac{0}{h} = 0$$

The derivative of a constant is 0.

EXERCISES (X-2)

1. Express the value of $f(x+h) - f(x)$ if
   a. $f(x) = x^5$
   b. $f(x) = 3x^2$
   c. $f(x) = \sqrt{x}$
   d. $f(x) = x + x^2$
   e. $f(x) = x^m + x^n$
2. Express the quotient

$$\frac{f(x+h) - f(x)}{h}$$

in simplest form for:
   a. $y = x^4$
   b. $y = 5x^2$
   c. $y = \dfrac{4}{x^2}$
   d. $y = \sqrt{x}$ (rationalize numerator)
   e. $y = \dfrac{1}{\sqrt{x}}$
3. Find $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for:
   a. $y = 5x^3$
   b. $y = \dfrac{1}{x}$
   c. $y = \dfrac{4}{x^2}$ at $(2, 4)$
   d. $y = \dfrac{1}{x^2}$ at $(2, \tfrac{1}{4})$
   e. $y = \sqrt{x}$ at $x = 4$
   f. $y = 1/\sqrt{x}$ at $y = \tfrac{1}{2}$
4. f is defined by the set $\{x, x^n\}$ and f' by $\{x, nx^{n-1}\}$. What would be the defining set for $(f')'$ and also of $[(f')']'$? The latter cases are written simply as f'' and f''' and are read as **the second and third derivatives,** respectively.
5. Find the second and third derivatives of x^6.

3. THE DERIVATIVE OF cx^n

We have formally developed the derivative of x^n for positive integral values of n. We carry this a step further by introducing a nonzero coefficient. We take $\boldsymbol{y = cx^n}$.

$$y' = \lim_{h \to 0} \frac{c(x+h)^n - cx^n}{h} = \lim c \lim \frac{(x+h)^n - x^n}{h}$$

$$y = cnx^{n-1}$$

Consequently the coefficient c remains as a coefficient for the derived function. We can highlight this by using the function notation and D as a symbol for the derivative. Thus, in the earlier case, we had

$$f' = Df$$

and now

$$D(cf) = c\,Df = cf'$$

It must be emphasized that in the notation Df, it is understood that the derivative is taken of the dependent variable with respect to the independent variable of the function f. The fuller and bulkier symbol would be $D_xf(x)$, which should be unnecessary by the stated convention.

EXERCISES (X-3)

1. Find the derivative of each of the following:

a. $y = 5x^4$ d. $M = 16w^3$
b. $y = -4x^3$ e. $v = 16t$
c. $h = 6t^2$

2. Find, where possible, the second and third derivatives of the following:

a. $y = 0.2x^5$ c. $y = 6x$
b. $A = 2\pi x^2$ d. $C = 2\pi x$

4. INTRODUCTORY APPLICATIONS

While, as is frequently the case, significant applications must await a fuller development of the theory, we may venture one or two. These, to be sure, will give only a limited glimpse of the great value of the derivative.

A simple and fairly obvious application has to do with the finding of the equation of a tangent to a curve at some point. For example, we take $y = x^4$ and the point (2, 16). Since the slope of the tangent and the derivative at the same point are equivalent, we have

$$m = \frac{dy}{dx} = 4x^3$$

at any point on the curve, and

$$m = 4(2^3) = 32 \qquad \text{at } (2, 16)$$

By the point slope formula of the straight line, which is

$$y - y_1 = m(x - x_1)$$

we have

$$y - 16 = 32(x - 2)$$

or

$$y = 32x - 48$$

By way of another application, consider the fact that the path of a point is described by the equation $s = 16t^2$, where t is in units of time (seconds) and s is in units of distance (feet). What is the **instantaneous** speed

when $t = 2\frac{1}{4}$ seconds? We saw that the derivative corresponds to the rate of change at a single point, rather than over a whole interval. This is precisely what we want here. Ordinary, average speed is the ratio of a distance interval to a time interval. We are in the position to get a *point* speed, an instantaneous speed.

$$s = 16t^2$$
$$\frac{ds}{dt} = 32t$$

for any t, for which the function is defined, and

$$\frac{ds}{dt} = \textbf{instantaneous speed}$$
$$= 32(2\tfrac{1}{4}) = 72 \text{ ft/sec at } t = 2\tfrac{1}{4}.$$

EXERCISES (X–4)

1. Find the slope of the curve at the indicated value:

a. $y = x^5, x = 1$ c. $y = \frac{1}{x^2}, x = 3$

b. $y = 3x^3, x = \frac{1}{3}$ d. $y = \sqrt{x}, x = 2$

2. Find the equation of the tangent to the curve at the indicated value:

a. $y = x^2, (2, 4)$ c. $y = \frac{1}{x}, x = -2$

b. $y = x^3, (-2, -8)$ d. $y = \frac{4}{\sqrt{x}}, x = 4$

3. The curve $y = kx^2$ has a slope of 4 at the point where the abscissa is 8. Find the value of k.

4. The distance s, in feet, that a freely falling body will cover in time t, in seconds, is given by $s = 16t^2$. Find the instantaneous velocity when $t = 1$, 3, and 8.

5. Represent each of the following symbolically:
   a. *Instantaneous power* which is the rate of change of work, W, done with respect to time.
   b. *Instantaneous angular velocity* is the instantaneous rate of change of angular displacement θ.
   c. *Instantaneous angular acceleration* which is the rate of change of angular velocity ω.

5. DERIVATIVE OF SUM OF TWO FUNCTIONS

It is not too difficult to anticipate the need for taking the derivatives of a polynomial function. Suppose that

$$y = f(x) + g(x)$$

where f and g are differentiable functions. What is the value of y'? We may be guided by the definition of the derivative. We have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\end{aligned}$$

We have used the theorem which states that the limit of a sum is equal to the sum of the limits of two or more functions. Thus

$$\frac{dy}{dx}=\frac{df(x)}{dx}+\frac{dg(x)}{dx}$$

or, more succinctly,

$$D(f+g)=Df+Dg$$

The derivative of the sum of two functions is equal to the sum of the derivatives of the functions. This could be extended inductively to any number of functions.

If $\qquad h = 4m^3 + 5m^2 + 6m + 9$

then $\qquad \dfrac{dh}{dm} = 12m^2 + 10m + 6$

Again, if $\qquad v = 6u^2 - 3u - 6$

then $\qquad \dfrac{dv}{du} = 12u - 3$

or $\qquad Dv = 12u - 3$

EXERCISES (X-5)

1. Each of the following represents a law of motion of a particle for $t > 0$. In each case find the equation that defines the function of the instantaneous speed:

a. $s = 5t + 1$ b. $s = 6t^2 - 10t + 4$

c. $s = \dfrac{1}{t} \quad (t \neq 0)$ d. $s = 12 - 7t$

2. Find the point(s) on the graph of $y = x^3 - 3x^2$ at which the slope of the tangent is (a) 9, (b) 0.

3. A body is thrown vertically upward in the air. The height of the body above the ground at any instant is given by

$$s = 48t - 16t^2 \qquad (0 \leq t \leq 3)$$

a. Determine the equation expressing the velocity of the body at any instant.
b. Find the velocity at $t = 1$ and $t = 2$.
c. Explain or interpret the answers in (b).
d. At what instant will the velocity be 0? Interpret.
e. Find the velocity of the particle when it strikes the ground.

4. The equation in the preceding example is a particular case of the general formula

$$s = v_0t - \tfrac{1}{2}gt^2$$

for the height of a body that is thrown vertically. v_0 is the initial velocity, and $g = 32$ feet per second per second is the acceleration due to gravity. Write the general equation of the instantaneous velocity v.

5. Should a projectile be fired at an angle θ with the horizontal, the vertical height s is given by

$$s = (v_0 \sin \theta)\, t - \tfrac{1}{2}gt^2$$

a. Write the equation for the instantaneous velocity.
b. If $v_0 = 1200$ feet per second, $\theta = 30°$, and $g = 32$ feet per second per second, find the instantaneous velocity $1\frac{1}{4}$ seconds after firing.
c. The velocity of the projectile will be 0 at the highest point in the trajectory. (The projectile is instantaneously at rest before beginning the descent.) After how many seconds will the projectile reach the highest point?
d. Using only the fact that $s = 0$ when the projectile strikes the ground, find the time it takes after firing for the projectile to return to the ground.
e. Interpret the answers of (c) and (d) with respect to each other.
f. Find the velocity of the projectile when it strikes the ground.

6. Find the coordinates of the points on the graph of $y = x^3 - 3x$ where the tangents are
a. Parallel to the X-axis.
b. Parallel to the line $y - 9x + 8 = 0$.

7. Find the points on the graph of $y = 2x^3 - 9x^2 - 24x + 100$ where the tangents are parallel to the X-axis.

8. a. Show how the identity $x^5 = x^3 \cdot x^2$ may be used to illustrate that $D(f \cdot g) \neq Df \cdot Dg$ and to prove that *the derivative of a product of two functions is not necessarily equal to the product of the derivatives of the functions.*
b. Illustrate as in (a) that the derivative of a quotient is not equal to the quotient of the derivatives.

9. Find the derivatives of each of the following expressions:

a. $x^3 + 5$	f. $x^3 - 5x^2$
b. $8x^4$	g. $\dfrac{1}{x} - x$
c. $5x^3 - 3x^2 + 7x - 6$	h. $\dfrac{1}{\sqrt{x}} - \sqrt{x}$
d. $16m^2 - 2m$	i. $t^3 - 3t^2$
e. $T^3 - 4T^2$	j. $\dfrac{5}{m^2} + \dfrac{1}{m}$

10. Consider the function described by $x = x$. Show that $dx/dx = 1$. (Note the consistency with the power law.)

X–5 REVIEW

1. Find the derivative of each of the following:

a. $y = 4x^3 - 3x^2 + x - 1$
b. $s = 2t^2 - 3$
c. $u = 3(2v^2 - 3)^3$
d. $u = \frac{1}{2}(v + 1)^3$
e. $V = \frac{4}{3}\pi r^3$
f. $A = \frac{1}{2}(x + x^2)$
g. $S = 4\pi r^2$

2. Find the equation of the tangent to each of the following curves at the indicated point:

a. $y = 3x^2 - 2x + 1$; (1, 2)
b. $y = 2x^3 - x^2 + 3$; (1, 4)
c. $y = (1 - x)^2$; at $x = 2$

3. Find the equations of lines tangent to $y = x^2 - 3x + 2$: (a) parallel to $2y - x - 3 = 0$; and (b) perpendicular to the same line.

4. Find the equations of the lines tangent to $y = x^3 - 2x + 1$: (a) parallel to $y = x$; and (b) perpendicular to $y = x$.

5. In the equation $s = kt - 16t^2$, if the speed v, is known to be 50 feet per second at $t = 1\frac{1}{2}$ seconds, find the value of k.

6. FUNCTION OF A FUNCTION

Situations such as $y = (x^3 + 5)^4$ will arise. One can use the binomial theorem and then take the derivative term by term. However, this is not only unnecessary but is not as yet always feasible, as in $y = \sqrt[4]{x^3 + 5}$.

Rather let us look at the concept of **function of a function,** also called the **composite of functions.** Specifically, suppose that we let $z = x^3 + 5$ in $y = (x^3 + 5)^4$, getting $y = z^4$. We have developed methods of getting dy/dz and dz/dx. May it not be possible to obtain from these the value of dy/dx?

The three equations in the preceding paragraph define three functions. The representations that follow do not depend on those specific equations which are used only as a guide to the thinking. Let

$y = z^4$	define	$f: \{z, y\}$	or	$y = f(z)$
$z = x^3 + 5$	define	$g: \{x, z\}$	or	$z = g(x)$
$y = (x^3 + 5)^4$	define	$F: \{x, y\}$	or	$y = F(x)$

By substitutions, we also have

$$y = f[g(x)]$$

and

$$F(x) = f[g(x)]$$

So, in function language,

$$F = f[g]$$

The brackets are used to highlight the fact that F is a function of a function. The brackets do not imply any multiplication or greatest integer in this context.

From the definition of continuity, one can deduce that if f is continuous at any point (z, y) and g is continuous at the point (x, z), then F is continuous at the point (x, y). As a result of continuity as $\Delta x \to 0$, we also have $\Delta y \to 0$ and $\Delta z \to 0$. Starting with Δz not 0, we write the identity

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x}$$

and so,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z} \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x}$$

Consequently

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

The form of this rule is responsible for the name of *chain rule of derivatives*. It is preferable to become familiar with it in the more general format of functional notation, that is,

$$Df[g] = Df \cdot Dg$$

The derivative of a function of a function is the product of the derivatives of the functions.

We return now to the original example to complete that illustration. We had

$$f(z) = z^4, \quad g(x) = x^3 + 5, \quad \text{and } z = x^3 + 5$$

Then

$$Df = 4z^3 = 4(x^3 + 5)^3$$

and

$$Dg = 3x^2$$

So,

$$DF = Df \cdot Dg = 12x^2(x^3 + 5)^3$$

With a little practice, the derivative is often accomplished at sight in one step. First we take the derivative as for the power law, treating the base as though it were a single independent variable, and then multiplying the result by the derivative of the base itself. We offer another illustration:

$$s = (t^2 - 3)^3 \qquad \text{(the base is } t^2 - 3\text{)}$$

$$\frac{ds}{dt} = 3(t^2 - 3)^2(2t)$$

$$\frac{ds}{dt} = 6t(t^2 - 3)^2$$

Another aspect of a function of a function involves a third variable which may be implied. If, for example, $y = x^2 + 6$ and x is some differen-

tiable function of t, then y is a differentiable function of t and the derivative of y with respect to t may be found. The condition may be generalized as follows:

$$y = f(x) \qquad \text{and} \qquad x = g(t)$$

then

$$y = f[g(t)] = F(t)$$

Thus

$$F = f[g]$$

where the independent variables are t, x, and t for F, f, and g, respectively. So,

$$DF = Df \cdot Dg \qquad \text{or} \qquad \frac{dF}{dt} = \frac{df}{dx}\frac{dg}{dt}$$

or

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Looking at the original example, we have for $y = x^2 + 6$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

since $dy/dx = 2x$. Without explicit knowledge of $x = g(t)$, we cannot go any further. We shall see shortly that when t represents time, this result has a meaningful interpretation. Other illustrations follow:

(1)
$$y = x^3 + 5x^2 - 6$$

where $y = f(t)$ and $x = g(t)$.

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 10x\frac{dx}{dt} \qquad [Df = D(g + g') = Dg + Dg']$$

(2)
$$p = 5q^3 - 7q^2$$

where $p = f(u)$ and $q = g(u)$.

$$\frac{dp}{du} = 15q^2\frac{dq}{du} - 14q\frac{dq}{du}$$

(3)
$$z = x^2 + y^2$$

where all are functions of t.

$$\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

EXERCISES (X-6)

1. Find the derivatives of each of the following. In each case check by expanding the binomial and taking the derivative term by term:

a. $y = (3x + 1)^2$
b. $y = (x^2 - x)^2$
c. $y = (x - 2)^3$
d. $y = (2 + 5x^2)^3$

2. In each of the following find dy/dx: (a) by using the chain rule, and then (b) by expressing y directly in terms of x and then differentiating:

a. $y = z^2; z = x^3$
b. $y = z^2 + 2z + 1; z = x + 1$
c. $y = u^3; u = x^2 + 1$
d. $y = u^3 - u^2; u = 2x^3$

7. DERIVATIVE APPLIED TO PARAMETRIC EQUATIONS

We have seen the values of parametric equations in describing functional relations. If, for example, we obtain in some context $x = t^3$ and $y = t^2$, we can determine now the function values dx/dt and dy/dt. Is it possible to determine the value of dy/dx from these two? This is likely to be very convenient and sometimes even necessary when it is difficult or impossible to eliminate the parameter.

Now, in the preceding article we developed the rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

By solving this for dy/dx, we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \left(\frac{dx}{dt} \neq 0\right)$$

In $x = t^3$ and $y = t^2$, we have

$$\frac{dy}{dt} = 2t \qquad \text{and} \qquad \frac{dx}{dt} = 3t^2$$

Thus

$$\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t} \qquad (t \neq 0)$$

EXERCISES (X-7)

1. Find the derivative of each of the following and check by deriving $y = f(x)$ and taking the derivative directly:

a. $x = 2t + 1; y = t^2$
b. $x = \frac{t}{3}; y = t^3$
c. $x = t^2 + 1; y = t^4$
d. $x = 3u^2 - 2; y = 5u^2 + 3$

2. Find the slope to the tangent of the curve $x = 1 - t$, $y = 3 - 2t - t^2$ at $t = 0$ and $t = -2$.

3. Find the rectangular coordinates of the point on $x = 2t - 1, y = 4t^2 - 6t - 4$, where the slope is 1.

4. Find dy/dx:

a. $x = t^2; y = t^3$
b. $y = t^3 - t^2; x = t - t^2$

8. DERIVATIVE OF AN IMPLICIT FUNCTION

We have seen that it is not always easy or even possible to express an implicit function explicitly. It is therefore of an advantage to know that the derivative may be found, if it exists, without the necessity of converting the form of the function.

The derivative of an implicit function value may be found by taking the derivative term by term. Thus,

a. $$D_x(x^2 + y^2) = D_x x^2 + D_x y^2 = 2x + 2y\frac{dy}{dx}$$

b. $$D_t(x^2 + y^2) = D_t x^2 + D_t y^2 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

In an equation we take the derivative of both members in the manner of the above.

c. $$\begin{aligned} y^3 + x^2 + x &= 12 \\ D_x(y^3 + x^2 + x) &= \mathrm{D}_x 12 \\ 3y^2\frac{dy}{dx} + 2x + 1 &= 0 \\ \frac{dy}{dx} &= -\frac{2x + 1}{3y^2} \end{aligned}$$

In taking the derivative of both members of the equation, we are actually assuming existence of a function implied by the equation in which x is the independent variable. We assume, therefore, the existence of some $f:\{x, y\}$. The validity of this assumption is justified in advanced books.

It is conceivable that one may select y as the independent variable This would be tantamount to assuming the existence of a function $g:\{y, x\}$.

In either case it will probably be necessary to stipulate certain restrictions on the values of the independent variable so that we will have functions rather than only relations. Under these circumstances the two functions will be inverse functions of each other. We may write $f = g^{-1}$ or $g = f^{-1}$.

It is interesting to return now to our last illustration (c) and apply the operator D_y.

$$\begin{aligned} D_y(y^3 + x^2 + x) &= D_y\, 12 \\ 3y^2 + 2x\frac{dx}{dy} + \frac{dx}{dy} &= 0 \\ (2x + 1)\frac{dx}{dy} &= -3y^2 \\ \frac{dx}{dy} &= -\frac{3y^2}{2x + 1} \end{aligned}$$

By comparison with the result in (c), we note that the results are reciprocals of each other. This indicates the true fact that

$$\frac{dy}{dx} = \frac{1}{dx/dy} \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{dy/dx}$$

EXERCISES (X–8)

1. Find dy/dx and dx/dy for each of the following:
   a. $5x - y^2 + 3 = 0$
   b. $y^3 + x^2 = x$
   c. $\frac{1}{y} = \frac{y}{x}$
   d. $x^2 + y^2 = 25$
   e. $y^2 = 3x + 6 - y$
   f. $y^q = x^p$
   g. $\sqrt{y} = x - 1$
   h. $y^2 - \sqrt{x^2 - 3} = 0$
   i. $\sqrt{x} = \sqrt[3]{y}$
2. Find the slope of $y^3 - x^2 = 0$ at $(1, 1)$.
3. Find the equation of the tangent to the curve $x^2 + 9y^2 = 36$ at $(3, \sqrt{3})$.
4. Find the equations of the tangents at the y-intercepts for $y^2 - 2x - 2y = 0$.
5. Find du/dv and dv/du for $3u^2 - 2v^2 + 2u + 3v + 8 = 0$.
6. Find dP/dQ for $P^3 - 3P + Q = 40$.

X–8 REVIEW

1. For $y = x^2 + 6$, we found earlier that $dy/dt = 2x(dx/dt)$. Suppose now that $x = 3t + 5$. Find dy/dt.

Express y as a function value of t, and find dy/dt.

2. Find the values of each of the following (the independent variable is indicated in every case):
   a. $D_t(x^3)$
   b. $\frac{d}{du}(5z^4)$
   c. $D_t(x^2 - 5)^3$
   d. $\frac{d}{dx}(x^5 - y^4)$
   e. $\frac{d(x^5 - y^4)}{dy}$
   f. $\frac{d}{dt}(x^5 - y^4)$
3. Find the values of each of the following:
   a. $D_x(x^2 - 5)^2$
   b. $D_t(x^2 - 5)^2$
   c. $\frac{d}{dx}(5x^2 + 4x^3)^3$
   d. $D_z(6z^3 + 9)^3$
   e. $D_y(y^9 + y^7)^5$
   f. $D_v(u^2 - u^3)^2$
4. If $y = x^3 + x^2$ and $x = t^4$, find dy/dt.
5. a. If $y = (x^3 + 1)^2$ and $x = t + 1$, find dy/dt.
   b. Find dy/dt when $t = \frac{1}{2}$.
6. If $u = v^3 + 3v$ and $v = (w^2 + 1)^4$, find
   a. $\frac{du}{dw}$
   b. $\frac{du}{dw}$ when $w = 1$
7. $y = 3x^2 - 2x + 1$. Find dy/dt if $dx/dt = \frac{1}{4}$ when $x = 2$.
8. $h = 5m^3 - 3m^2$. Find dh/dk if $dm/dk = -2$ when $m = -1$.
9. Given: $f = g + h$, where the functions are determined by the following sets: $\{u, f(u)\}$, $\{u, g(u)\}$, and $\{u, h(u)\}$. Further, $u = F(v)$. Complete each of the following, with one correct response for each:
   a. $f[F] =$
   b. $Df[F] =$
   c. $Df \cdot DF =$
   d. $\frac{df}{du}\frac{du}{dv} =$

10. $z^3 = x^2$. Find dz/dx and dx/dz, and compare results.

11. $x^2 + y^2 = z^2$, and each variable is a function value of t. Find dz/dt.

12. The indicated derivatives may be found more easily if both members of each of the following equations are raised to an appropriate power. Find the indicated derivative.

a. $y = \sqrt{x^3 + 1}$, $\dfrac{dy}{dx}$ c. $z = \sqrt{x^2 + y^2}$, $\dfrac{dz}{dt}$

b. $y = \sqrt[3]{x - 1}$, $\dfrac{dy}{dx}$

13. Find the equation of the tangent to the curve $y^3 = x^2$ at $x = 8$.

14. Find the equation of the tangent to the curve of $x^2 + y = 1$ that is parallel to $2y + x = 10$.

15. The center of a circle is at the origin and its radius is 6. Find the equations of the tangents to the circle at $x = 3$.

16. The center of an ellipse is at the origin and the semi-axes terminate at (6, 0) and (0, 4). Find the equation of the tangent to the ellipse in the first quadrant at the point on the ellipse where $x = 3$.

17. Find the equation of the line which is tangent to the curve $y^2 - 3x - 4y = 0$ at $(-1, 1)$.

18. Find the equation of the line that is normal to the curve $y^2 - 3x - 4y = 0$ at $(-1, 1)$. (The normal is the line perpendicular to the tangent at the point on the curve.)

9. IRRATIONAL ROOTS BY NEWTON'S METHOD

An important problem in modern science concerns the evaluation of irrational roots of equations. We saw earlier how this was possible only for quadratic equations. In Newton's method we have the means of approximating the roots to any desired degree of accuracy.

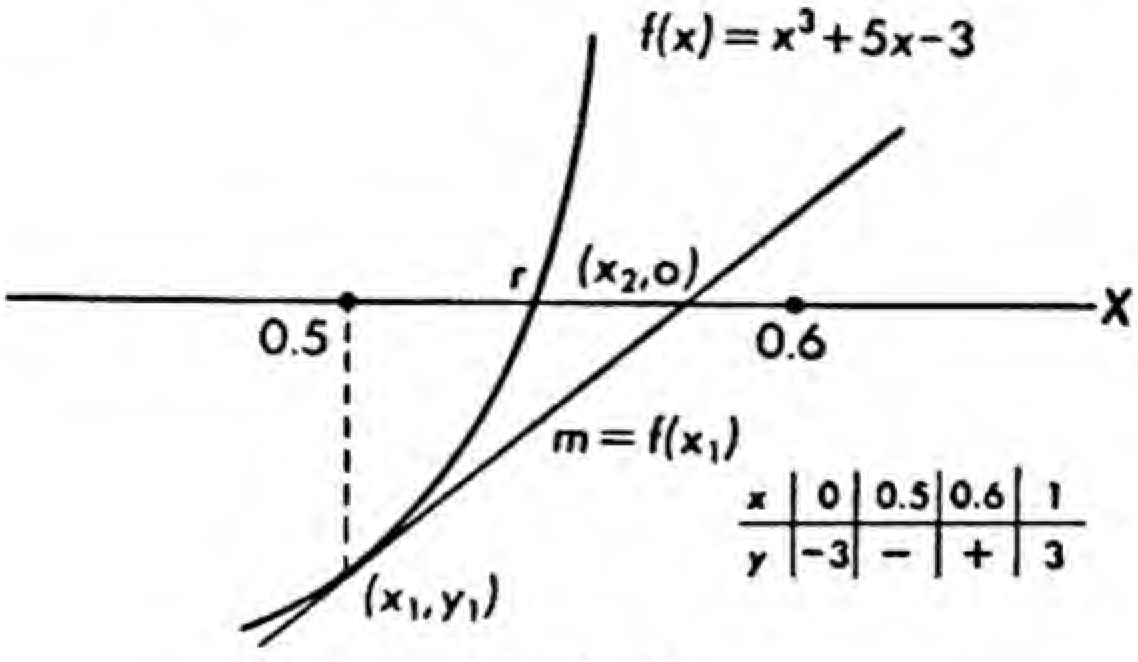


Fig. X–6

Consider the equation $y = f(x) = x^3 + 5x - 3 = 0$. By means of the graph and synthetic division, we not only locate a root r between 0 and 1 but also determine that the root lies between 0.5 and 0.6, Fig. X–6. This follows from the fact that $f(0.5)$ is negative and $f(0.6)$ is positive. Let us

call x_1 the first approximation to the root; that is, $x_1 = 0.5$. By synthetic division we get $y_1 = f(x_1) = -0.375$. Should y_1 or some subsequent y_i turn out to be 0, it would indicate that we had stumbled on a rational root.

The gist of Newton's method can be seen now. We have located a point $P_1(x_1, y_1)$ on the curve whose abscissa is near r. We are in the position of writing the equation of the tangent line to the curve at P_1. We have the coordinates of a point on this prospective tangent line, and we can calculate the slope via the first derivative at that point. Barring exceptional circumstances, the x-intercept (x_2) of this tangent line will be even closer to r than x_1. By means of the slope definition, we write

$$\frac{y_1}{x_1 - x_2} = m = f'(x_1)$$

or

$$\frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

where $f'(x_1) = f'(x)$ when $x = x_1$.

Solving for x_2, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The value of x_2 can be calculated according to the formula:

$$x_2 = 0.5 - \frac{-0.375}{5.75} = 0.5 + 0.065 = 0.565$$

The procedure can be continued indefinitely. We may now calculate $f(x_2)$ and $f'(x_2)$. Assuming $(x_3, 0)$ to be the new intercept of another tangent line to the curve at $P_2(x_2, y_2)$, we obtain an equation as above, with each subscript increased by 1.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

We get

$$x_3 = 0.565 - \frac{0.00536}{5.95768} = 0.565 - 0.000900 = 0.5641$$

There is no question but that we have approximated the root correctly to four significant figures:

$$r \approx 0.5641$$

We generalize the foregoing considerations inductively and write for the $(n + 1)$ approximation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f'(x_n) = f'(x)$ when $x = x_n$.

In using the preceding method, should it happen that our first approximation is taken where the absolute value of the derivative at P_1 is very small, we should find that the x-intercept (x_2) of the tangent line at P_1 is very far from our first approximation. Often this can be remedied by shifting the first approximation slightly.

EXERCISES (X-9)

Find an irrational root for each of the following:

1. $x^3 + x - 11 = 0$
2. $3x^3 + x - 1 = 0$
3. $x^3 - 4x - 6 = 0$
4. $2x^3 + 3x^2 - 5x - 7 = 0$
5. $x^3 - 2x^2 - 8 = 0$
6. Find the cube root of three without using tables.

10. ANGLE BETWEEN TWO CURVES

Intuition suggests that intersecting curves meet at some sort of *angle.* We have already identified the direction of a curve at a point on the curve with the slope of the tangent at the point. It is a small matter to go a step further and define that

> *an angle between two curves at their point of intersection is the angle between their tangents, taken in the same order, at the point of intersection.*

The angle is measured counterclockwise for a positive result (see Fig. X-7).

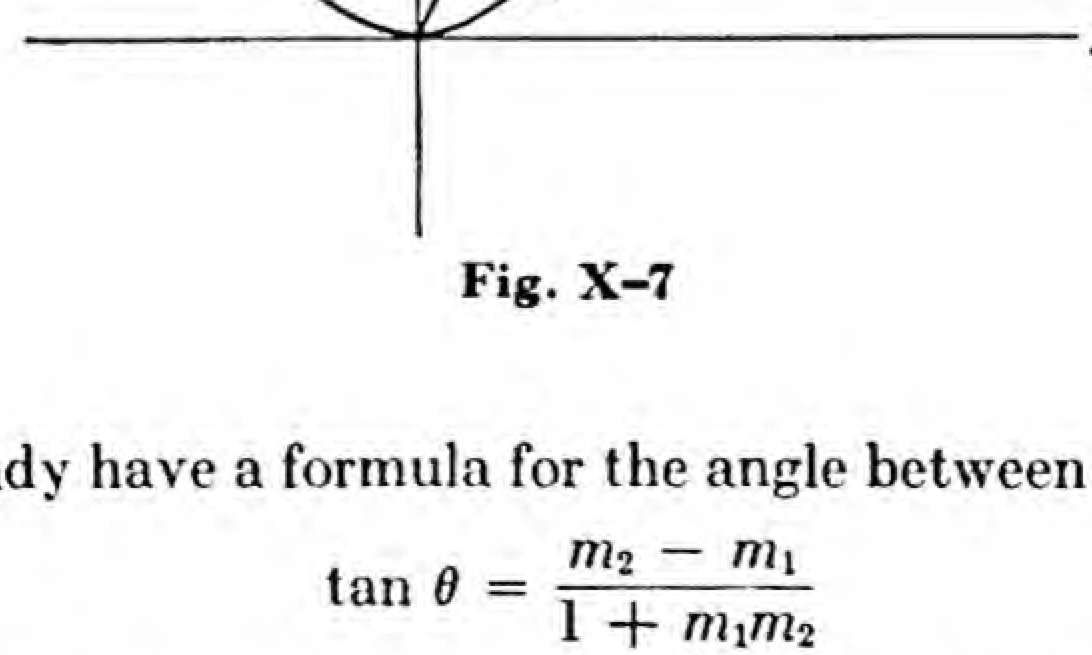


Fig. X-7

Now, we already have a formula for the angle between intersecting lines:

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 is the slope of the first line and m_2 the slope of the other.

The two parabolas, $y = x^2$ and $y = \sqrt{x}$, intersect at (1, 1), as can be determined by simultaneous solution. At (1, 1) the slopes of the respective tangents are 2 and ½, as may be obtained from their derivatives. Then by substitution,

$$\tan\theta = \frac{2 - \frac{1}{2}}{1 + 2(\frac{1}{2})} = \tfrac{3}{4} = 0.75$$
$$\theta = 37°$$

This measures the angle from the parabola with the slope ½, at the given point, to the other. Should we desire the angle from the second curve to the first, all we need do is to interchange the terms in the numerator, which would give a result that is the negative of the former answer. The new angle would then be the supplement of the first. This is in clear agreement with an examination of the geometry of the situation.

EXERCISES (X-10)

In exercises 1 to 3, find in the given order the angles of intersection between the curves in quadrant I.

1. $y = x^2$ and $x^2 + y = 8$.
2. $x^2 + y^2 = 13$ and $2y^2 = 9x$.
3. $x^2 + y^2 = 17$ and $x^2 - y^2 = 15$.
4. Show that the curves $2x^2 + y^2 = 1$ and $y^2 = x$ are *orthogonal* to each other. (Curves are orthogonal to each other when they intersect at right angles.)

11. RATES OF CHANGE

We have seen that the derivative represents a rate of change of the dependent variable with respect to the independent variable of a function at a particular member of the set for which the function is defined. The value dy/dt, if y is measured in inches and t in seconds, represents a rate of change in inches per second. This is the instantaneous speed referred to earlier. However, the scope is greater than this.

Consider a continuously expanding circle whose radius is increasing uniformly at the rate of 3 inches per minute. What can we find out about the rate of change of the area?

Well, the time rate of change of the radius is symbolized by dr/dt, and that is 3 inches per minute. The time rate of change of the area is symbolized by dA/dt, which is to be found. These indicate that what we need as a starting point is a function whose variables are A and r, where both are explicit functions of t. Of course we have

$$A = \pi r^2$$

from which we get
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We see now that the rate of change of the area depends not only on the rate of change of the radius dr/dt but also on the radius itself. For a specific numerical answer, we take the moment when the radius is 5 inches. Then

$$\frac{dA}{dt} = 2\pi(5\text{ in.})(3\text{ in./min}) = 30\pi\frac{\text{sq in.}}{\text{min}} \approx 94\frac{\text{sq in.}}{\text{min}}$$

All the units were shown to indicate that the final product represents a change in area in unit time, as was to be expected.

For another illustration, consider a weight W (Fig. X–8) being lowered by two ropes through the openings S and T, which are 10 feet apart. Each rope is being lowered at the uniform rate of 6 feet per minute. We wish to find the rate at which W is descending at the instant that WT is 12 feet long.

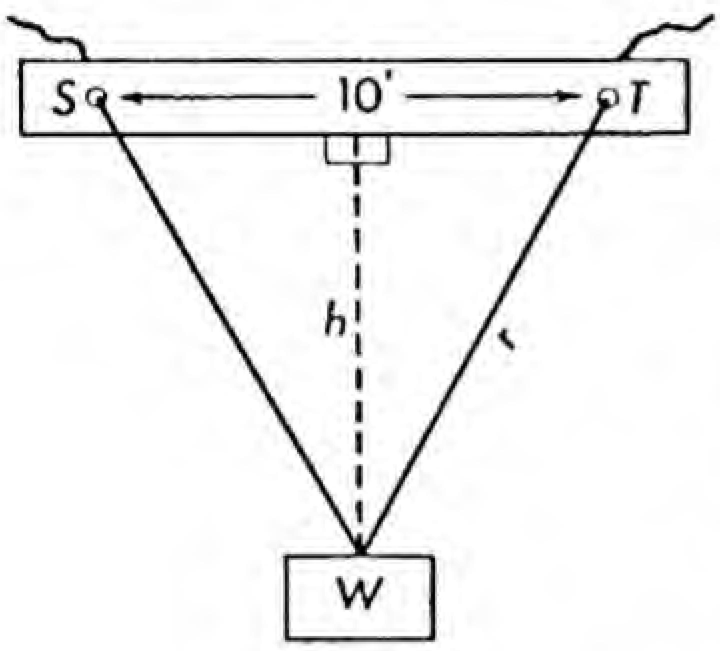


Fig. X–8

Since the weight is descending vertically, we seek dh/dt, knowing dr/dt and r. This requires a function where h and r are the variables and t is a parameter for both. This is supplied by the Pythagorean formula:

$$h^2 = r^2 - 25$$

Then

$$2h\frac{dh}{dt} = 2r\frac{dr}{dt}$$

and

$$h\frac{dh}{dt} = r\frac{dr}{dt}$$

This is the general equation involving the relevant rates of change. We must now specify the case. When $r = 12$, $h = \sqrt{r^2 - 25} = \sqrt{119}$. Substituting the known facts, we have

$$\sqrt{119}\frac{dh}{dt} = (12)(6)$$

$$\frac{dh}{dt} = \frac{72}{\sqrt{119}} \approx 6.6\text{ ft/min}$$

EXERCISES (X–11)

1. The 6 inch arm of a right triangle is kept constant while the other arm is allowed to increase at the uniform rate of 2 inches per minute. Find the rate of change of the hypotenuse when the hypotenuse is 10 inches.

2. Consider a chord in a 26 inch circle whose distance from the center is increasing at the rate of 2 inches per minute. Find the rate of change in the length of the chord when it is 5 inches from the center.

3. A 25-foot ladder is leaning against a wall. Its foot is being pulled away from the building at the rate of $3\frac{1}{2}$ feet per minute. How fast is the top descending the wall at the instant when it is 20 feet from the ground?

4. Two people leave the same spot at the same time. One travels easterly at the uniform rate of 4 miles per hour, while the other proceeds northerly at the rate of 8 miles per hour. How fast are they separating at the end of $1\frac{1}{2}$ hours?

5. The side of a square is increasing at the rate of 2 inches per second. How fast is the area changing when the side is 8 inches?

6. At noon, a car leaves the intersection on one of two straight roads that intersect at 60°. The car is driven at 40 miles per hour. A marker is situated on the other road, 30 miles from the intersection. How fast is the car separating from the marker at 12:45 P.M.?

7. At noon a ship going due east at 12 knots is 50 nautical miles due north of a ship that is sailing north at 9 knots. How fast are the ships separating at 1 P.M., 2:30 P.M .? Interpret the signs of the answers. (Note: The speed of one ship must be considered as negative. Why?)

8. The volume of a sphere is expanding at the rate of 500 cubic feet per minute.
   a. Find the rate at which the radius is changing when $r = 12$ feet.
   b. Find the rate at which the surface is changing. ($V = \frac{4}{3}\pi r^3$; $S = 4\pi r^2$)

9. A conical filter is 5 centimeters in radius and 15 centimeters deep. A liquid that is being filtered is passing through at the rate of 2 cubic centimeters per second. Determine the rate of change of the level of the liquid when the depth is 9 centimeters. ($V = \frac{1}{3}\pi r^2 h$)

10. A 6-foot man is walking away from a street lamp at the rate of 5 feet per second. The lamp is suspended 30 feet above the ground. How fast is the length of his shadow changing?

11. Sand poured from a spout will form a conical pile on the ground. If the diameter of the base of the pile is always three times its height:
   a. Express the volume in terms of the height.
   b. If the sand is being poured at the rate of 45 cubic feet per minute, how fast is the height changing when the pile is 5 feet high?

12. ROUNDING OUT THE DERIVATIVE PICTURE

So far the exponents involved in derivative taking have been positive and integral. It is time we investigated the possibility of the positive rational exponent. Suppose

$$y = x^{m/n} \qquad \text{where } m > 0,\ n > 0$$

Then

$$y^n = x^m$$

Taking the derivative of both members, we have

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1} \qquad\qquad y^{n-1} = (x^{m/n})^{n-1}$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{nx^{m-(m/n)}} \qquad\qquad y^{n-1} = x^{m-(m/n)}$$

$$\mathbf{\frac{dy}{dx} = \frac{m}{n}x^{(m/n)-1}} \qquad \text{(subtraction of exponents)}$$

The result is identical in form with the power formula

$$\frac{dy}{dx} = px^{p-1}$$

when $y = x^p$ for positive integral values of p. Further, if y and x were related parametrically to another variable (say, u), then by previous determinations we would have for $y = x^{m/n}$,

$$\frac{dy}{du} = \frac{m}{n}x^{(m/n)-1}\frac{dx}{du}$$

The following are illustrative of these results:

(a)
$$y = x^{3/2}$$
$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

(b)
$$u = \sqrt{6v^5 + 1} = (6v^5 + 1)^{1/2}$$
$$\frac{du}{dt} = \frac{1}{2}(6v^5 + 1)^{-1/2}(30v^4)\frac{dv}{dt}$$
$$\frac{du}{dt} = \frac{15v^4}{\sqrt{6v^5 + 1}}\frac{dv}{dt}$$

We saw earlier that another approach is to raise both members of the equation to the appropriate power, thereby obtaining implicit equations with integral positive powers. The formula itself was developed by just this approach.

Our differentiation has been restricted thus far to addition and subtraction of function values. We have gone far enough for the reader to know that sooner or later we must investigate the matter of the other basic operations. Let us turn our attention to the derivative of a product of two polynomial expressions in the same unknown. The reader may prefer to initiate the study with an illustrative case which, on examination, yields some valuable clues. However, we shall take it on the general level immediately.

$$y = f(x)g(x)$$
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$

Two extra terms will be inserted in the numerator, the sum of which will be zero, so that the value of the fraction will be unchanged.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - g(x)f(x + h) + g(x)f(x + h) - f(x)g(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \left[f(x + h)\frac{g(x + h) - g(x)}{h} + g(x)\frac{f(x + h) - f(x)}{h}\right]$$

This result was obtained from the previous line by writing it as two terms and doing some partial factoring. According to the theory of limits

we can, as we discovered, find the limit of the whole expression by taking the limit of each term and of each factor separately. Doing so, we get

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

or

$$D\,(f \cdot g) = f(x)Dg + g(x)Df$$

> *The derivative of a product of two functions is equal to one function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.*

Note that the notation for the product of two functions differs emphatically from that of the function of a function.

(a)

$$y = (x^2 + 3)(x^2 + x^2)$$

$$\frac{dy}{dx} = (x^2 + 3)(3x^2 + 2x) + (x^3 + x^2)(2x)$$

(b)

$$z = xy$$

$$\frac{dz}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

Finally, we take up the quotient of two functions:

$$y = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

In order to utilize the previous finding, we multiply both members of the equation by $g(x)$ and take the derivative of both members of the equation:

$$yg(x) = f(x)$$

$$yg'(x) + y'g(x) = f'(x)$$

We replace y by its equivalent $f(x)/g(x)$ and solve the equation for y'. We get

$$y' = D\left(\frac{f}{g}\right) = \frac{g(x)\,Df - f(x)\,Dg}{[g(x)]^2}$$

We note that the result is a fraction whose denominator is the original one squared. The numerator, excepting for the minus sign, resembles the procedure in taking the derivative of the product of two function values. Specifically the numerator is obtained by taking the product of the denominator by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator.

Illustrations:

a. $y = \dfrac{2x}{x^3 + 3}$

$$\frac{dy}{dx} = \frac{(x^3 + 3)(2) - 2x(3x^2)}{(x^3 + 3)^2}$$

$$\frac{dy}{dx} = \frac{6 - 4x^3}{(x^3 + 3)^2}$$

b. $z = \dfrac{x}{y}$

$$\frac{dz}{dt} = \frac{y\dfrac{dx}{dt} - x\dfrac{dy}{dt}}{y^2}$$

The power formula can now be expanded to include the negative exponent. Suppose that $m > 0$; then

$$y = x^{-m} = \frac{1}{x^m}$$

$$\frac{dy}{dx} = \frac{x^m(0) - 1(mx^{m-1})}{x^{2m}}$$

$$\frac{dy}{dx} = -\frac{mx^{m-1}}{x^{2m}}$$

$$\frac{dy}{dx} = -mx^{-m-1}$$

If a parameter is involved, we would have

$$\frac{dy}{du} = -mx^{-m-1}\,\frac{dx}{du}$$

Thus, the power law holds for negative exponents as well.

EXERCISES (X–12)

1. Find dy/dx in each of the following:

a. $y = x^3(x^2 + 1)$

b. $y = x\sqrt{x}$

c. $y = \dfrac{x^2}{\sqrt{x}}$

d. $y^3 = \sqrt{x - 1}$

e. $y = 6\sqrt{x} - \dfrac{8}{\sqrt[3]{x}}$

f. $y = (x + 1)\sqrt{x^2 + 1}$

g. $4y^2 - x = 8$

h. $x^{1/2} + y^{1/2} = 1$

i. $y = \sqrt{t + 1}$, $x = \sqrt{t} + \dfrac{1}{\sqrt{t}}$

j. $x^2 + y^2 - 2xy + 8 = 0$

k. $y = \dfrac{x^2}{(x + 1)^3}$

l. $y = (x^2 + 1)^{3/2}$

m. $y = \dfrac{x + 2}{x + 1}$

2. Find the equations of the tangents to the following curves at the indicated points:

a. $y^2 = x^3$, $(4, -8)$

b. $x^{2/3} + y^{2/3} = 4$, $(3\sqrt{3}, -1)$

c. $y = \sqrt{1 - t^2}$, $x = \sqrt{1 + t^2}$, $(1, 1)$

3. Find the angle of intersection of the cubical parabola $y^3 = x$ and the hyperbola $xy = 16$.

4. A spherical iron ball, 10 inches in diameter, is uniformly coated with ice. Assume that the ice is melting steadily at the rate of 6 cubic inches per minute.

a. How fast is the thickness of the ice decreasing when it is 1 inch thick?

b. How fast is the outer surface of the ice changing at that moment?

5. Complete the development of the formula for the derivative of

$$y = \frac{g(x)}{f(x)}$$

by writing this as $yf(x) = g(x)$.

6. Use the same approach as in the preceding example on $y = x^{-m}$ to derive dy/dx.

X-12 REVIEW

1. Show that $f(x) = f'(x)(x - a)$ is the equation of the tangent line to $f(x)$ at its x-intercept $(a, 0)$, if $f'(x)$ exists at $(a, 0)$.

2. Find the positive root of $x^3 - x^2 - 2x - 1 = 0$ to three significant figures.

3. Find the angle between $y = x^3 - 3x^2 + 5x$ and $y = 2x^2 - x + 2$ at the rational point of intersection.

4. a. Find the values of k for which $y = x^2 - kx + 1$ and $y = x^2$ intersect orthogonally.

b. Determine the points of intersection and sketch the curves.

5. Take the derivatives of each of the following with respect to t:

a. $y = x^3 - 3x + 6$

b. $y = \sqrt{x}$

c. $u = (v^2 - 1)^3$

d. $y^2 = \dfrac{1}{x - 1}$

e. $\dfrac{1}{w} = (z^2 - 1)^{1/3}$

f. $xy + y^2 + x = 5$

6. The volume of a cube is increasing at the rate of 40 cubic feet per minute at the instant the edge is 12 inches. How fast is the edge changing?

7. The volume of a quantity of gas is given by $V = 400/p$. How fast is V changing if p is decreasing at the rate of 0.3 at the instant when $p = 30$?

8. A kite is flying horizontally at the rate of 4.5 miles per hour away from a boy flying it. At a particular instant the kite is 80 feet high with 100 feet of cord let out. Assuming no drag exerted by the boy, how fast is he releasing the cord?

9. The circumference of an expanding circle is increasing at the rate of 2 inches per minute. Find the rate of increase of the area when the radius is 5 inches.

10. Find the angle of intersection in quadrant I of $y = x^2$ and $xy = 8$.

11. Find the slope of $x^{2/3} + y^{2/3} = 1$ at $x = \frac{1}{8}$.

12. Find the equation of the tangent to the curve $y = x^2/(x - 1)$ at $x = 3$.

13. If $x = \sqrt{t}$ and $y = t/(t - 1)$, find dy/dx.

14. At noon two cars leave the crossing of two straight roads that intersect at 60°. One is driven uniformly at 40 and the other at 60 miles per hour. How fast are they separating at 12:45 P.M.?

13. CRITICAL VALUES

We turn now to another type of application of the derivative. Consider the case of a man who has available 60 feet of wire fencing with which he intends to make a rectangular enclosure along a straight river bank. The river bank is to serve as one side of the enclosure.

The enclosure can be made in various dimensions. The side AB (Fig. X–9) can be made as short as he pleases or up to 30 feet in length. An infinite assortment of rectangular enclosures are possible, all utilizing the full 60 feet of fencing. But the consequence of the varying shapes is that the areas will vary too. In the two cases where AB is very small or as large as possible, the areas will be minute. Somewhere in between is a happy mean of a "maximum" area enclosure.

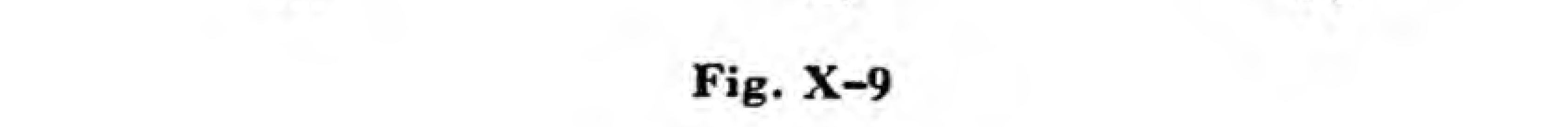


Fig. X–9

Let us examine this analytically. If we indicate the length of AB as x, in feet, then DC is also x ft, so that these two use up $2x$ feet together. This leaves $(60 - 2x)$ feet for BC. If K represents the area of the rectangle in square feet, we may write

$$K = x(60 - 2x) = 60x - 2x^2$$

The graph of the equation (Fig. X–10), where K is the dependent variable, bears out what we sensed intuitively or arithmetically. The area varies symmetrically, yielding results for small values of x equal to those values of x near 30 feet. In between there is a maximum K, 450 square feet, which occurs when the width is 15 feet.

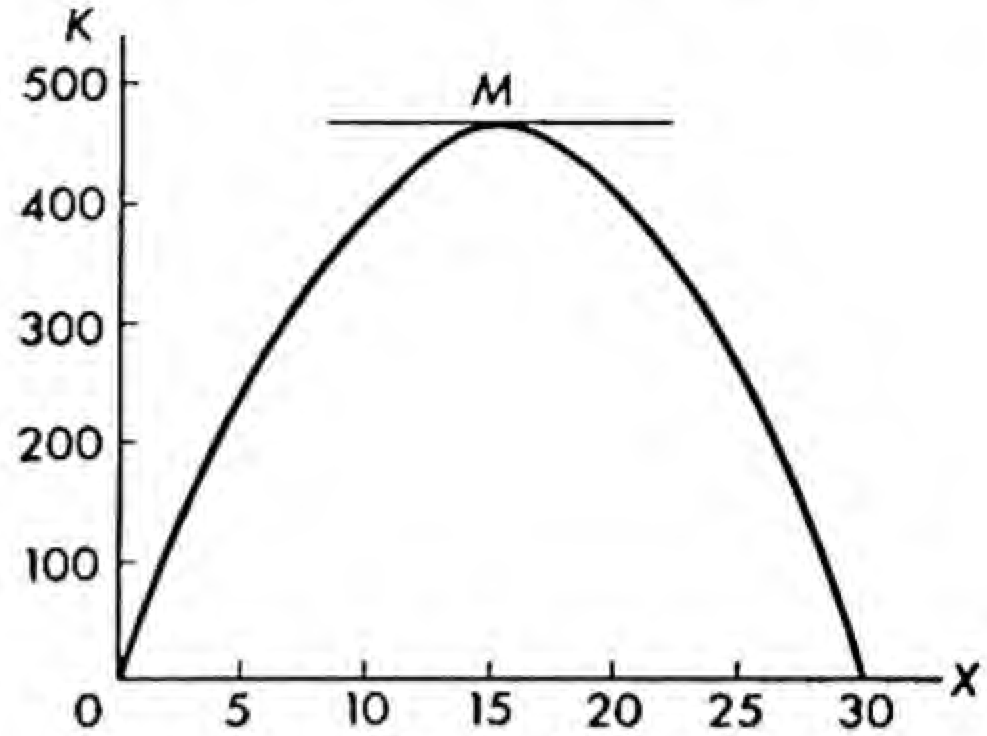


Fig. X–10

It may not always be desirable or feasible to make such determination graphically. Fortunately the calculus provides means of exact determination. The significant observation is the fact that *at M the slope of the curve is* 0. That is, $Df = 0$, or $f'(x) = 0$. We note, too, in passing, that *the slope of the curve is positive to the left of M and negative to the right of M.* Or, equivalently, $f(x)$ is increasing to the left of M and decreasing to the right of M. So, if we take our function value, $K = 60x - 2x^2$, and find where the derivative is 0, we shall locate the point M.

$$K = 60x - 2x^2$$
$$\frac{dK}{dx} = 60 - 4x$$
$$60 - 4x = 0$$
$$x = 15 \text{ ft}$$
$$K = 450 \text{ sq ft} \qquad \text{(by substitution)}$$

Now let us take the same problem and turn it about. Suppose that the man of our example had decided to enclose a rectangular area of 450 square feet. How could this be done most economically, that is, with the least amount of fencing material?

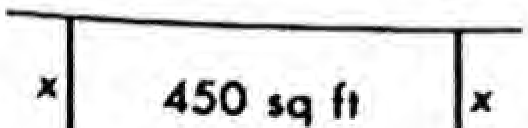


Fig. X–11

If x is the number of feet in the width (Fig. X–11) then, to assure us of an area of 450 square feet, $450/x$ must be the number of feet in the length. By addition we get that the total length of fencing needed is

$$L = 2x + \frac{450}{x} \qquad (x \neq 0)$$

or

$$L = 2x + 450x^{-1}$$

At the minimum value, m in Fig. X–12, the slope will again be 0.

$$\frac{dL}{dx} = 2 - 450x^{-2} = \frac{2x^2 - 450}{x^2}$$

For the derivative, the numerator must be 0 if the fraction is to be zero.

$$2x^2 - 450 = 0$$
$$x^2 = 225$$
$$x = 15 \text{ ft}$$
$$\frac{450}{x} = 30 \text{ ft}$$

The same dimensions as before yield a three-sided rectangle of minimum perimeter for the given fixed area. We call attention to the fact that *the slope is negative to the left of the minimum point and positive to the right of this point* as seen in the graph of the function (Fig. X–12). This is the reverse of the case that was noted for the maximum point.

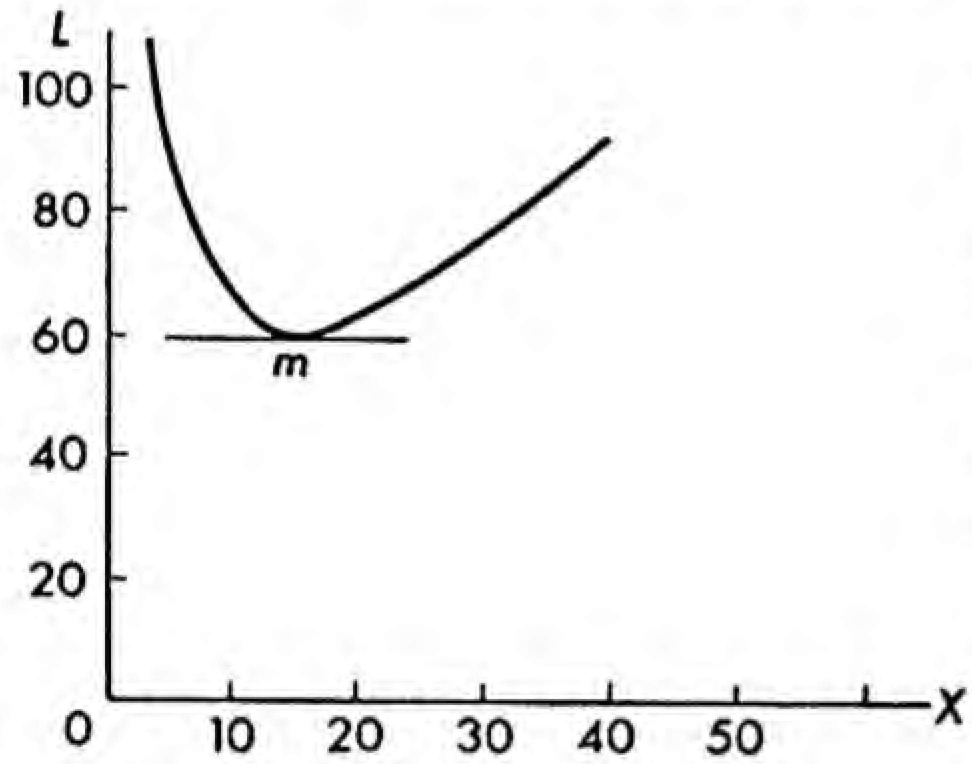


Fig. X–12

As a further variation, suppose that for some peculiar reason the fence along the width (in the last problem) were to cost \$10 a foot, while the fence along the length were to cost \$5 a foot. The problem now is to find the dimensions so that the cost is a minimum. To do this, we need a cost equation. The $2x$ feet along the two widths cost together $20x$ dollars, while the length $(450/x)$ at \$5 a foot, costs $2250/x$ dollars. So, the total cost C, in dollars, is

$$C = 20x + \frac{2250}{x}$$

$$\frac{dC}{dx} = 20 - \frac{2250}{x^2} = \frac{20x^2 - 2250}{x^2}$$

Then,
$$20x^2 - 2250 = 0$$
$$x^2 = 112.50$$
$$x \approx 10.6 \text{ ft}$$
$$\frac{450}{x} \approx 42.5 \text{ ft}$$
$$C = \$424.26$$

The derivative is 0 for the extreme values, be they maximum or minimum. This raises the question as to whether this may not cause confusion at times. In practical situations, as in the ones we have just discussed, there is no question usually as to whether we are finding one rather than the

other. However, for special cases, and for a broad view of the situation as a whole, we need to take a closer look.

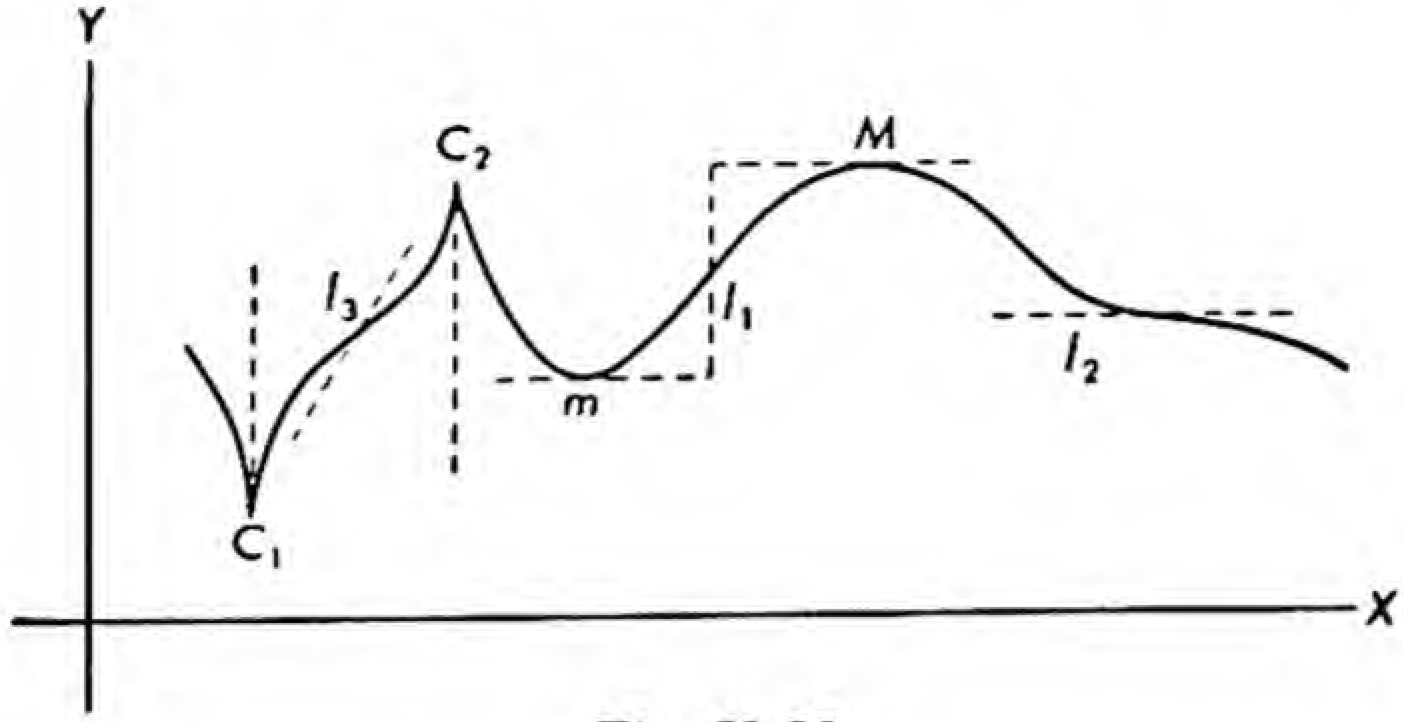


Fig. X–13

The curve shown in Fig. X–13 is admittedly an extreme and unlikely situation. Yet it will highlight various possibilities. The points M and m will be called **relative maximum and minimum points,** respectively. The points M and m are local phenomena. They are the high and low points, respectfully, in their immediate vicinity only.

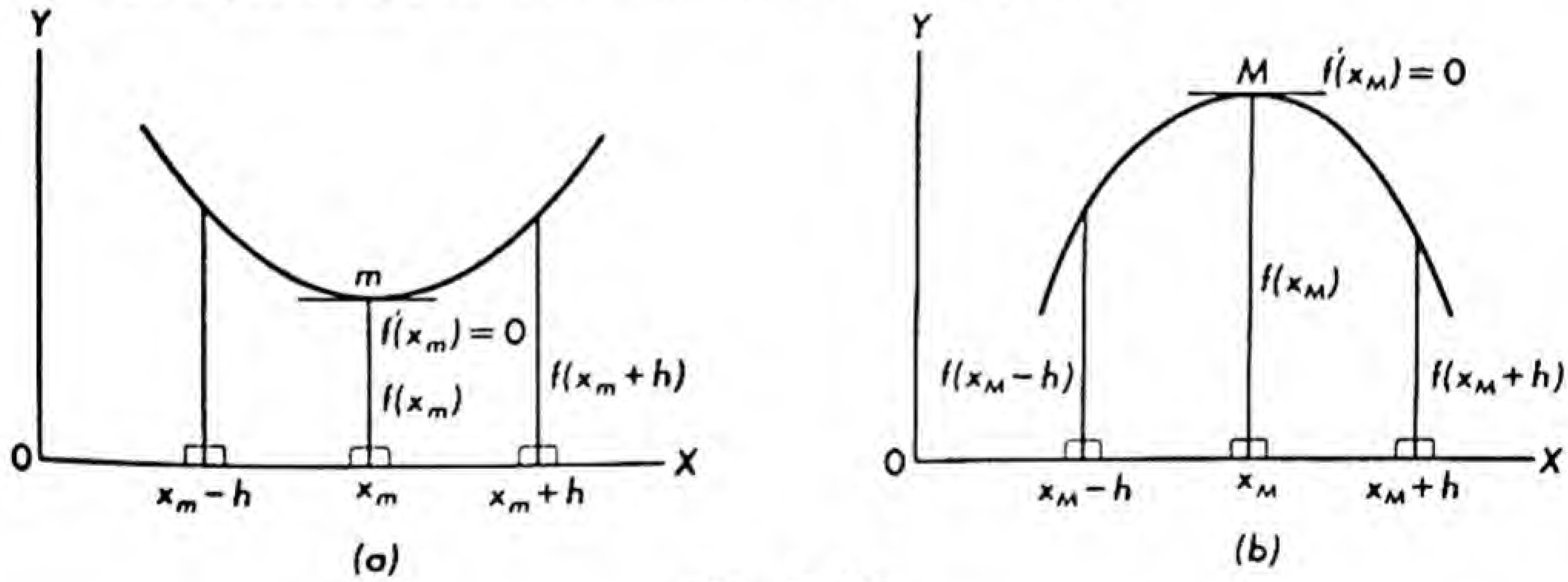


Fig. X–14

Let us see what this means symbolically (refer to Fig. X–14). At m, $f(x_m)$ is less than any $f(x_m + h)$ for small values of $|h|$. Thus

$$f(x_m + h) > f(x_m)$$

or
$$f(x_m + h) - f(x_m) > 0 \qquad h > 0$$

Then
$$\frac{f(x_m + h) - f(x_m)}{h} > 0$$

The numerator of the fraction is positive for every value of h. Thus, if h is positive, the fraction is positive, as indicated. If h is negative, on the other hand, the fraction is negative.

The limit of the fraction is the same whether we take the limit through negative or positive values of h. But in one case we are approaching the limit through persistently negative values and in the other through positive values of the fraction. The only possible limit of a fraction under these circumstances is 0. We have then

$$\lim_{h \to 0} \frac{f(x_m + h) - f(x_m)}{h} = f'(x_m) = 0$$

Thus, *for any point of a defined interval, if the derivative exists it is 0 for any interior minimum point.* In the same fashion it can be seen that *the derivative is 0 for any interior maximum point,* Fig. X–14b. The only difference in the discussion would be a change in the direction of the inequality symbol.

$$\frac{f(x_m + h) - f(x_m)}{h} < 0 \qquad h > 0$$

At I_1, I_2, and I_3 of Fig. X–13, we have three other critical values. All are called *inflection points*. In each case the curve crosses the tangent.

At C_1 and C_2, vertical tangents appear but in a different manner than at I_1. At these critical points, the slopes on either side of the point are opposite in sign, being positive on one side and negative on the other. The slope of the curve in the immediate vicinity of an inflection point and on either side of the point are the same in sign.

Because of the signs, there is no limit to the slope at C_1 and C_2. Yet these must be considered as relative minimum and maximum points, respectively.

It would be well to become more familiar with the foregoing observations through a few illustrations.

In subsequent text we shall be concerned only with those cases where f' is defined at the point in question and where both f and f' exist in at least a small neighborhood about the point in question. We consider first (Fig. X–15)

$$y = (x - 1)^2$$

Then

$$f'(x) = 2(x - 1)$$

so

$$f'(1) = 0$$

and so

$$(1, 0) \text{ is a critical point.}$$

By observation or actual substitution we note that $f(x)$ is greater than $f(1)$ on either side of $f(1)$ in its immediate vicinity. This, together with the fact that the derivative is 0 at (1, 0), is sufficient to recognize (1, 0) as a minimum point.

An alternative procedure consists of noting by observation or actual

substitution that the derivative is negative to the left of the critical point and positive to the right of it.

These two tests, which we shall call A and B, may be summarized as follows: (See Fig. X–15).

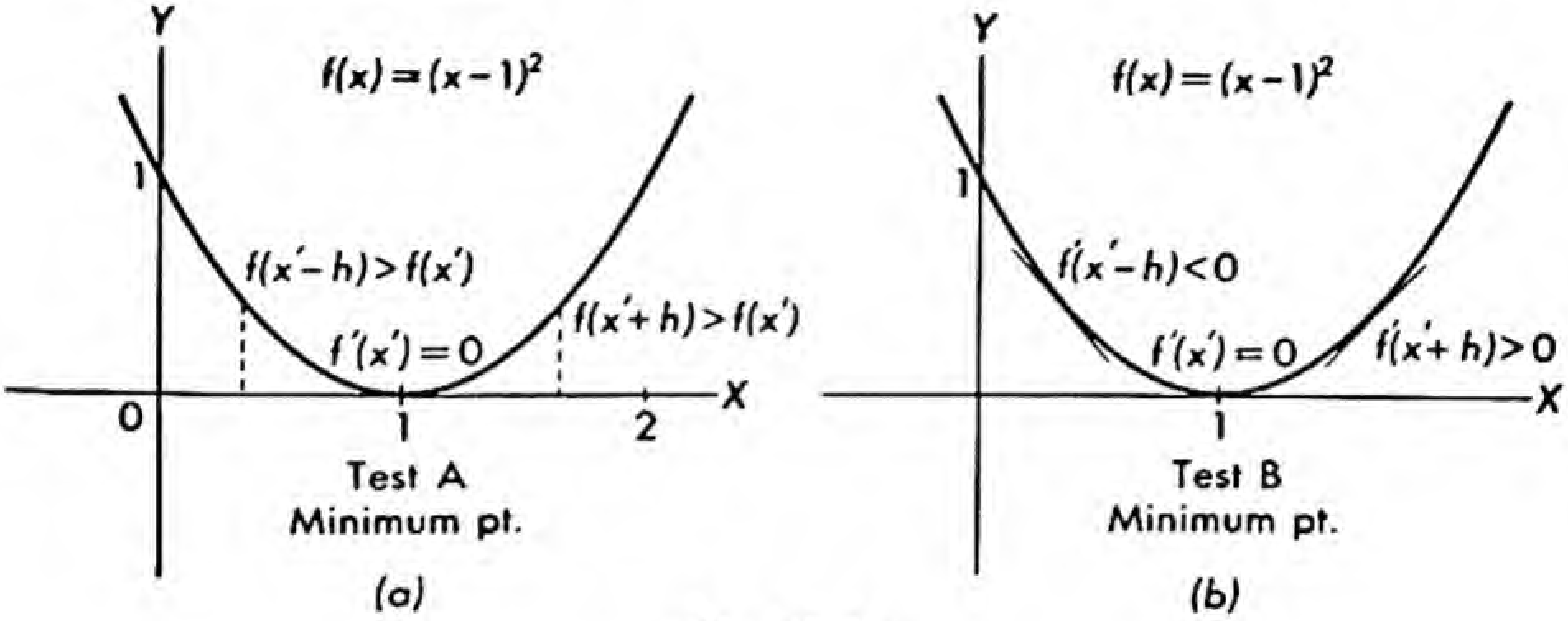


Fig. X–15

Minimum point at x'

Test A: $f(x' - h) > f(x')$ $\quad f'(x') = 0 \quad$ $f(x' + h) > f(x')$
Test B: $f'(x' - h) < 0$ $\quad f'(x') = 0 \quad$ $f'(x' + h) > 0$

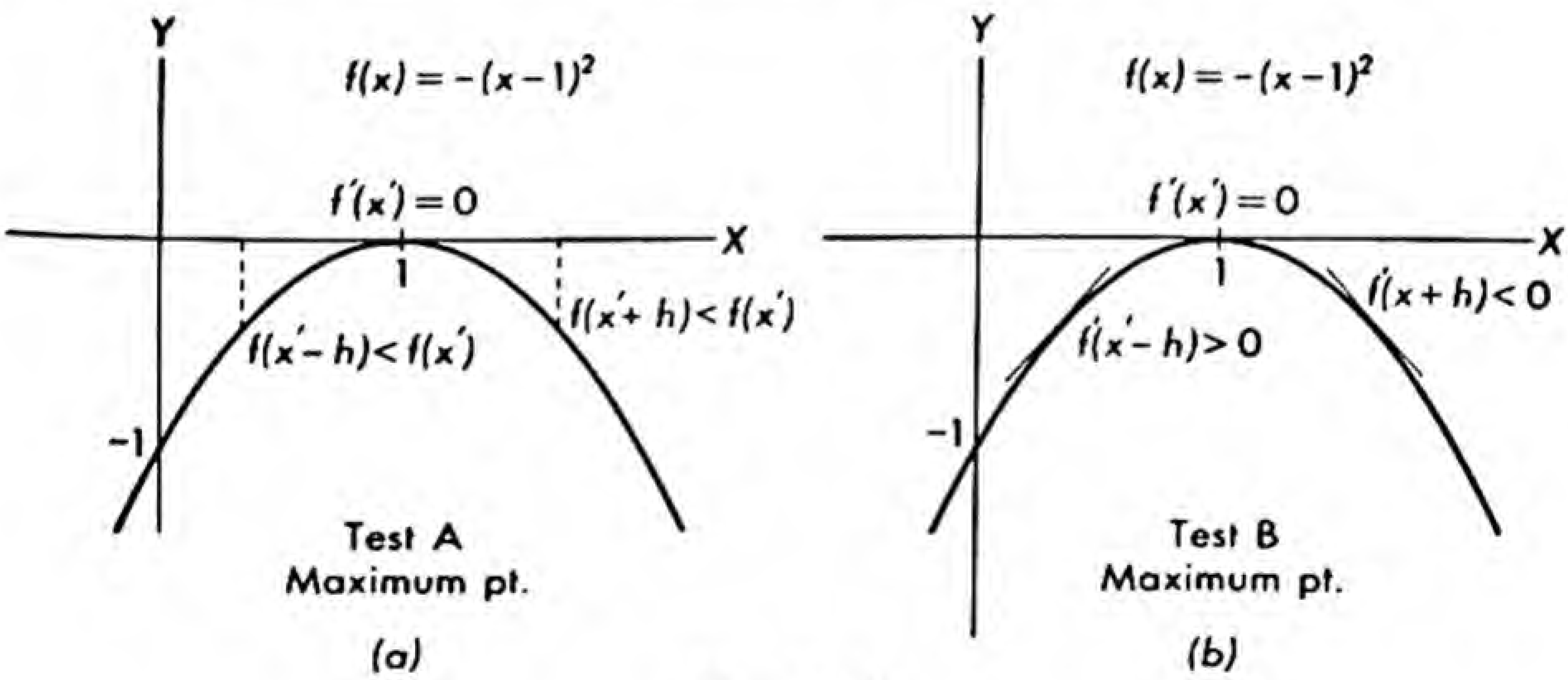


Fig. X–16

We take as a second illustration (Fig. X–16)

$$\begin{aligned} y &= -(x - 1)^2 \\ f'(x) &= -2(x - 1) \\ f'(1) &= 0 \end{aligned}$$

Here we note that to the immediate left of $f(1)$, as well as to the immediate right, the function values are less than at $f(1)$. This, with the knowledge

that $f'(1) = 0$, indicates the presence of a maximum point. Of course the fact that the slope is positive to the left and negative to the right of the critical point, together with the fact that the derivative is 0, constitutes another test for the maximum point. These observations are summarized in the table.

For maximum point at x'

Test A:	$f(x' - h) < f(x')$	$f'(x') = 0$	$f(x' + h) < f(x')$
Test B:	$f'(x' - h) > 0$	$f'(x' = 0)$	$f'(x' + h) < 0$

By way of contrast with the previous cases, we take a look at a case where the first derivative is 0, too, but the critical point is an inflection point. We take (Fig. X–17)

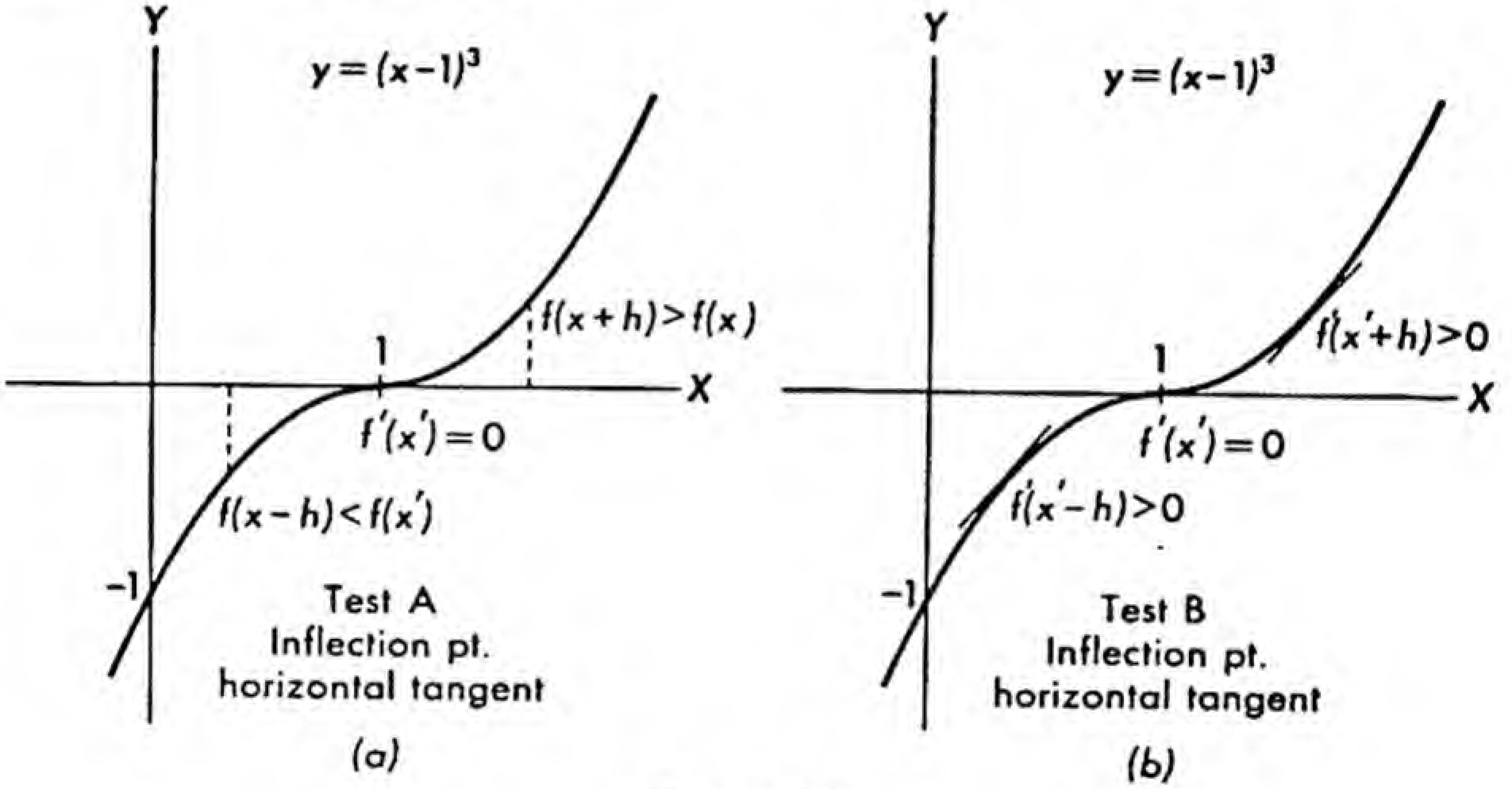


Fig. X–17

$$y = (x - 1)^3$$
$$f'(x) = 3(x - 1)^2$$
$$f'(1) = 0$$

Here the function values to the left of the critical point are less than at the critical point, and they are greater at points to the right of the critical point. Also, the derivatives are positive on either side of the critical point. We summarize:

Inflection point at x' with horizontal tangent

Test A:	$f(x' - h) < f(x')$	$f'(x') = 0$	$f(x' + h) > f(x')$
Test B:	$f'(x' - h) > 0$	$f'(x') = 0$	$f'(x' + h) > 0$

If we studied $y = -(x - 1)^3$, our function would be rotated 180° about the X-axis. The result, as far as the table is concerned would be the

change in sense of all the inequality symbols in the last case. The greater symbol would be replaced by the smaller symbol, and conversely. Thus, the last table is amended by:

Test A:	$f(x' - h) > f(x')$	$f'(x') = 0$	$f(x' + h) < f(x')$
Test B:	$f'(x' - h) < 0$	$f'(x') = 0$	$f'(x' + h) < 0$

EXERCISES (X–13)

1. Examine each of the following for relative maximum, minimum, and inflection points with horizontal tangents:

a. $y = x^2 - 5x - 6$
b. $y = x(x - 1)^2$
c. $y = -(x - 1)^3$
d. $y = 5 - 4x - x^2$
e. $y = x^3 + 3x^2 - 9x - 11$
f. $y = x^4 - 2x^2 + 2$
g. $y = \sqrt[3]{(x + 2)(x - 4)}$
h. $y = x^2(x - 1)$
i. $y = x^3 - 12x + 8$
j. $y = x^{2/3}$
k. $y = x^4$
l. $y = x^{1/3}$
m. $y = x^3 - 3x^2 + 3x$
n. $y = \dfrac{1 + x^2}{x}$
o. $y = |x| + 2$
p. $y = x^2$ for $x \leq 0$
   $y = x$ for $x > 0$
q. $y = |1 - x^2|$

2. An open box is to be made from a rectangular sheet of tin, 12 by 18 inches, by cutting out equal squares from each of the four corners and turning up the sides and ends. Find the size of the squares to be cut out so that the volume of the box shall be a maximum.

3. A rectangle may be inscribed in a 10-inch circle. Find the dimensions so that (a) the area is a maximum; (b) the perimeter is a maximum.

4. A rectangle is inscribed in a semicircle that has a 6-inch radius. (a) Find the dimensions so that the area is a maximum. (b) Find the dimensions so that the perimeter is a maximum.

5. The following equations are derived from some problem:

$$A = xy \qquad \text{and} \qquad y = 24 - 4x$$

a. Determine the conditions for which $A' = 0$ by expressing A in terms of x.
b. Determine the conditions for which $A' = 0$ by expressing A in terms of y.

6. The arms of a right triangle are 6 inches and 8 inches. A rectangle is inscribed in the triangle in such manner that two sides of the rectangle lie along the arms of the triangle, and one vertex of the rectangle is on the hypotenuse. Find the dimensions of the rectangle for maximum area

7. A point moves in a plane so that at time t its coordinates are given by $x = 3t - 1$ and $y = 12t - t^2$. Find the conditions for the maximum (x, y).

8. A rectangular enclosure is planned to cover 500 square feet. In addition to being fenced all around, the enclosure is to be divided into three equal parts by fences parallel to one side. Find the dimensions so that the least amount of material will be needed.

9. A closed rectangular box is to be made from 360 square inches of material with the dimensions of the base in the ratio of 3 to 4. Find the dimensions of the box that will have the greatest capacity.

10. A rectangular box with square base and open top is to have a capacity of 108 cubic feet.
    a. Find the dimensions for minimum surface.
    b. What would be the dimensions if one desired to keep the cost to a minimum, knowing that it costs 5 cents per square foot for the sides and 10 cents per square foot for the base?

11. The contents of a closed, right cylindrical can is planned for 16π cubic inches. What should be the specifications so that it will take the least amount of material? (Volume is equal to the area of the base multiplied by the height. The lateral area is equal to the circumference of the base multiplied by the height.)

12. Two roads meet at right angles. A bicyclist is on one of the roads, 15 miles from the intersection. His destination is 12 miles on the other road from the point of intersection. This other road is passable only on foot. He can average 5 miles per hour on the bicycle and walk at 3 miles per hour. He realizes that if he bicycles part of the way and walks across the meadow in a straight line to his destination, he can save time.
    a. How far will he ride before taking off on foot in order to reach his destination in the least time possible?
    b. What will be the least time?

13. The center of an ellipse is the origin and $(2, 0)$ and $(0, \sqrt{2})$ represent the coordinates of the positive ends of the semimajor and semiminor axes, respectively.
    a. Write the equation of the ellipse.
    b. Consider an inscribed rectangle whose sides are parallel to the axes. Taking one vertex of the ellipse as (X, Y), express Y in terms of X.
    c. Express the value of the area of the rectangle in terms of X.
    d. Find the dimensions of the rectangle with maximum area.

14. If $f(x_M) > f(x_M + h)$ for all small values of $|h|$, and if $f'(x_M)$ exists, show that $f'(x_M) = 0$.

14. A NEW INVERSE OPERATION

We have considered inverses of all operations thus far. It is time that we consider the inverse of differentiation. Of course a symbol is needed. We may take D^{-1} to connote this and refer to it as the **inverse derivative** or the **antiderivative**. The -1 is part of the complex and is not to be construed as a negative power. After all, D is an operator and not a quantity to be raised to any power. If

$$y = x^3$$

for example, then we know that

$$Dy = 3x^2$$

and so

$$D^{-1}(3x^2) = x^3$$

to complete the return to the starting point.

However, $y = x^3 + 8$ or $y = x^3 - 4$ or $y = x^3 + c$, where c is any constant, would have led to the same $Dy = 3x^2$. Now, if $D^{-1}(3x^2)$ is our starting point, we have no way, barring extra information, of predicting the constant. Thus $D^{-1}(3x^2)$ represents an infinite set of solutions, of which x^3 is but one possibility. Any one solution defines a **primitive function,** while $x^3 + c$, where c is any constant, is called a **general solution.** Symbolically, if

$$DF = f$$

then

$$F = D^{-1}f$$

or if

$$\frac{dF(x)}{dx} = f(x)$$

then

$$F(x) = D^{-1}f(x)$$

The function F is called the "primitive function."

The multiplicity of solutions is nothing new to us. The square of a number, for example, consisted of a single number, yet the square root of a number offered two possibilities. The nth root, similarly, has n possibilities. The inverse trigonometric function led to infinite possibilities until certain restrictions and principal values were imposed. So, primitive functions analogously represent a set of functions.

Now, how do we obtain the primitive functions? In the case of the power formula,

$$y = cx^n \qquad \text{and} \qquad \frac{dy}{dx} = ncx^{n-1}$$

the derivative was obtained by: (a) multiplying the coefficient by the exponent, and (b) reducing the exponent by 1.

For the inverse operation, we need only invert the two operations and reverse the order. We obtain the primitive function by (a) increasing the exponent by 1, and (b) dividing the coefficient by the (new) exponent. The following illustrates this:

$$D^{-1}(cx^n) = \frac{c}{n+1}x^{n+1} \qquad n \neq -1$$

is a primitive solution, and the general solution is

$$D^{-1}(cx^n) = \frac{c}{n+1}x^{n+1} + k \qquad (k \text{ is any constant})$$

Check:

$$D\left(\frac{c}{n+1}x^{n+1} + k\right) = cx^n$$

The k represents a set of constants whose specific determination is not possible without further data. We note from the preceding solution that a

constant factor may be placed in front of the operator in inverse differentiation, as in differentiation.

$$D^{-1}\,cx^n = cD^{-1}\,x^n$$

The following are specific illustrations of the above:

a. $D^{-1}(5x^2) = \frac{5}{3}x^3 + k$

b. $D^{-1}(0) = D^{-1}(0x^\circ) = \frac{0}{1}x^1 + k = k$

c. $D^{-1}(5) = D^{-1}(5x^\circ) = \frac{5}{1}x^1 + k = 5x + k$

d. $D^{-1}(2y^3 - 3y^2 + 2y + 6) = \frac{1}{2}y^4 - y^3 + y^2 + 6y + k$

e. $D^{-1}(\sqrt[3]{m}) = D^{-1}m^{1/3} = \frac{1}{4/3}m^{4/3} + k = \frac{3}{4}m^{4/3} + k$

f. $D^{-1}\left(\frac{1}{x}\right) = D^{-1}(x^{-1}) = \frac{1}{0}x^\circ$ (impossible)

Consider a function wherein the rate of change of the dependent variable with respect to the independent variable is always twice the independent variable. If the set of points so defined is indicated by $\{x, y\}$, then we are given that

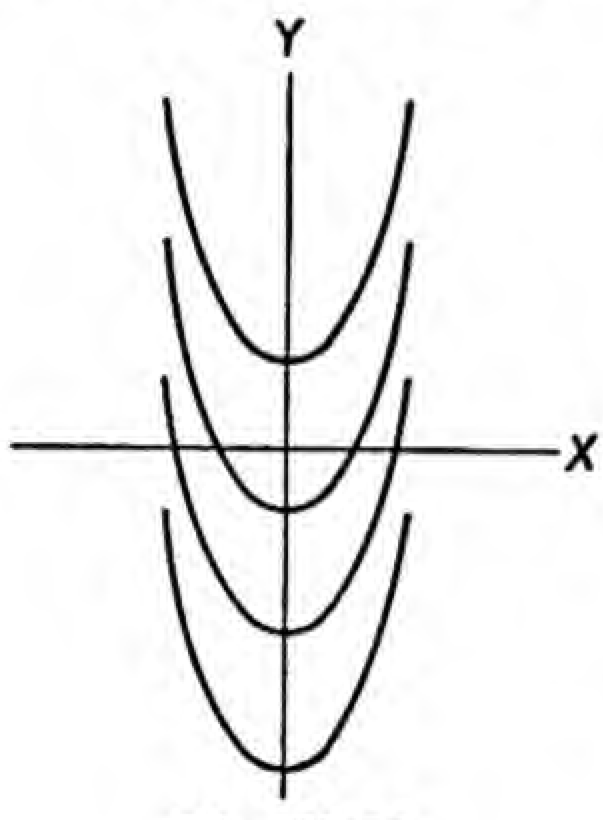


Fig. X–18

$$\frac{dy}{dx} = 2x$$

An equivalent statement of the given might be that the slope is always twice the independent variable. What primitive function is so defined? Well, since

$$Dy = 2x$$

then

$$y = D^{-1}(2x) = x^2$$

and the general solution is

$$y = x^2 + c$$

This indicates that we have a family of parabolas (Fig. X–18) that satisfies the condition of the problem. The slope of the tangent at any point on any one of these parabolas is always $2x$. If, somehow, we knew additionally that (1, 2) is a member of a particular set in which we are interested, then a particular solution is possible. For, by substitution, we have

$$2 = (1)^2 + c$$

and so

$$c = 1$$

yielding the particular primitive solution

$$y = x^2 + 1$$

EXERCISES (X–14)

1. Find the antiderivatives of each of the following and check in every case (the independent variable appears in each expression):

a. $2x^4 + 3x^2 - 2x$

b. $3x - 2x^2$

c. $5x^3 - 7$

d. 6; $\{x, y\}$

e. $9\sqrt{x}$

f. $8x^{1/3}$

g. $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

h. $x^2 + \dfrac{1}{x^2}$

i. $2y^3 - \dfrac{4}{y^3}$

j. $2x(x^2 + 1)^3$

k. $\dfrac{-6x^2}{\sqrt{1 - x^3}}$

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2. Find the particular solution, given the additional data as shown:

a. $Df(x) = x + 1$; $(2, 5)$

b. $Df(x) = 3\sqrt{x} - x$; $(1, \frac{1}{2})$

c. $y = D^{-1}\left(\dfrac{1}{x^2} + 1\right)$; $(1, 0)$

d. $\dfrac{dy}{dx} = x^2 - 3x$; $(1, -1)$

e. $\dfrac{dy}{dx} = 2x(x^2 + 1)^{3/2}$; $(0, 0)$

3. If $Df = h$ and $Dg = k$, then $f = D^{-1}h$ and $g = D^{-1}k$. Supply the reasons for the following argument, which shows that inverse differentiation is distributive over a sum.

$$\begin{aligned} D(f + g) &= Df + Dg \\ f + g &= D^{-1}(Df + Dg) \\ D^{-1}(Df + Dg) &= f + g \\ D^{-1}(h + k) &= D^{-1}h + D^{-1}k \end{aligned}$$

4. If the slope of a curve at any point is given by $4x + 2$ and the curve passes through $(-1, 3)$, find the equation of the curve and the coordinates of any extreme point.

5. If $D_x y = k$ and k is a constant, then the graph of the equation $y = f(x)$ is a straight line. Why?

6. We recall that motion in a plane may be described as a function of the set $\{t, s\}$, where t represents time and s represents distance. We have also seen that ds/dt represents instantaneous speed. Suppose that we know that the instantaneous speed is $20t + 5$. Express s in terms of t, knowing too that when $t = 0$, $s = 0$.

7. Instantaneous acceleration a was defined as the rate of change of the instantaneous velocity v. Thus $a = dv/dt$. Find the unknown information in each of the following:

a. $a = 32$, $v = ?$, $(v = 0, t = 0)$

b. $a = 32$, $v = ?$, $(v = 60, t = 0)$

c. $a = 32$, $v = ?$, $s = ?$, $(v = 60, t = 0)$ and $(s = 0, t = 0)$

8. Near the earth's surface the acceleration due to gravity may be represented by $-g$ for a particle that is thrown vertically upward. Write an equation in v; then express the motion of the particle as s in terms of t. Use v_0 to represent the initial velocity (when $t = 0$) and s_0 to represent the initial height.

9. Explain why instantaneous acceleration may be represented by $D^2_t s$ or d^2s/dt^2.

10. Equations such as $dy/dx = 3x + 5$, which contain derivatives (first, second, or any other *order*) and in which the derivatives are to any degree [first as in the last illustration, second as in $(dy/dx)^2 = 7x$] are called *differential equations*. We have actually solved some simple differential equations. We have gone far enough to recognize some other cases that we are in a position to solve. For example, the solution of

$$3y\frac{dy}{dx} + 2\frac{dy}{dx} + x + 5 = 0$$

is

$$\tfrac{3}{2}y^2 + 2y + \tfrac{1}{2}x^2 + 5x + c = 0.$$

Find the general solutions of each of the following:

a. $y^2\dfrac{dy}{dx} - y\dfrac{dy}{dx} = x$

b. $\dfrac{dy}{dx} - 5x^2 + 2 = 0$

c. $y\dfrac{dy}{dx} + \dfrac{x}{(x^2 - 1)^2} = 0$

11. Prove that $dy/dx = -(x/y)$ for any circle with center at the origin.

12. Find an equation, as in exercise 11, for the ellipse with center at the origin.

13. Prove that for any straight line $x \neq c$, through the origin $dy/dx = y/x$.

14. Prove that for any straight line, $x \neq c$, $d^2y/dx^2 = 0$.

15. MORE INVERSE DERIVATIVES

We turn our attention now to the matter of antiderivatives where a third variable is present explicitly or implicitly. Consider

$$\frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

or, equivalently,

$$D_t x^3 = 3x^2 D_t x$$

Then, by definition,

$$D_t^{-1}(3x^2 D_t x) = x^3 + c$$

The derivative of x^3 is $3x^2$ multiplied by some differential coefficient which depends on the independent variable. If the independent variable is x itself, then the differential coefficient is $dx/dx = 1$. So, the antiderivative of $3x^2\, D_* x$ [which is $D_*^{-1}(3x^2\, D_* x)$, in which the asterisk subscript represents any related independent variable, even x itself] is always $x^3 + c$. Thus

$$D_x^{-1}(3x^2) = D_t^{-1}(3x^2\, D_t x) = D_*^{-1}(3x^2\, D_* x) = \cdots = x^3 + c$$

It follows, too, that if functions are equal, their antiderivatives can differ by at most an additive constant. Thus if,

$$3y\frac{dy}{dt} = x\frac{dx}{dt}$$

then

$$\tfrac{3}{2}y^2 = \tfrac{1}{2}x^2 + c'$$

or

$$3y^2 = x^2 + c \qquad (c = 2c')$$

(The replacement of one constant by another is just a matter of convenience, since both are arbitrary in this context. This explains too why it is immaterial whether we place the constant on one or the other side of the equation. Further, this indicates that it is unnecessary to place a constant on each side of the equation.)

Similarly, if

$$y^2\frac{dy}{dt} - x^3\frac{dx}{dt} = 0$$

then

$$\tfrac{1}{3}y^3 - \tfrac{1}{4}x^4 = k$$

or

$$4y^3 - 3x^4 = c$$

Suppose that we have

$$\frac{dy}{dx} = x(x^2 - 3)^5$$

Enough experience may have accumulated already for us to determine the answer at sight. It looks as though the answer ought to be some numerical multiple of $(x^2 - 3)^6$. Once this is anticipated, the mental process of checking, by taking the derivative of the last expression, will indicate the numerical coefficient that is needed to obtain the original derivative above. However, for this purpose as well as others, a systematic approach can be recommended.

We set $u = x^2 - 3$, which gives us $du/dx = 2x$ and then that $x = (1/2)(du/dx)$. Substituting these facts, we see that the problem is converted to a power-law case. (If it did not so convert, then this method would be of no aid.) We have

$$\frac{dy}{dx} = \frac{1}{2}u^5\frac{du}{dx}$$

which, by the previous method, yields

$$y = \frac{1}{12}u^6 + c$$

and so

$$y = \frac{1}{12}(x^2 - 3)^6 + c$$

As another illustration consider:

$$\frac{dy}{dx} = 2(3x^2 - 1)\sqrt{x^3 - x}$$

$$\frac{dy}{dx} = 2u^{1/2}\frac{du}{dx} \qquad \text{Let } u = x^3 - x \text{ so } \frac{du}{dx} = 3x^2 - 1$$

$$y = \tfrac{4}{3}u^{3/2} + c$$

$$y = \tfrac{4}{3}(x^3 - x)^{3/2} + c$$

Had the coefficient of the radical term been anything but $3x^2 - 1$ or a numerical multiple thereof, the solution would have become quite difficult.

EXERCISES (X–15)

1. Evaluate or solve each of the following:

a. $\dfrac{dy}{dx} = x(4 - x^2)^3$

b. $\dfrac{dy}{dx} = \dfrac{5x}{(x^2 - 1)^2}$

c. $D^{-1}3x^3(3x^4 + 1)^6$

d. $D^{-1}x\sqrt{x^2 + 1}$

e. $D^{-1}\dfrac{x^2}{(x^3 + 1)^{3/2}}$

f. $y\dfrac{dy}{dt} + x\dfrac{dx}{dt} = 0$

g. $5y\dfrac{dy}{dt} = 3x^2\dfrac{dx}{dt}$

h. $\dfrac{dy}{dt} + \dfrac{4x^2}{y}\dfrac{dx}{dt} = 0$

2. The subject of antiderivatives would be unnecessarily complicated if we were to admit discontinuous primitives. Thus $D^{-1}x$ has the discontinuous primitive

$$y = \begin{cases} \tfrac{1}{2}x^2 & x < 0 \\ \tfrac{1}{2}x^2 + 1 & x \geq 0 \end{cases}$$

a. Show that $Dy = x$ for all real values of x.
b. Graph the primitive solution.
c. Write two other discontinuous primitives for $D^{-1}x$.

3. a. Find three primitives of $D^{-1}x^2$ that are continuous for all real values of x.
b. Show that the solutions in (a) differ by an additive constant, as mentioned in the text.
c. Illustrate (b) by graphing your solutions.

4. Show that continuous primitives defined on the same interval differ only by an additive constant.

X–15 REVIEW

1. Find the extreme values for $y = x^2(x - 3)^2$.
2. Study $y = x^4 - 2x^2 - 2$ for extreme points.
3. Show that for a given hypotenuse, the isosceles right triangle has the maximum area.
4. Find two numbers whose sum is 17, such that the product of one by the square of the other is a maximum.

5. A rectangular box with open top and square base is to be lined with material that costs 10 cents a square foot. The capacity of the box must be 25 cubic feet. Find the minimum cost.

6. The base of an isosceles triangle is 8 inches and its altitude is 12 inches. A rectangle is inscribed in this triangle so that a vertex lies on each of the arms of the triangle and a side of the rectangle rests on the base of the triangle. Find the dimensions of the rectangle for maximum area.

7. The sum of two numbers is m. Find the value of each so that the sum of their squares shall be a minimum.

8. Investigate $y = x^3/(2 - x)$ for extreme values.

9. Find the general solution for each of the following:

a. $D^{-1}(6x^2 - x + 3)$

b. $D^{-1}(\sqrt{x} + \sqrt[3]{x})$

c. $D^{-1}(x - 1)^{-2}$

d. $D^{-1}5x^2(x^3 + 1)^2$

e. $D^{-1}\dfrac{x}{\sqrt{x^2 - 1}}$

10. Find the general solution:

a. $\dfrac{dy}{dx} = x^2 - x$

b. $x^2\dfrac{dy}{dx} + 2xy = 0$

c. $y\dfrac{dy}{dt} - x\dfrac{dx}{dt} = 0$

d. $\sqrt{y}\dfrac{dy}{dt} + \sqrt{x}\dfrac{dx}{dt} = 0$

11. Find the antiderivatives of each of the following, wherein the letter given is the independent variable:

a. $x^3 - 5x^2 + 1$

b. $\dfrac{6z}{(z^2 - 1)^3}$

c. $u\sqrt{1 - u^2}$

d. $2v(v^2 + 1)^3$

12. If $f'(x) = 3x^2 - 1$ and $(1, 2)$ belongs to f, find $f(x)$.

13. If

$$g'(x) = \frac{5x}{(x^2 - 1)^2}$$

and $(-4, 1)$ is an element of g, find $g(x)$.

XI

INTEGRATION

1. THE FUNDAMENTAL THEOREM

The solution of the tangent problem, which we met in the preceding chapter, was one of the two basic problems that motivated the creation of the calculus. The second problem concerned areas. In an earlier chapter, we developed the postulational basis for the concept of area in connection with rectangles, triangles, and other polygons. But what of *areas* enclosed totally or in part by curves, as in Fig. XI–1? We shall witness in this chapter something of the solution of this as well as related problems. In defining this new area, we shall employ once again the method of extension, guided by the principle of consistency, which we have found so useful heretofore.

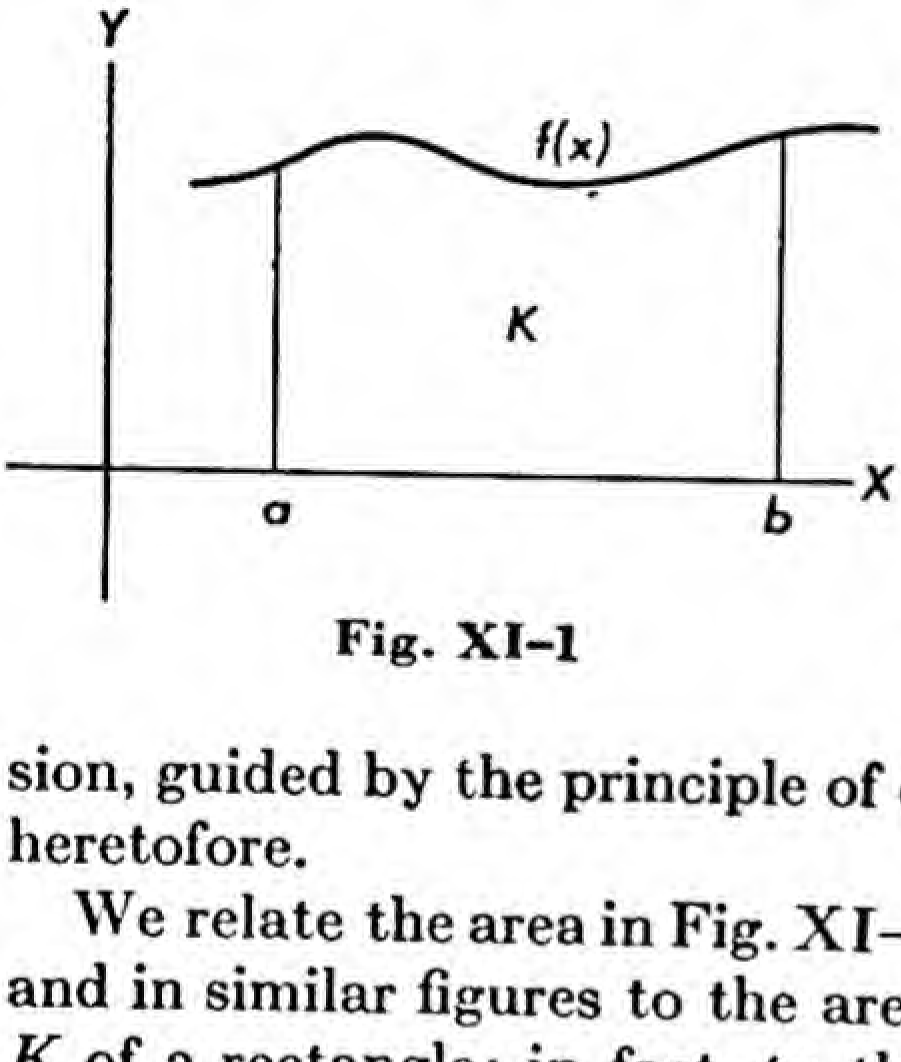

Fig. XI–1

We relate the area in Fig. XI–1 and in similar figures to the area K of a rectangle; in fact, to the sum of the areas of a great many *inscribed* rectangles. Indeed we shall define a sequence of such sums whose limit, when it exists, will constitute the definition of the area of the figure. We turn to this presently.

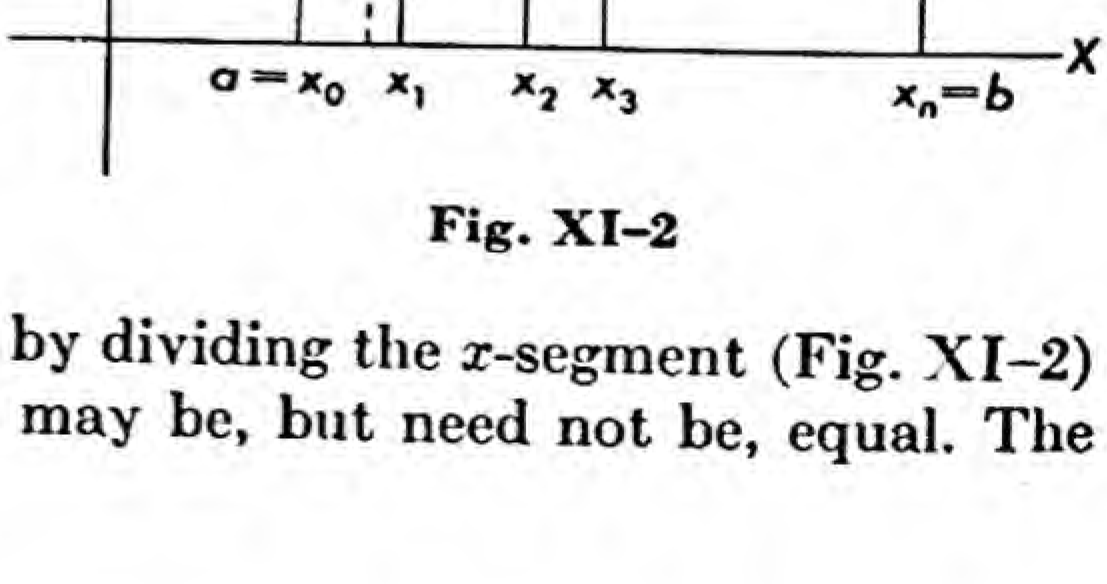

Fig. XI–2

We make a start in this direction by dividing the x-segment (Fig. XI–2) from a to b into n segments which may be, but need not be, equal. The

points of demarcation can be indicated by:

$$a = x_0, x_1, x_2, \cdots, x_n = b$$

Let $f(x_1)$ be any ordinate in the first interval or, better still, the largest ordinate in that interval. We build a rectangle of width $(x_1 - x_0)$, which for convenience we label as Δx_1. The area of this rectangle is $f(x_1)\Delta x_1$. Similarly the areas of the succeeding rectangles are

$$f(x_2)\Delta x_2, f(x_3)\Delta x_3, \cdots, f(x_n)\Delta x_n$$

The sum of the areas of these rectangles is our first approximation to the area K, as indicated in Fig. XI–1. We represent this sum by S_1.

We increase the number of intervals on the X-axis, and thereby we get an increased number of rectangles whose sum is indicated by S_2, which is a closer approximation to K. We can continue this indefinitely, in the light of our original assumption, and get a sequence of sums of areas of rectangles

$$S_1, S_2, S_3, \cdots, S_p, \cdots$$

where

$$S_p = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_p)\Delta x_p$$

This can be more conveniently represented by a new symbol

$$S_p = \sum_{i=1}^{p} f(x_i)\Delta x_i$$

The capital Greek letter $\sum$ (sigma) is generally used to denote summation. Here the summation is understood to proceed with $i = 1$, then $i = 2, 3, 4, \cdots$, and so forth until the integer on top of the sigma sign is reached. Thus, for the sake of illustration,

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$$

$$\sum_{i=1}^{5} x^i = x^1 + x^2 + x^3 + x^4 + x^5$$

$$\sum_{i=1}^{3} (x_i + 1)^i = (x_1 + 1) + (x_2 + 1)^2 + (x_3 + 1)^3$$

If we permit p in S_p to increase indefinitely, while the largest $\Delta x_i \to 0$, then the definition of the area K is given by

$$K = \lim_{p \to \infty} S_p = \lim_{p \to \infty} \sum_{1}^{p} f(x_i)\Delta x_i$$

We assume that this limit exists for any continuous function $f(x)$. Further, $f(x)$ need not be everywhere positive in the interval, as our first diagram

suggests. That is merely a matter of convenience. If $f(x)$ is negative throughout, our sums will be negative. If $f(x)$ is positive as well as negative in the interval, our sums will merely be the algebraic addition of the terms. We could, if we desired, take the negative and positive portions separately and then deal with their absolute values. We shall see this concretely soon.

We introduce still another symbol

$$K = \lim_{p \to \infty} \sum_{1}^{p} f(x_i)\, \Delta x_i = \int_a^b f(x)\, dx$$

The new **integral** symbol is defined in the equality represented in the preceding line. The term on the right represents the limit of the indicated sum, where at least the largest interval Δx_i approaches 0 and $f(x_i)$ is any arbitrary value in the corresponding interval. The term on the right is read as *the integral from a to b of* $f(x)$. The a and b are the *lower and upper limits*, respectively, and $f(x)$ is the *integrand*. The entire term on the right is a constant and is called a *definite integral*.

The symbol dx is an outgrowth of the earlier symbol Δx, which indicates that x is the independent variable of the *integrand* function f. The operation of summing and limit taking (or, briefly, of *integration*) is to be taken only with respect to the indicated independent variable. The integral symbol, $\int$, is an elongation of the symbol S which suggests the originating concepts of sum and limit.

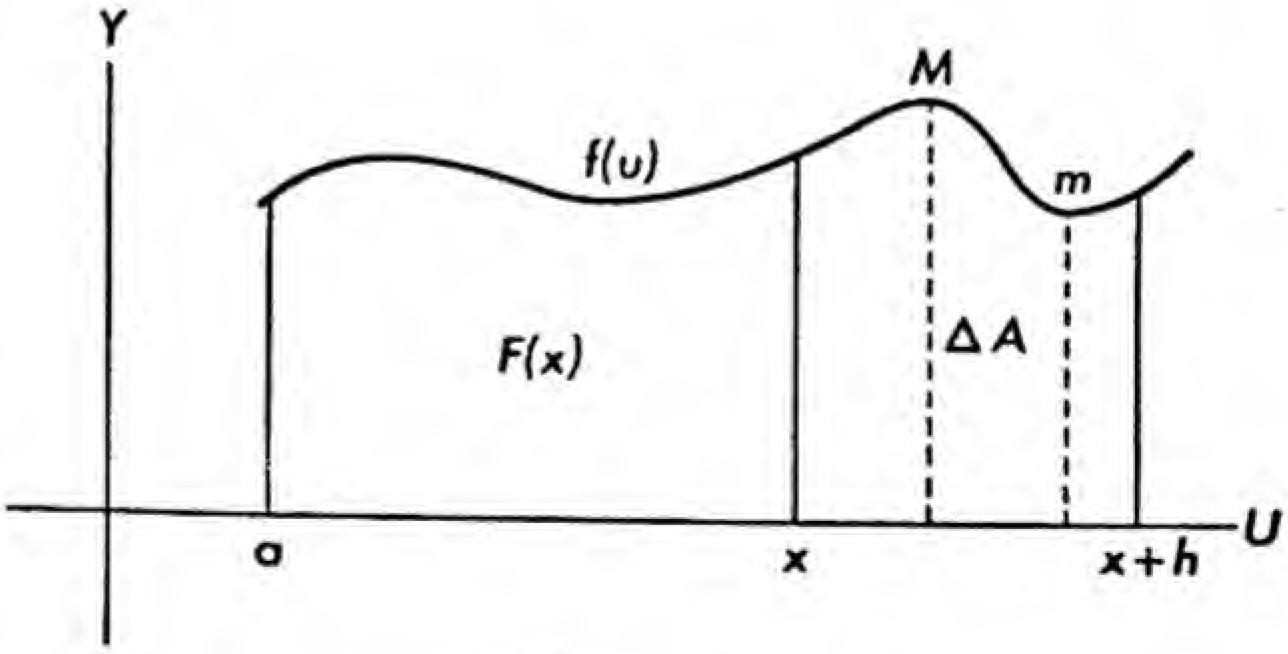

Fig. XI–3

Let us take a section of area under the continuous curve $f(u)$ which is bounded by the curve, $u = a$, $u = x$, and the U-axis. (The switch to u is only for convenience.) We have defined the area K of Fig. XI–3 as

$$K = \int_a^x f(u)\, du$$

Since the upper limit x is a variable, the area depends on this variable, and so may be expressed as some function of x, say, $F(x)$. The integral is no

longer a constant and is called, by way of contrast, an **indefinite integral.** We have

$$F(x) = \int_a^x f(u)\,du$$

Let us increase our upper bound by h; that is, to $x + h$. The total area is now $F(x + h)$, and the increase in area ΔA is indicated by

$$\Delta A = F(x + h) - F(x)$$

Suppose that we let M represent the largest value of $f(u)$ in this interval and let m be the smallest value of $f(u)$ in the same interval. If we draw rectangles, one with the height M and the other with height m on the base from x to $x + h$ (Fig. XI–4), then Mh is no less than $\Delta\ A$ and mh is no greater than $\Delta\ A$.

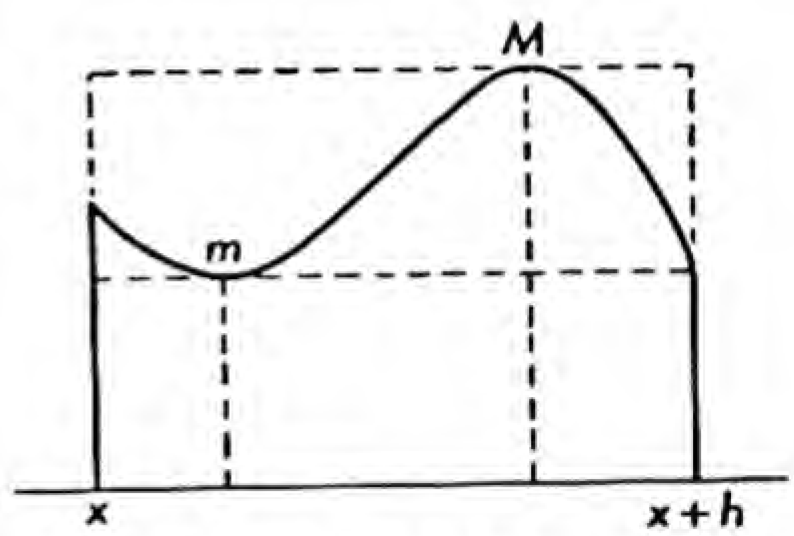

Fig. XI–4

$$mh \le \Delta A \le Mh$$

or

$$mh \le F(x + h) - F(x) \le Mh$$

Since h has been taken as a positive quantity, we can divide through by h without change in the sense of the inequalities:

$$m \le \frac{F(x + h) - F(x)}{h} \le M$$

Because of continuity, as $h \to 0$, M and m both will approach the value of $f(u)$ at x; that is, they will approach $f(x)$. Thus

$$f(x) \le \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \le f(x)$$

The middle quantity, of course, is the familiar $F'(x)$ which is caught between two equal quantities $f(x)$ and so must be equal to them. We have, then, that

$$F'(x) = f(x)$$

or

$$F(x) = D^{-1}f(x)$$

This means that $F(x)$ is a primitive function of $f(x)$. Since $F(x)$ has a derivative $f(x)$, it is a continuous primitive over the domain of definition of $f(u)$. We recall that

$$F(x) = \int_a^x f(u)\,du$$

Consequently the indefinite integral, by substitution, is a primitive function of $f(x)$.

$$\int_a^x f(u)\,du = D^{-1} f(x)$$

Suppose that $H(x)$ is another continuous primitive function of $f(x)$ over the same interval. Since continuous primitive functions can differ from each other only by an additive constant, as was seen at the end of Chapter X, we write that

$$F(x) = H(x) + c$$

We should not lose sight of the fact that $F(x)$ and the integral represent the area under the curve bounded by the X-axis and the ordinates $x = a$ and $x = b$. We assume that all the functions are defined over $a \leq x \leq b$. If we take the upper limit of the integral to be a, the same as the lower limit, then $K = F(x) = F(a) = 0$. So, by substitution in the preceding equation, we have

$$0 = H(a) + c$$

or

$$c = -H(a)$$

This means that

$$F(x) = H(x) - H(a)$$
$$\int_a^x f(u)\,du = H(x) - H(a)$$

Now, if the upper limit is fixed at $x = b$, we have, by substitution, the definite integral

$$\int_a^b f(u)\,du = H(b) - H(a)$$

where H is a continuous primitive function, antiderivative, of f over the domain $a \leq x \leq b$.

Thus *a link has been forged between an integral and the antiderivative.* While much of our thinking has rested intuitively on the concept of area, it is possible, as is done in advanced works, to develop the whole matter as pure analysis.

Let us give a concrete example of this. Suppose that we are interested in the value of

$$\int_2^3 x^2\,dx$$

This indicates that $f(x) = x^2$. $H(x)$ is a primitive function of this. That is,

$$\begin{aligned} H(x) &= D^{-1}x^2 \\ H(x) &= \tfrac{1}{3}x^3 + c \\ H(b) &= H(3) = \tfrac{1}{3}(3)^3 = 9 + c \\ H(a) &= H(2) = \tfrac{1}{3}(2)^3 = 2\tfrac{2}{3} + c \end{aligned}$$

Thus, $$\int_2^3 x^2\,dx = H(3) - H(2) = 9 - 2\tfrac{2}{3} = 6\tfrac{1}{3}$$

All this could be condensed as follows:

$$\int_2^3 x^2\,dx = \tfrac{1}{3}x^3\Big]_2^3 = 9 - 2\tfrac{2}{3} = 6\tfrac{1}{3}$$

In finding the value of the integral, we first find the antiderivative of the integrand, disregarding the constant (since this disappears after the subtraction). A half-bracket is placed on this antiderivative, with the limits of integration indicated on it as $]_2^3$. The value of this antiderivative is found for each of the values of the two limits by substitution, and the result of the substitution of the lower limit is subtracted from that of the upper limit.

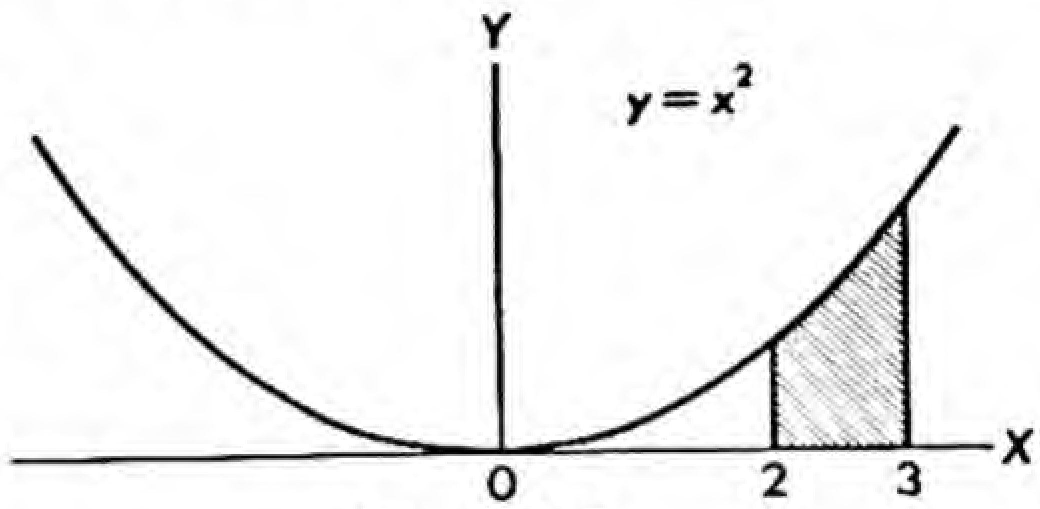

Fig. XI–5

If we refer the foregoing illustration to the matter of areas, we have in effect found the area bounded by the parabola $y = f(x) = x^2$, the X-axis, $x = 2$, and $x = 3$ (refer to Fig. XI–5).

The independent variable of the integrand is of the nature of a **dummy variable,** since it acts only as a place holder. We have met other instances of this before. Specifically,

$$\int_2^3 x^2\,dx = \int_2^3 u^2\,du = \int_2^3 t^2\,dt = \cdots$$

Or, in general,

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(*)\,d*$$

Let us find the area between the X-axis and the parabola $y = x^2 - x - 6$. By setting $y = 0$, we determine the points of intersection of the curve and the X-axis (refer to Fig. XI–6).

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \qquad \text{and} \qquad x = -2$$

So, $$K = \int_{-2}^3 (x^2 - x - 6)\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 6x\Big]_{-2}^3$$

$$K = (9 - 4\tfrac{1}{2} - 18) - (-2\tfrac{2}{3} - 2 + 12)$$

$$K = -20\tfrac{5}{6} \text{ sq units}$$

The negative result is something that was anticipated some time ago, since the integrand is negative throughout the interval of integration.

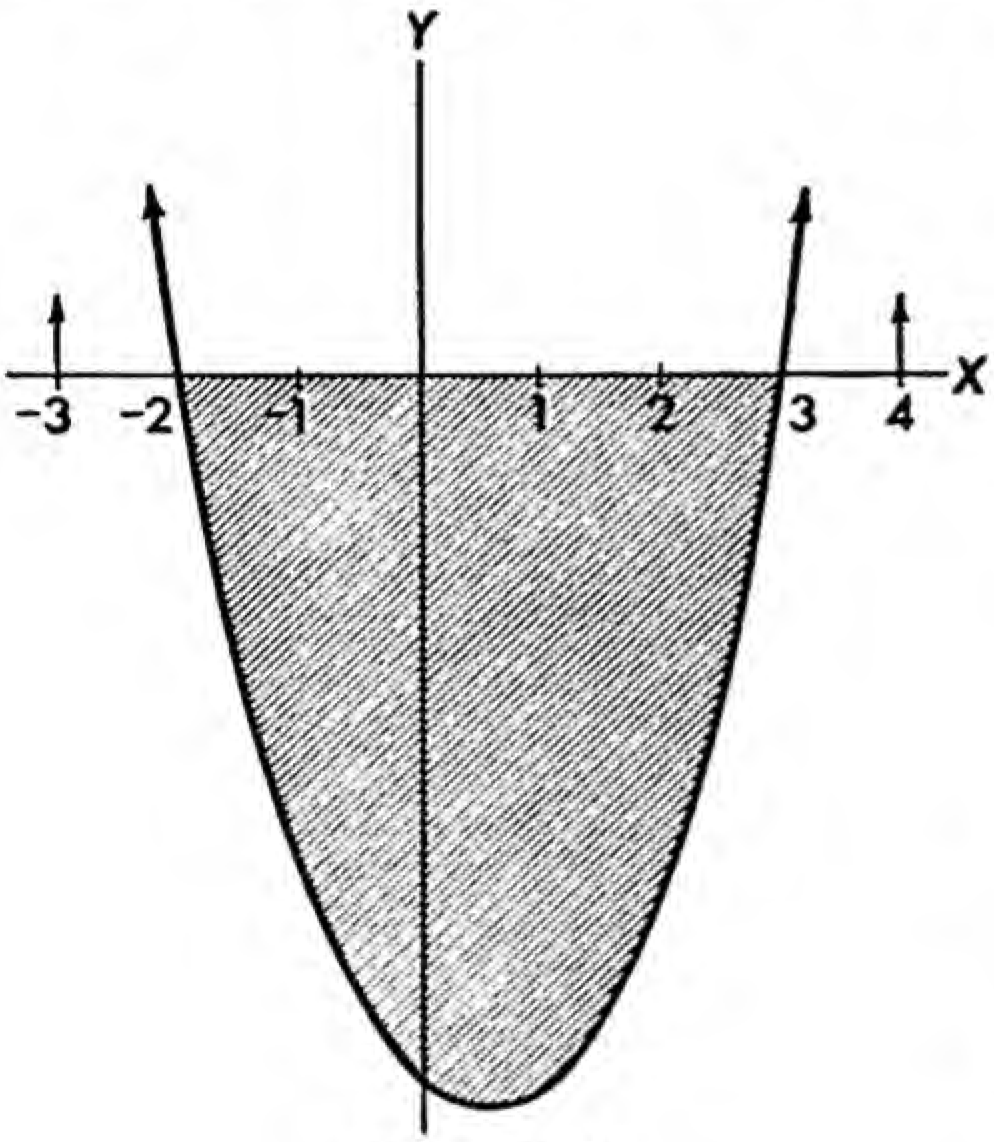

Fig. XI–6

Suppose that we consider the same integrand between the bounds -3 and 4. This will add two more area sections, both above the X-axis to that previously computed (Fig. XI–6).

$$K = \int_{-3}^{4} (x^2 - x - 6)\, d_x =$$

$$\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 6x\Big]_{-3}^{4} = -15\tfrac{1}{6}$$

In terms of absolute value, this area is less than that of the preceding case which is supposedly only a portion of this. The enigma is quickly resolved if we note that by taking the area from $x = -4$ to $x = 3$, we have combined positive and negative areas, getting an algebraic sum. To avoid this, it is necessary to obtain the positive and negative areas separately and then to combine their absolute values. The following illustrates this further:

$$\int_0^3 (2x - 4)\, dx$$

Since the integrand is 0 when $x = 2$ only, it is necessary to write

$$\int_0^3 \left| (2x - 4) \right| dx = \left| \int_0^2 (2x - 4)\, dx \right| + \left| \int_2^3 (2x - 4) \right|$$

$$= \left| x^2 - 4x\Big]_0^2 \right| + \left| x^2 - 4x\Big]_2^3 \right|$$

$$= 4 + 1 = 5$$

As an alternative we could place a negative sign in front of the first integral as follows:

$$\int_0^3 |(2x - 4)|\, dx = -\int_0^2 (2x - 4)\, dx + \int_2^3 (2x - 4)\, dx = 4 + 1 = 5$$

It is possible by the above method to find the area included between two curves, as in the case of the following parabolas:

$$f(x) = y = x^2 - 4x + 8$$
$$g(x) = y = 6x - x^2$$

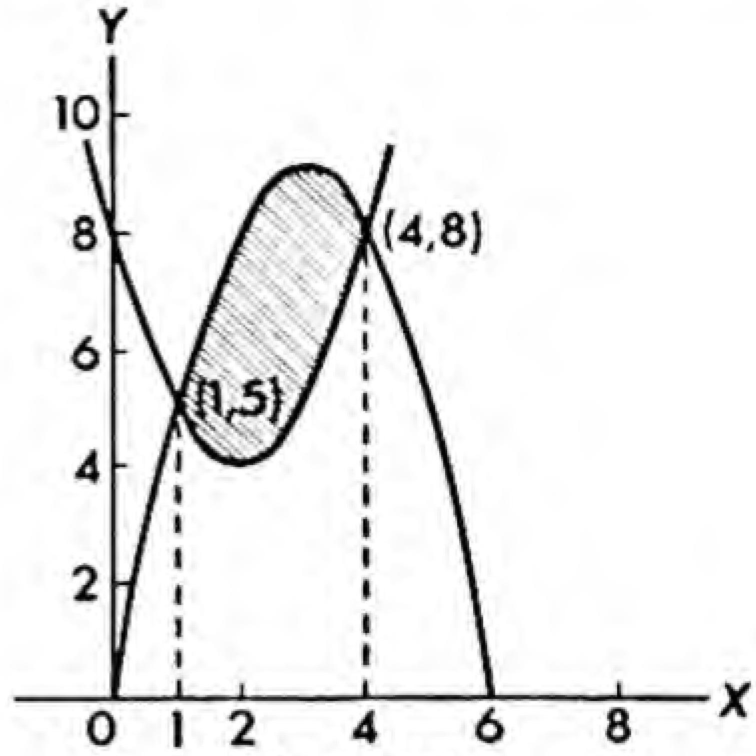

Fig. XI–7

By means of substitution we solve the equations simultaneously to get the points of intersection:

$$6x - x^2 = x^2 - 4x + 8$$
$$2x^2 - 10x + 8 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x = 4 \quad \Big| \quad x = 1$$
$$y = 8 \quad \Big| \quad y = 5$$

The curves intersect at (4, 8) and (1, 5). Figure XI–7 shows that $6x - x^2 > x^2 - 4x + 8$ in this interval. We are concerned with two areas here: (1) the area included by $x = 1$, $x = 4$, $y = 0$, and $g(x)$; and (2) the area included by $x = 1$, $x = 4$, $y = 0$, and $f(x)$. The desired area is found by taking the difference of these two.

$$K = \int_1^4 (6x - x^2)\, dx - \int_1^4 (x^2 - 4x + 8)\, dx$$

The integrations can be performed and the results subtracted to get the result sought. However, we take this occasion to point out a valuable simplification. The problem calls for, essentially, the finding of antiderivatives of $g(x)$ and $f(x)$ and subtracting the results for certain values. We have seen that the taking of antiderivatives is a distributive process. Thus, if the limits are the same, we could take instead the antiderivative of the difference, as is shown now:

$$K = \int_1^4 \{6x - x^2 - (x^2 - 4x + 8)\}\, dx$$
$$= \int_1^4 (-2x^2 + 10x - 8)\, dx = -\tfrac{2}{3}x^3 + 5x^2 - 8x \Big]_1^4 = 9 \text{ sq units}$$

EXERCISES (XI–1)

1. Find the value of each of the following:

   a. $\sum_1^4 x^i$, when $x = 2$

   b. $\sum_1^3 (x + 1)^i$, when $x = 3$

   c. $\sum_1^3 3^k$

2. Evaluate each of the following:

a. $\int_1^3 x^3\,dx$

b. $\int_1^4 \sqrt{x}\,dx$

c. $\int_0^2 (t^2 - 3t + 4)\,dt$

d. $\int_1^4 \frac{1}{x^2}\,dx$

e. $\int_0^{-4} x(x^2 + 1)\,dx$

f. $\int_{-2}^1 x^2(5 - x^3)\,dx$

g. $\int_0^a (5 - h)dh$

h. $\int_1^3 \left(z + \frac{1}{z^2}\right) dz$

i. $\int_0^5 \frac{u}{\sqrt{u^2 + 1}}\,du$

j. $\int_0^4 y\sqrt{y^2 + 9}\,dy$

3. Evaluate each of the following area integrals, wherein the integrand has to be scrutinized carefully within the bounds of integration. The integral may have to be replaced with two integrals or banished entirely as being meaningless.

a. $\int_2^6 (x - 4)\,dx$

b. $\int_1^6 (x^2 - 5x + 4)\,dx$

c. $\int_{-4}^4 x\sqrt{x^2 - 9}\,dx$

d. $\int_0^6 (x^3 - 9x^2 + 18x)\,dx$

4. a. Find the area bounded by $y = 2x + 3$, $x = 1$, $x = 4$, and the X-axis.
   b. Find the area by geometric means alone.

5. Find the area bounded by $y = 4x - x^2$ and the X-axis. (A sketch of the curve should always be made.)

6. Find the area between $y = x^3 - 9x$ and the X-axis.

7. Find the area between $y = -x^2 + 2x + 8$, $x = 1$, $x = 3$, and the X-axis.

8. Find the area included between the curves in each case:
   a. $y = x$ and $6y = x^2$
   b. $y = 3x^2$ and $y = 4 - x^2$
   c. $y = 3x - x^2$ and $y + x = 3$
   d. $y = x\sqrt{2x^2 + 1}$ and $y = 3x$

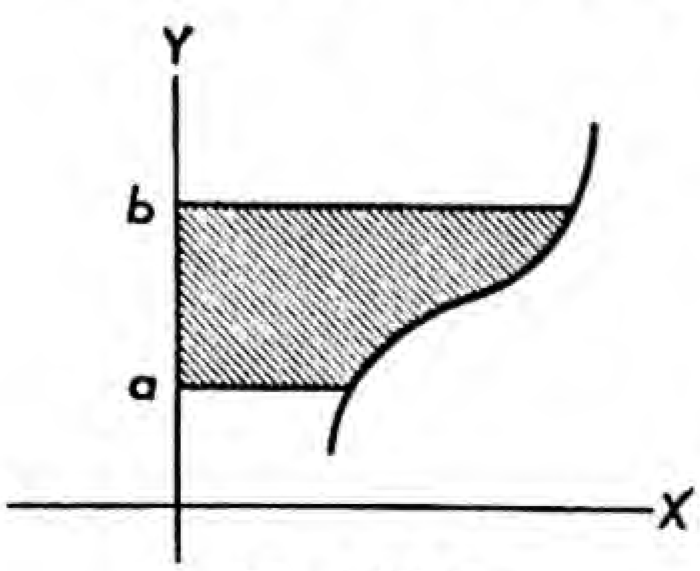

Fig. XI-8

9. Using the fact that

$$\int_a^b f(x)\,dx = H(b) - H(a)$$

where $f(x) = H'(x)$, show that

a. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

b. $\int_a^m f(x)\,dx + \int_m^b f(x)\,dx = \int_a^b f(x)\,dx$

c. Devise a geometric illustration for each of (a) and (b).

10. The entire discussion concerning areas could have been made from the view of the area included by a curve, the Y-axis, and two lines of the abscissa, as in Fig. XI–8. One can well imagine that the net effect would merely be a wholesale substitution of y-values for x-values, so that the area formula would become

$$K_y = \int_{y=a}^{y=b} f(y)\, dy$$

By way of an illustration (Fig. XI–9), let us find the area bounded by the Y-axis and $x = y^3 - y^2$. To get the y-intercepts, we set $x = 0$.

$$y^2(y - 1) = 0$$
$$y = 0 \qquad y = 0 \qquad y = 1$$
$$x = 0 \qquad x = 0 \qquad x = 1$$
$$K = \int_0^1 x\, dy = \int_0^1 (y^3 - y^2)\, dy = \tfrac{1}{4}y^4 - \tfrac{1}{3}y^3\Big]_0^1$$
$$= \tfrac{1}{4} - \tfrac{1}{3} = -\tfrac{1}{12} \text{ sq units}$$

Interpret the negative answer geometrically.

11. Find the area bounded by $x = y^2 + y$ and the Y-axis.

12. Find the area bounded by $x = y^2$, $y = -2$, $y = 2$, and the Y-axis.

2. VOLUMES OF REVOLUTION

The integral as a limit of a sequence of sums continues to offer additional opportunities. Another application concerns the volumes of surfaces of revolution.

If the curve $y = f(x)$ is revolved around the X-axis, we get the surface of revolution shown in Fig. XI–10.

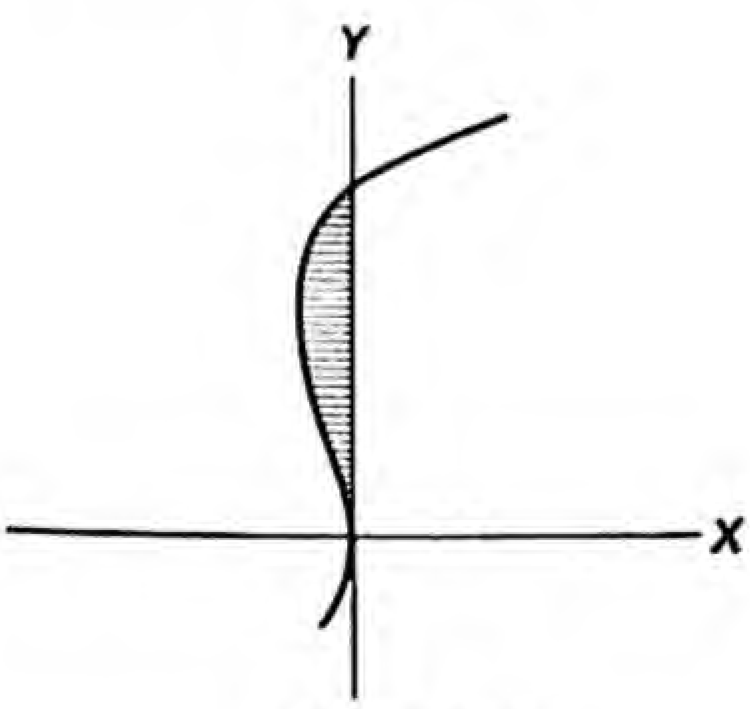

Fig. XI–9

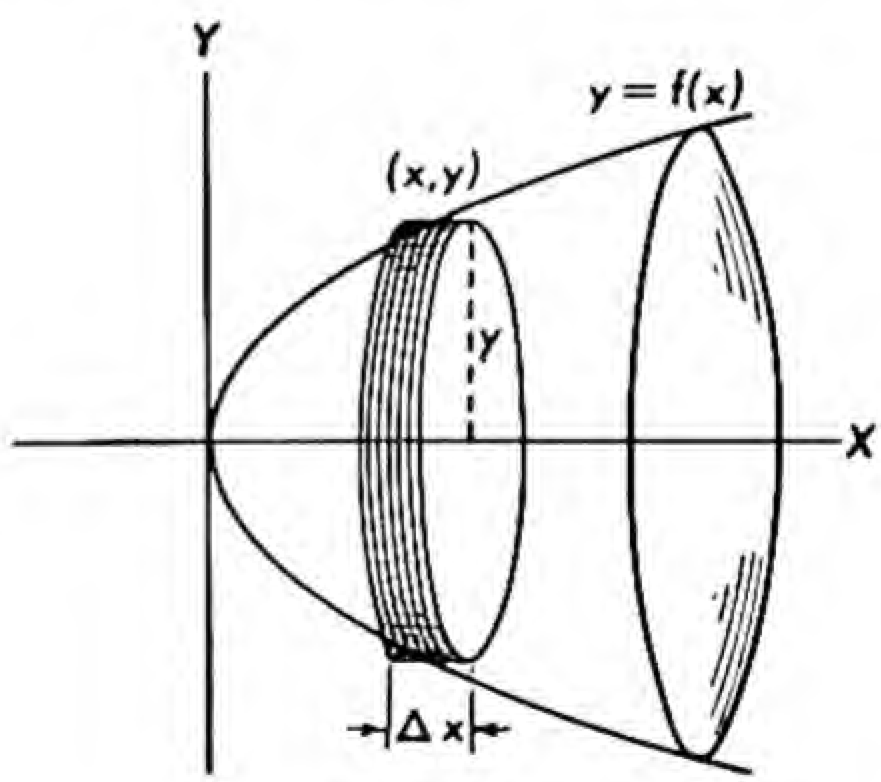

Fig. XI–10

The volume, of some restricted portion (say, $a \leq x \leq b$), can be approximated by the sum of the volumes of a finite number of successive circular cylinders, one of which is shown. The thickness of the cylindrical disc is Δx, and the radius of the base is y, so that the area of the base is πy^2.

Since the volume of a cylinder is given by $V = Bh$ = area of base multiplied by the height, the volume of the disc is $\pi y^2 \,\Delta x$. Hence the volume is approximately equal to

$$\sum_1^p \pi y_i^2 \,\Delta x_i$$

Just as in the case of areas, we now take the limit as $p \to \infty$ and as $\Delta x_i \to 0$. This leads to an integral, as before. We therefore define the volume of the solid of revolution by either of the equivalent expressions:

$$V = \lim_{p\to\infty} \sum_1^p \pi y_i^2 \,\Delta x_i = \int_a^b \pi y^2 \,dx$$

When $y = f(x)$ is the equation of the curve, this becomes

$$V = \int_a^b \pi[f(x)]^2 \,dx$$

Example 1. Find the volume generated by revolving the region bounded by $y = x^2$, $x = 1$, $x = 3$, and $y = 0$ around the X-axis (refer to Fig. XI–11).

$$V = \pi \int_a^b y^2 \,dx$$

$$V = \pi \int_1^3 x^4 \,dx = \pi\left[\frac{1}{5}x^5\right]_1^3 = \frac{\pi}{5}\left[x^5\right]_1^3$$

$$= \frac{\pi}{5}(243 - 1) = \frac{242\pi}{5} = 48\tfrac{2}{5}\pi \quad \text{cu units}$$

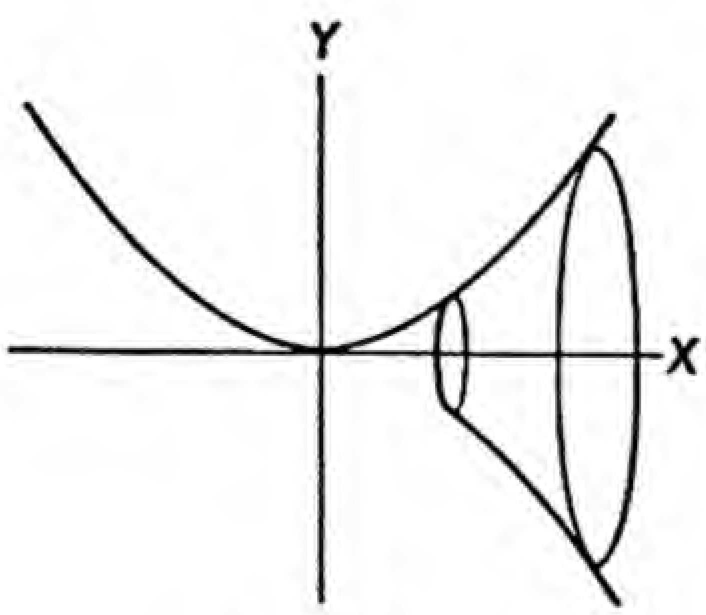

Fig. XI–11

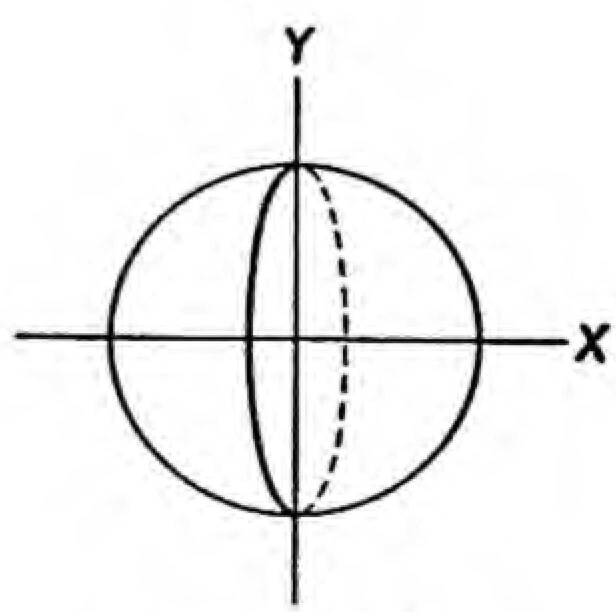

Fig. XI–12

Example 2. Find the volume generated by revolving the region $x^2 + y^2 = r^2$ around the X-axis. (The integral could be taken from $-r$ to r or from

0 to r. In the latter case the result would have to be doubled to get the full volume desired.)

$$V = \pi\int_a^b y^2\,dx$$

$$V = 2\pi\int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi\left[r^2x - \tfrac{1}{3}x^3\right]_0^r = 2\pi(r^3 - \tfrac{1}{3}r^3)$$

$$= \frac{4}{3}\pi r^3$$

Example 3. Find the volume generated by revolving the region included by $y = \sqrt{x}$ and $y = \frac{1}{2}x$ about the X-axis.

As in the case of areas, we can find the difference of two volumes, or like areas, since the bounds are the same, we can incorporate the difference under one integral sign (Refer to Fig. XI-13.)

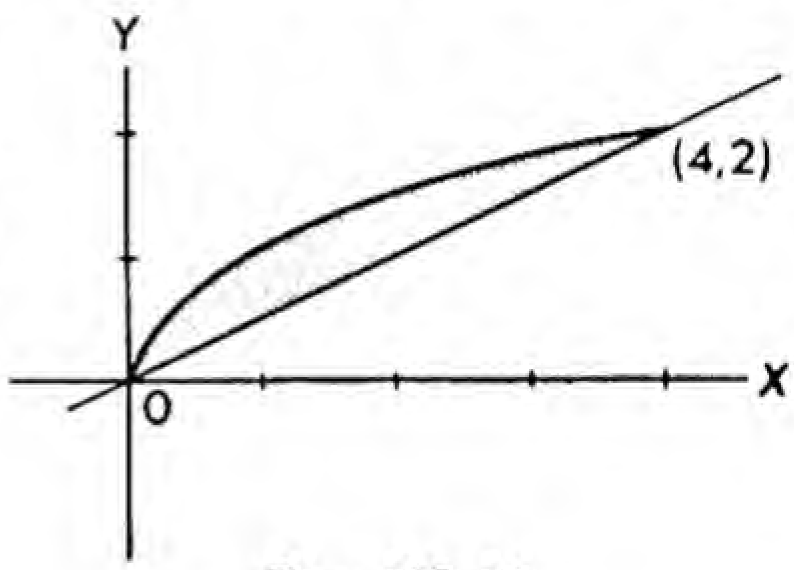

Fig. XI-13

$$V = \pi\int_0^4 (y^2_1 - y^2_2)\,dx$$

$$= \pi\int_0^4\left(x - \frac{1}{4}x^2\right)dx$$

$$= \pi\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \frac{8}{3}\pi \quad \text{cu units}$$

EXERCISES (XI-2)

1. Find the volume of the solid obtained by revolving the region bounded by $y = x^2$, $y = 0$, and $x = 6$ about the X-axis.

2. The region bounded by $y = ax$, the X-axis, and $x = h$ is revolved around the X-axis. Find the volume of the cone thus formed, and show that it leads to the formula $V = \frac{1}{3}Bh$, where B represents the area of the base.

3. A paraboloid is formed by revolving around the X-axis the region bounded by $y = \sqrt{x}$, $x = a$, and the X-axis. Show that $V = \frac{1}{2}\pi a^2$.

4. The region bounded by $y = 3 + x^2$, $x = 0$, $x = 3$, and $y = 0$ is revolved around the X-axis. Find the volume of the solid thus formed.

5. The region bounded by $y = x^2 - 4x$ and the X-axis is revolved around the X-axis. Find the volume thus formed.

6. The region bounded by $y = x^2 - 4x$ and $y = -3$ is revolved about the X-axis. Find the volume of the solid formed.

7. Find the volume of the zone formed by revolving the region bounded by $x^2 + y^2 = 25$, $x = 2$, and $x = 4$ about the X-axis.

8. Find the volume of the ellipsoid formed by revolving $x^2 + 4y^2 = 36$ around the X-axis.

9. As with areas, we can view the formation of solids of revolution by reference to the Y-axis. Solids formed by revolving around the Y-axis will yield

$$V = \pi \int_a^b x^2\,dy.$$

Find the volumes of the following, which are revolved around the Y-axis within the indicated bounds.

a. $y = x^2, y = 0, y = 4$
b. $y = x, y = -2, y = 2$
c. $x = y^2 - 1, y = -1, y = 1$

10. The preceding computation of volumes suggests an approach to the calculation of other volumes that are not volumes of revolution. Consider the wedge-shaped solid in Fig. XI–14, where every cross-section in the direction shown is a triangle. The altitude of these triangles are each one-half the base length.

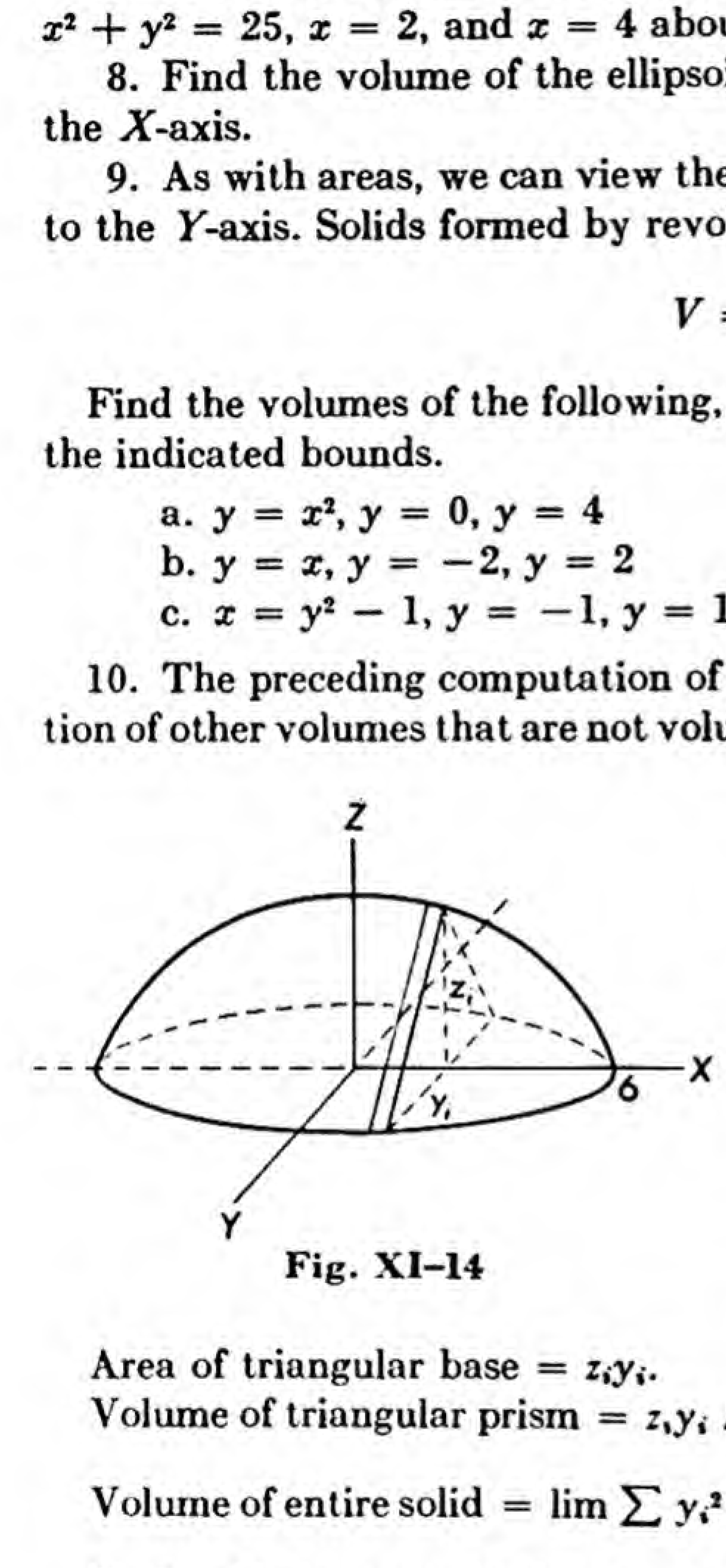

Fig. XI–14

We take a slice of this volume bounded by two such triangular sections and select axes as shown. The following are our facts:

Each slice is a triangular prism.
Thickness of any slice $= \Delta x_i$.
Extent of solid in x-direction $= 12$ units.
Height of triangular prism $= z_i$.
Height is one-half base of triangle, $z_i = y_i$.

Area of triangular base $= z_i y_i$.
Volume of triangular prism $= z_i y_i\,\Delta x_i = y_i^2\,\Delta x_i$

$$\text{Volume of entire solid} = \lim \sum y_i^2\,\Delta x_i = \int_{-6}^{6} y^2\,dx$$

To be able to compute the volume, it is necessary now to be able to represent y in terms of x. This is possible if it is also given that the base of the solid is a circle, that $x^2 + y^2 = 36$. Then

$$V = \int_{-6}^{6} (36 - x^2)\,dx = 2\int_0^6 (36 - x^2)\,dx$$

$$= 2\left[36x - \tfrac{1}{3}x^3\right]_0^6 = 2 \cdot 6(36 - 12) = 288 \text{ cu in.}$$

11. In the solid $ABCD$ (Fig. XI–15), ADC and ABC are semicircles whose planes are perpendicular to each other and intersect in AC. Every cross-section

formed by a plane perpendicular to AC is an isosceles triangle. $AC = 12$ inches. Find the volume of the solid.

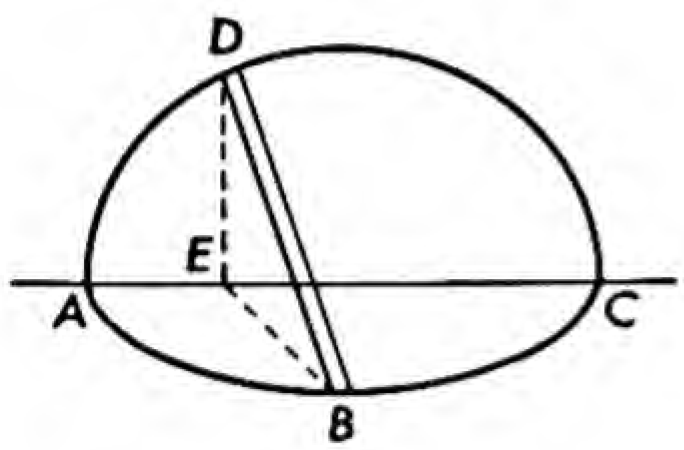

Fig. XI–15

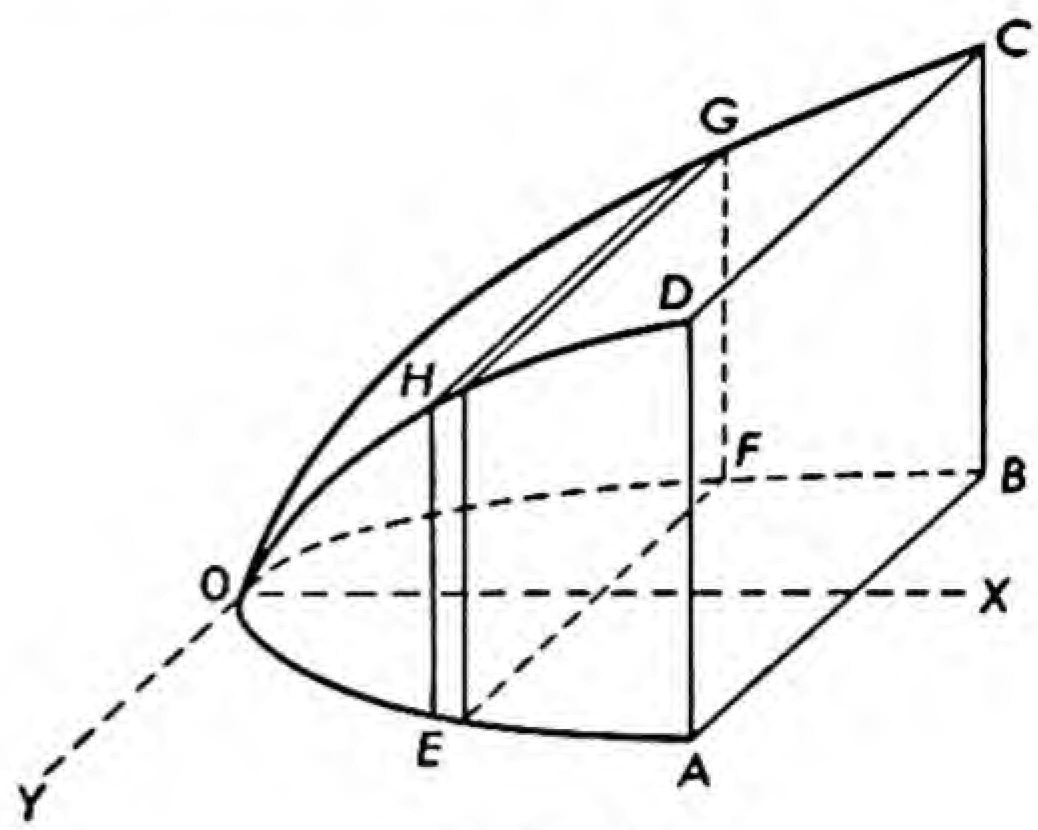

Fig. XI–16

12. AOB is a parabola $y^2 = x$; AB is $x = 1$. (Refer to Fig. XI–16). Every section $EFGH$ formed by a plane perpendicular to the X-axis, is a square. Find the volume.

13. We have seen that the volume of a circular cone is given by $V = \frac{1}{3}Bh$. The formula actually holds for any solid with a circular base in which every cross-section parallel to the base is a circle and such that the ratio of the radii of the circles equals the ratio of the corresponding distances from the apex. Show that this is so in Fig. XI–17.

3. WORK

Another application of integration may be found in connection with a host of physical problems dealing with the concept of *work*.

As defined in the physical sciences, the work done by a force of 20 pounds acting through a distance of 5 feet is

$$\text{Work} = \text{force} \times \text{distance}$$
$$W = (20 \text{ lb}) \times (5 \text{ ft}) = 100 \text{ ft-lb}$$

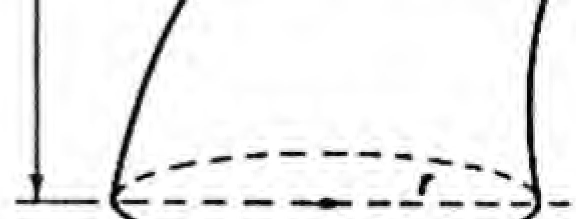

Fig. XI–17

where the unit of work, *foot-pounds*, is a description of the operation performed. For example, a 160-pound man climbing a flight of stairs which lifts him 10 feet vertically does 1600 foot-pounds of work. (The man must exert a force of 160 pounds to lift himself vertically against gravity. The distance, or displacement, in the formula must be in the direction of the force.)

Suppose that we have to deal with a variable force. For example, two electrons repel each other with a force that is inversely proportional to the

square of the distance between them. If F is the symbol for the force, s is the distance between them, and k is the proportionality factor, which will depend on the units used then

$$F = \frac{k}{s^2}$$

Clearly, as s gets larger, F gets smaller, and conversely.

Now suppose that the electrons are one unit apart and that one of them is being repulsed to a distance of three units from the other. How much work is done?

Let us assume an arbitrary position (Fig. XI–18) for the moving electron, which we shall take to be the one on the right. Suppose, somehow, that it is, in terms of distance, at the position x_i. The force that this electron experiences at this point, according to the formula, is k/x_i^2. Assuming further that this force is momentarily constant, at least for the additional small distance Δx_i, the work necessary to move the electron from x_i to $x_i + \Delta x_i$ is

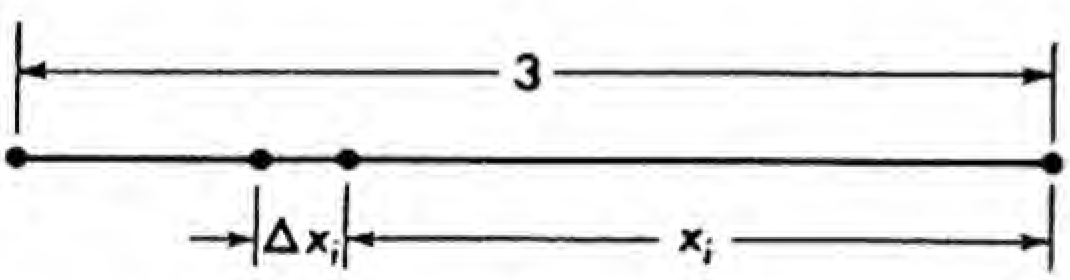

Fig. XI–18

$$W = \text{force} \times \text{distance} = \frac{k}{x_i^2}\,\Delta x_i$$

As with areas and volumes, we may consider the distance through which the electron will move to be subdivided into a finite number of small intervals. Hence the work is approximately

$$\sum_1^p \frac{k}{x_i^2}\,\Delta x_i$$

As before, we take the limit as $p \to \infty$ and $\Delta x_i \to 0$. We therefore define the work by either of the equivalent expressions:

$$W = \lim \sum \frac{k}{x_i^2}\,\Delta x_i = \int_a^b \frac{k}{x^2}\,dx$$

In the foregoing illustration the intervals will start with $x = 1$ and terminate when $x = 3$; in that case we have

$$W = \int_1^3 \frac{k}{x^2}\,dx = k\Big[-x^{-1}\Big]_1^3 = k\Big[-\frac{1}{x}\Big]_1^3 = k\Big(-\frac{1}{3} + 1\Big) = \frac{2}{3}k$$

As a second illustration of a variable force (Fig. XI–19), consider the problem of calculating the work required to empty a hemispherical reservoir of water by raising the water over the rim. The bowl has a 10-foot radius, and water weighs approximately 62.5 pounds per cubic foot. If we think of the water as being composed of consecutive layers of cylindrical discs (Fig. XI–19), we see that each disc varies in volume, and so, in weight. The calculation of the work is also approached as a limit of a sequence of sums of work done on such discs. The work required to lift a disc is equal to its weight multiplied by the vertical distance through which it is moved.

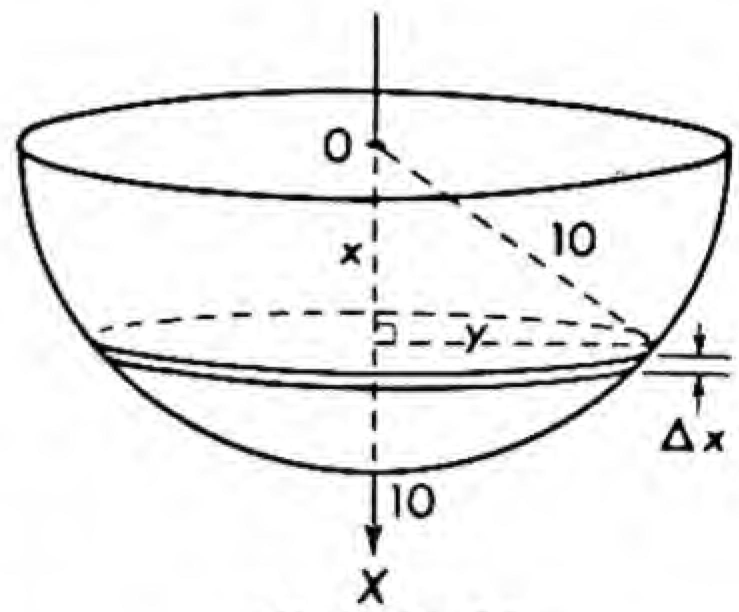

Fig. XI–19

The direction of the force will be vertical, and so, the X-axis is taken that way, with the origin at the center of the hemisphere. Now, ΔW refers to the work in lifting the indicated disc of water (Fig. XI–19) to the top, a distance of x feet.

$$\Delta V = \text{volume of disc} = \pi y^2\,\Delta x = \pi(100 - x^2)\,\Delta x$$
$$\Delta F = \text{weight of the disc} = 62.5\pi(100 - x^2)\,\Delta x$$
$$\Delta W = \Delta F \cdot x = 62.5\pi(100 - x^2)x\,\Delta x$$

In the manner of previous applications, the work done is obtained by either of the following equivalent expressions:

$$W = \lim_{p\to\infty} \sum_{1}^{p} 62.5\pi(100 - x^2)x\,\Delta x = \int_a^b 62.5\pi(100 - x^2)x\,dx$$

The discs will be raised as little as zero feet and as much as 10 feet. So,

$$\begin{aligned} W &= \int_0^{10} 62.5\pi(100 - x^2)x\,dx = 62.5\pi\int_0^{10}(100 - x^2)x\,dx \\ &= 62.5\pi\Big[50x^2 - \tfrac{1}{4}x^4\Big]_0^{10} \\ &= 62.5\pi(2500) = 156{,}250\pi \text{ ft-lb} = 78\pi \text{ ft-tons} \end{aligned}$$

EXERCISES (XI–3)

1. Find the work necessary to move one electron which is 5 inches from another to a position 3 inches from it.

2. Two stationary electrons are situated at (0, 0) and (3, 0). Find the work done in moving another electron from (6, 0) to (4, 0).

3. Suppose that the hemispherical reservoir of water in the text illustration is filled to a depth of 5 feet. Find the work done in emptying it in the same manner.

4. A conical reservoir, 12 feet across the top and 18 feet deep, is full of water. Find the work done in pumping the water to the top.

5. The work done in compressing a spring is also governed by the conclusion $W = \int_a^b F\,dx$, and the analysis strongly resembles our first illustration. Here, $F = kx$. That is, the force required to hold a spring under compression is proportional to the distance x that the spring has already been compressed.

Suppose for example, a spring whose natural length is 12 inches is held at 10 inches under a compression force of 8 pounds. This means that $x = 12 - 10 = 2$ inches, and since $F = kx$, so $8 = 2k$ and $k = 4$. Then, for this spring, $F = 4x$.

To find the work needed to compress this spring from the 10 inches at which it is held to 5 inches means that we are going from a 2-inch compressed distance to a 7-inch compressed distance.

$$W = \int_2^7 F\,dx = \int_2^7 4x\,dx = \Big[2x^2\Big]_2^7 = 90 \text{ in.-lb}$$

6. A force of 20 pounds keeps a spring stretched to 16 inches from its natural length of 12 inches. What force will keep the spring stretched from 16 inches to 18 inches, assuming that the spring remains within the elastic limits.

7. A spring whose natural length is 7 inches is compressed to $6\frac{1}{2}$ inches by a weight of 150 pounds. Find the work done in:
   a. Compressing the spring from 7 to 5 inches.
   b. Stretching the spring from 8 to 10 inches. [Assume same k as in (a).]

8. A 10-foot cylindrical tank is 9 feet high and is two-thirds full of water. Find the work done in bringing all the water to the top.

4. THE LENGTH OF AN ARC

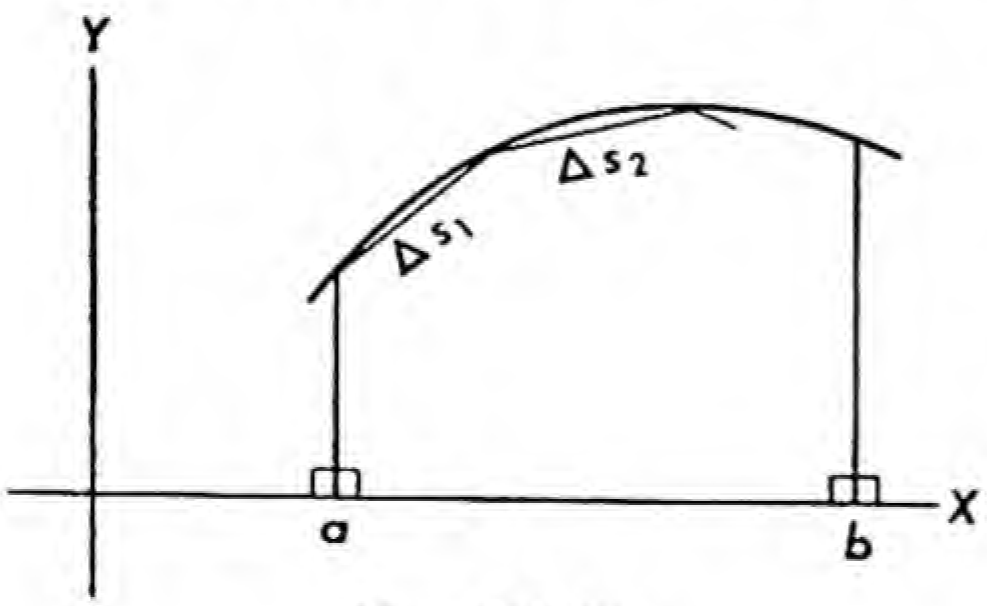

Fig. XI–20

At the moment the only lengths that we can find are straight line segments. We can also find the length of an arc of a circle but only because we have assumed the value of $2\pi r$ for the circumference of a circle.

Let f be a function whose derived function f' is continuous in the closed interval from $x = a$ to $x = b$. Can we find the length of the arc of its graph? (Refer to Fig. XI–20.)

We take a succession of points (Fig. XI–20), and draw the chords Δs_i, each as small as we please. The length of the curve is then approximately

$$L \approx \sum_{i=1}^{p} \Delta s_i$$

If the number of division points is increased indefinitely, and if the largest of the chords approaches 0, then, in the spirit of earlier commitments, we define the length of an arc as

$$L = \lim_{p\to\infty} \sum_1^p \Delta s_i = \int_a^b ds$$

We note that for any Δs_i, Fig. XI–21,

$$(\Delta s_i)^2 = (\Delta x_i)^2 + (\Delta y_i)^2$$

$$(\Delta s_i)^2 = (\Delta x_i)^2\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)$$

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

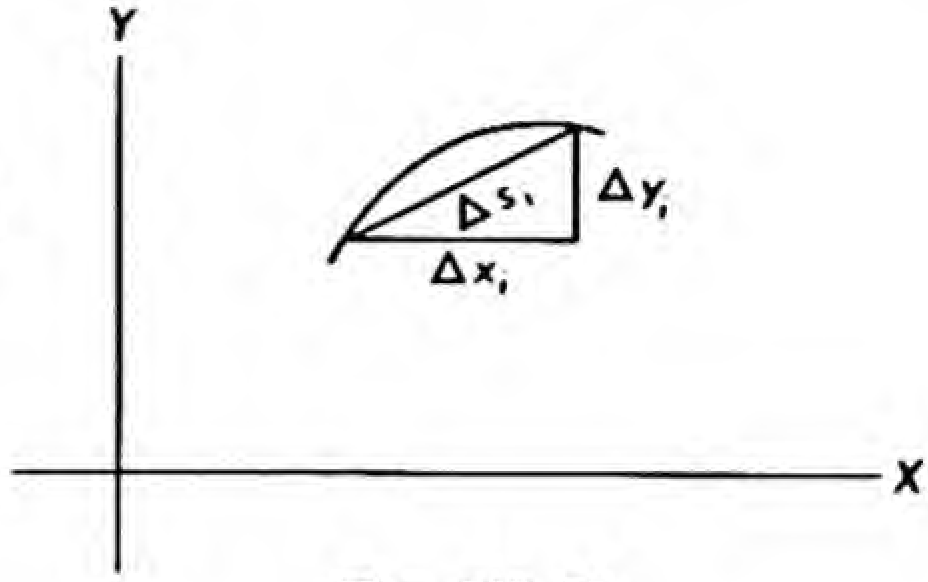

Fig. XI–21

We could write instead that

$$L = \lim_{p\to\infty} \sum \Delta s_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Let us apply this to finding the length of the arc of $y = x^{3/2}$ from $x = 0$ to $x = 4$.

$$L = \int_0^4 \sqrt{1 + \frac{9x}{4}}\,dx \qquad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$= \frac{2}{3}\cdot\frac{4}{9}\left(1 + \frac{9x}{4}\right)^{3/2}\Big]_0^4 \qquad \left(\frac{dy}{dx}\right)^2 = \frac{9x}{4}$$

$$= \frac{8}{27}(10^{3/2} - 1) = \frac{8}{27}(10\sqrt{10} - 1)$$

EXERCISES (XI–4)

1. Find the length of the graph of $y = 3x + 4$ from $x = 0$ to $x = 4$.
2. Find the length of the arc $4y^2 = x^3$ from $x = 1$ to $x = 9$ in the first quadrant.
3. Find the length of the arc of $y^2 = x^3$ between $x = 0$ and $x = 4$ in the first quadrant.

5. AREA OF A SURFACE OF REVOLUTION

The concept of *area*, just like that of *length* in Art. 4, requires an extension so that we can include surface areas beyond that of the plane which we met some time ago. We shall extend the concept to include only those surfaces formed by rotating a plane curve about a line in the plane.

We take for consideration a continuous function f whose derived function f' is continuous between $x = a$ and $x = b$. On the arc that is a graph of

f, we take a sequence of consecutive chords Δs_i (Fig. XI–22). If an arc length, approximately Δs_i, is rotated around the X-axis, the figure formed will approximate a frustrum of a cone. (See exercise 5.) Its surface is given by $2\pi y_i \Delta s_i$, which is the circumference of its midsection multiplied by the slant height. (y_i is the radius of the midsection.) The whole surface (Fig. XI–22), consisting of an infinite sum of such consecutive discs is defined to be (in a spirit now familiar)

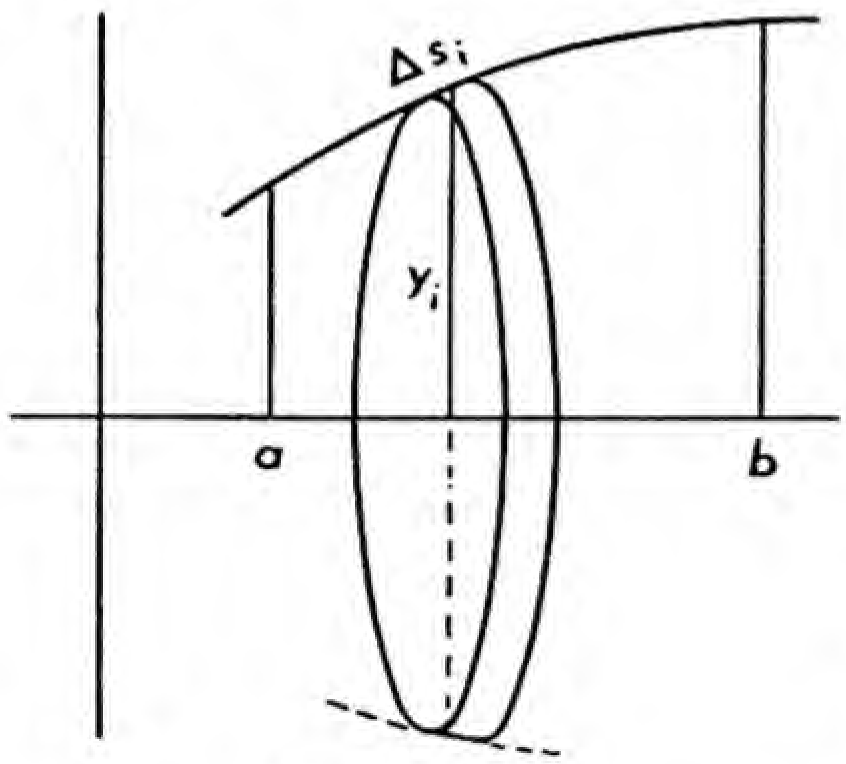

Fig. XI–22

$$S = \lim_{p\to\infty} \sum_{i=1}^{p} 2\pi y_i \Delta s_i = 2\pi \int_a^b y\, ds$$

In the preceding article we saw that $ds = \sqrt{1 + (dy/dx)^2}\ dx$, so that, by substitution

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

As an illustration, we shall revolve the circle $x^2 + y^2 = r^2$ or $y = \sqrt{r^2 - x^2}$ about the X-axis, forming a sphere. We take the bounds of x from $-r$ to $+r$ or get the surface of the hemisphere from $x = 0$ to $x = r$ and double the result:

$$S = 2\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$= 2\pi \int_0^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int_0^r r\, dx$$

$$S = 2\pi \Big[rx \Big]_0^r = 2\pi r^2$$

Thus the surface of the entire sphere is $4\pi r^2$.

EXERCISES (XI–5)

Find the surface formed by rotating the following arcs about the X-axis:

1. $y = x$ from $x = 0$ to $x = 4$
2. $y = x^3$ from $x = 0$ to $x = 1$
3. Write the formulas for the length of an arc and the surface of revolution with respect to the Y-axis instead of the X-axis.

4. Show that the lateral area of a right circular cylinder is equal to the circumference of its base multiplied by the height. Relate the figure to a rectangle.

5. A frustrum of a cone is the two-base figure that is cut off from a cone by a plane parallel to the base of the cone.

a. The lateral area of a frustrum of a cone is given by one-half the sum of the circumferences of the bases multiplied by the slant height of the frustrum. Using S for the area, s for the slant height, and R and r for the radii of the respective bases, express as a formula the lateral area of a frustrum of a cone.

b. A plane cross-section of the frustrum which is perpendicular to the bases and which passes through their centers is a trapezoid. A line segment that joins the midpoints of the nonparallel sides of a trapezoid is called a *median*. Prove that the median r' is equal to $\frac{1}{2}(R + r)$. (Draw perpendiculars to the bases through the midpoints of the arms.)

c. Use the facts in (b) to show that the lateral area of a frustrum of a cone is given by $2\pi r's$.

d. Show how the formula in (c) relates to that used in the preceding article.

XI-5 REVIEW

1. Verify $\sum_{p=1}^{n} p = \dfrac{n(n+1)}{2}$ for $n = 5, 6$.

2. Verify $\sum_{p=1}^{n} p^2 = \dfrac{n(n+1)(2n+1)}{6}$ for $n = 4, 5$.

3. Verify $\sum_{p=1}^{n} p^3 = \dfrac{n^2(n+1)^2}{4}$ for $n = 6$.

4. Evaluate each of the following:

a. $\displaystyle\int_0^1 u(u-1)\,du$

b. $\displaystyle\int_0^2 u(u-1)\,du$

c. $\displaystyle\int_{-1}^2 \sqrt{1-x}\,dx$

d. $\displaystyle\int_0^3 \frac{x}{(x^2+1)^2}\,dx$

e. $\displaystyle\int_1^4 \frac{x-1}{x^3}\,dx$

f. $\displaystyle\int_{-2}^3 (6 + x + x^2)\,dx$

5. Find the area included between $y = x^3 - x^2 - 6x$ and the X-axis.

6. Find the area between the Y-axis and $x = y^2 - 4y$.

7. Find the area between $y = \frac{1}{4}x^2$ and $y = x$.

8. Find the area between $y^3 = x^2$, $y = 0$, and $x = 8$.

9. The region formed by $x^{1/2} + y = 3$ and the coordinate axes is revolved around the X-axis. Find the volume.

10. Find the volume formed by revolving $4y^2 - x^2 = 16$ (between $x = -3$ and $x = 3$) around the X-axis.

11. A cylindrical tank with open top and circular base with a 6-foot radius, height 15 feet, is half-full of water. Find the work done in pumping the water to the top of the tank.

12. A hemispherical tank of 6-foot diameter contains water up to 2 feet from the top. Find the amount of work that is necessary to raise the water to the top.

13. Find the length of the graph of $y = 2x - 4$ from $x = 1$ to $x = 3$.

14. Find the length of $y = 4x^{3/2}$ from $x = 0$ to $x = 2$.

15. Find the area of the surface generated by rotating about the X-axis the arc of the curve $y = 2x^3$ between $x = 0$ and $x = 1$.

16. Find the area of the surface generated by revolving around the X-axis the curve $y^2 = 9x$ from $x = 0$ to $x = 4$.

17. Find the area generated by revolving around the Y-axis the arc of the curve $y = x^2$ between $y = 0$ and $y = 4$.

XII

TRANSCENDENTAL FUNCTIONS, SERIES

1. TRIGONOMETRIC FUNCTIONS AGAIN

It has been some time since mention was made of the trigonometric functions. Surely, as part of our mathematical system, they, too, have an important role in the calculus. We turn immediately to a determination of the derivative of the sine function.

We recall first that the quantity u in $\sin u$, which began as a measure of an angle, was later incorporated into the real number system through radian measure. In the unit circle we saw that u was an arc length on the circle which contained 2π radians. The trigonometric functions were also identified with certain line segments, as in Fig. XII–1 of this chapter. We saw, too, that all these were in one-to-one correspondence with the real numbers on a line. Thus $\sin u$ is a function value of the set of real numbers $\{u, \sin u\}$. Consequently the operations that are to follow, which have been employed with real numbers before, continue to be employed with real numbers.

By our basic definition, we have

$$Df(u) = \lim_{h \to 0} \frac{f(u+h) - f(u)}{h}$$

and with $f(u) = \sin u$, we also have

$$D \sin u = \lim_{h \to 0} \frac{\sin (u+h) - \sin u}{h}$$

To determine the limit of the fraction in the preceding line, we shall have

to engage in some simplifications of that term. We start with the numerator and a substitution for the $\sin(u+h)$ by a formula we developed earlier.

$$\begin{aligned}\sin(u+h) - \sin u &= \sin u \cos h + \cos u \sin h - \sin u \\ &= \cos u \sin h - \sin u\,(1 - \cos h)\end{aligned}$$

Then $$\frac{\sin(u+h) - \sin u}{h} = \cos u \frac{\sin h}{h} - \sin u \frac{1 - \cos h}{h}$$

We assume two valuable limits that will be discussed further in the exercise section to follow.

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

and $$\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

where h is in radian measure.

We now return to the last trigonometric identity and take the limit of both members, keeping in mind that the limit of a difference is the difference of the limits, and the limit of a product of functions is the product of the limits of the functions.

$$\lim_{h \to 0} \frac{\sin(u+h) - \sin u}{h} = \lim_{h \to 0} \left\{\cos u \frac{\sin h}{h} - \sin u \frac{1 - \cosh}{h}\right\} = \cos u$$

Consequently,

$$D \sin u = \cos u$$

where u is the independent variable.

If u is a function of another variable (say, x), we have a function of a function, for which we know

$$Df[g] = Df\,Dg$$

or, to make explicit the independent variable

$$D_x f[g] = D_u f\, D_x g$$

For example, in $\sin u$, where $u = x^3$, we have

$$\begin{aligned}D_x \sin x^3 = D_x \sin u &= D_u \sin u\, D_x u \\ &= \cos u(3x^2) \\ &= 3x^2 \cos x^3\end{aligned}$$

Similarly, $$D \sin \frac{x}{5} = \cos \frac{x}{5} \cdot \frac{1}{5} = \frac{1}{5} \cos \frac{x}{5}$$

Also $$D \sin^3 x = D(\sin x)^3 = 3 \sin^2 x \cos x$$

according to the power formula.

Taking an antiderivative of a conclusion above, we have

$$D^{-1} \cos x = \sin x + c$$

and so

$$\int_a^b \cos x \, dx = \sin x \Big]_a^b = \sin b - \sin a$$

A specific illustration of this important conclusion is

$$\int_0^{\pi/2} \cos x \, dx = \sin x \Big]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$$

We can develop the derivative of the cosine function, starting with a well-known identity:

$$\cos u = \sin \left(\frac{\pi}{2} - u\right)$$

$$D \cos u = D \sin \left(\frac{\pi}{2} - u\right) = \cos \left(\frac{\pi}{2} - u\right) \cdot (-1)$$

$$= -\cos \left(\frac{\pi}{2} - u\right) = -\sin u$$

Thus

$$D \cos u = -\sin u$$

and

$$D_x \cos u = -\sin u \frac{du}{dx}$$

The following examples illustrate these conclusions:

a. $D \cos 2x = D_{2x} \cos 2x \, D_x 2_x = -\sin 2x(2) = -2 \sin 2x$
b. $D \cos x^3 = -\sin x^3(3x^2) = -3x^2 \sin x^3$
c. $D (\cos x)^3 = 3 \cos^2 x(-\sin x) = -3 \cos^2 x \sin x$

Since $Dkf(x) = kDf(x)$, we can write

$$D(-\cos x) = \sin x$$

and so

$$D^{-1}(\sin x) = -\cos x + c$$

or

$$\int_a^b \sin x \, dx = -\cos x \Big]_a^b = -(\cos b - \cos a)$$

These latter cases indicate how it is possible to derive other trigonometric derivatives and integrals, one from the other. This, of course, should not be surprising, for we have seen that all the trigonometric functions are intimately related.

EXERCISES (XII–1)

1. Corroborate each of the following limits (the assumed limits in preceding text may be used):

a. $\lim\limits_{h \to 0} \dfrac{\tan h}{h} = 1$

b. $\lim\limits_{h \to 0} \dfrac{\sin h}{\cos h + 1} = 0$

c. $\lim\limits_{h \to \pi/2} \dfrac{\sin h}{1 + \cos h} = 1$

d. $\lim\limits_{h \to 0} \dfrac{\sin 2h}{h} = 2$

e. $\lim\limits_{x \to 0} \dfrac{\sin ax}{x} = a$ (substitute $ax = z$)

f. $\lim\limits_{h \to 0} \dfrac{\sin (h + \pi)}{h} = -1$

2. Show that:
a. $D_x \tan x = \sec^2 x$ and b. $D_x \tan u = \sec^2 u \, D_x u$ by starting with $\tan \theta = \sin \theta / \cos \theta$ and using the quotient or product rule of differentiation.

3. The knowledge of the derivatives of the sine, cosine, and tangent function is adequate for most purposes. However, in the manner already indicated, find the rules for the other functions. Develop the following:

a $D_x \cot x$ b. $D_x \sec x$ c. $D_x \csc x$

4. Find the derivatives of each of the following with respect to x:

a. $5 \sin 2x$

b. $8 \cos 3x$

c. $\left(\cos \dfrac{x}{2}\right)^2$

d. $\sin \dfrac{3x}{5}$

e. $\sin x + \cos x$

f. $\sin x \tan x$

g. $\cos^2 2x$

h. $\sec^2 x$

i. $4 \sin \left(x + \dfrac{\pi}{2}\right)$

j. $x \sin x$

5. Show that $D \sin x° = \dfrac{\pi}{180} \cos x°$

6. Find the slope of each of the following graphs at the indicated values:

a $y = \sin x;\ \pi/2,\ \pi$ b. $y = \cos 2x;\ \pi/4,\ 2\pi/3$

7. Find the maximum value of:

a. $\sin x + \cos x$ b. $2 \cos x - \sin 2x$

8. Two sides of a triangle are 12 inches and 16 inches and include an angle that is increasing at the rate of $\pi/90$ radians per second.
a. How fast is the area changing when the angle is $\pi/4$?
b. How fast is the third side changing then?

9. Show that the graphs of $\cos x$ and $\tan x$ meet orthogonally.

10. Evaluate each of the following:

a. $\displaystyle\int_0^{\pi/2} \sin x \, dx$

b. $\displaystyle\int_0^{\pi/6} \cos 3x \, dx$

c. $\displaystyle\int_0^{\pi/6} \sec^2 x \, dx$

d. $\displaystyle\int_0^{\pi/2} \sin 2x \, dx$

11. Find the area between one arch of the sine curve and the X-axis.

12. Draw the graph of $(\sin x)/x$.

13. Find the value of $\int_0^{\pi} \sin^2 x\, dx$. (Use $\cos 2x = 1 - 2\sin^2 x$.)

14. Find the volume of revolution formed by revolving one arch of $\sin x$ around the X-axis.

15. If $h > 0$, and h is the central angle in Fig. XII–1, explain or justify each of the following (the radius of the circle is taken as 1. CB and DA are perpendicular to OB):

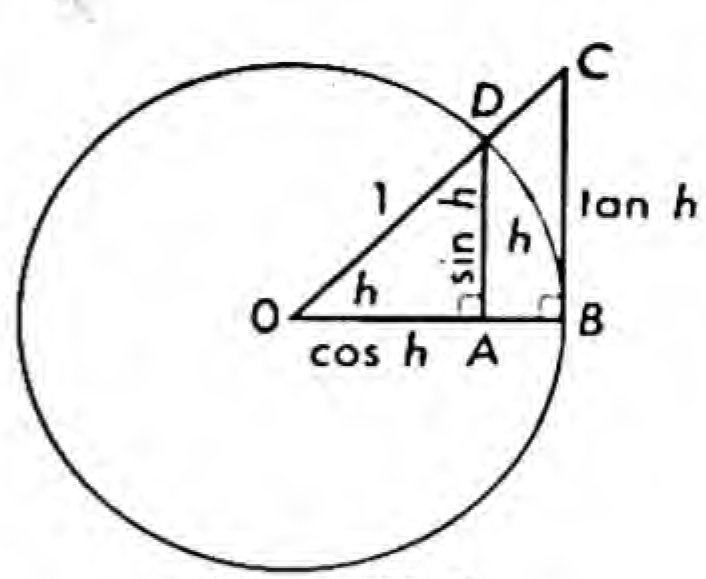

Fig. XII–1

a. $OA = \cos h$
b. $DA = \sin h$
c. $\widehat{DB} = h$
d. $CB = \tan h$
e. Area of $\triangle OBC = \frac{1}{2}\tan h$
f. Area of sector $OBD = \frac{1}{2}h$
g. Area of $\triangle OAD = \frac{1}{2}\sin h \cos h$

16. Using the results of the preceding exercise, justify each of the following steps:

$$\triangle OAD < \text{sector } OBD < \triangle OBC$$

$$\tfrac{1}{2}\sin h \cos h < \tfrac{1}{2}h < \tfrac{1}{2}\tan h$$

$$\cos h < \frac{h}{\sin h} < \frac{1}{\cos h} \qquad \left(h < \frac{\pi}{2}\right)$$

$$\frac{1}{\cos h} > \frac{\sin h}{h} > \cos h \qquad \left(\text{suppose } h > \frac{\pi}{2}?\right)$$

$$\lim_{h \to 0} \frac{1}{\cos h} \geqslant \lim_{h \to 0} \frac{\sin h}{h} \geqslant \lim_{h \to 0} \cos h$$

$$1 \geqslant \lim_{h \to 0} \frac{\sin h}{h} \geqslant 1$$

Therefore

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

17. Justify each of the following steps:

$$1 - \cos^2 h = \sin^2 h$$

$$(1 - \cos h)(1 + \cos h) = \sin^2 h$$

$$1 - \cos h = \frac{\sin^2 h}{1 + \cos h}$$

$$\frac{1 - \cos h}{h} = \frac{\sin h}{h} \cdot \frac{\sin h}{1 + \cos h}$$

Thus

$$\lim_{h \to 0} \frac{1 - \cos h}{h} = 1 \cdot 0 = 0$$

2. LOGARITHMS AND EXPONENTIALS

Having made a start in the calculus with transcendental functions, we can go on to some of the others. Our first point of departure will seem a bit far-fetched, but its interesting relevancy and contribution will more than compensate for the seeming digression.

Space does not permit the development of the Compound Interest Formula, which is

$$A = P\left(1 + \frac{r}{k}\right)^{kn}$$

where P represents the principal; r, the rate of interest; k, the frequency of compounding the interest per year; n, the number of years; and A, the amount that will accrue as a result of the investment.

Suppose now that P, r, and n are all equal to 1. That is, the formula is being specialized to an investment of \$1.00 at 100 per cent for 1 year. We have, then,

$$A = \left(1 + \frac{1}{k}\right)^{k}$$

The quantity A is now a function value depending only on k, which represents a positive integer. We envisage a sequence of increasing values for k, with k approaching infinity. Thus

$$A = \lim_{|k| \to \infty} \left(1 + \frac{1}{k}\right)^{k}$$

It can be proved that this limit exists. It is universally represented by e. It has also been proved that e is an irrational, transcendental number.

$$e \approx 2.71828 \cdots$$

We shall develop an infinite series shortly which will permit us to approximate e to any desired degree of accuracy. Under the peculiar circumstances that we imposed on the interest formula, the \$1.00 will be worth about \$2.72 at the end of the year. At the moment, getting away from banking, we have

$$\lim_{|k| \to \infty} \left(1 + \frac{1}{k}\right)^{k} = e \approx 2.71828 \ldots$$

where k is now any number.

We substitute u/h for k, which gives us

$$\left(1 + \frac{1}{k}\right)^{k} = \left(1 + \frac{h}{u}\right)^{u/h}$$

So, $$e = \lim_{|k| \to \infty} \left(1 + \frac{1}{k}\right)^{k} = \lim_{h \to 0} \left(1 + \frac{h}{u}\right)^{u/h} \qquad u \neq 0$$

Strangely enough, this background of information is essential for the development of the derivative of the logarithm function. Let us start with $y = \log u$, where this time the base of the logarithm may be any positive number b, which will be omitted below for the sake of simplicity.

$$y = \log u \qquad u > 0$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\log (u + h) - \log u}{h}$$

The last line is in conformity with the definition of the derivative. Now,

$$\log (u + h) - \log u = \log \frac{u + h}{u} = \log \left(1 + \frac{h}{u}\right)$$

So,
$$\frac{\log (u + h) - \log u}{h} = \frac{1}{h} \log \left(1 + \frac{h}{u}\right) = \frac{1}{u} \cdot \frac{u}{h} \log \left(1 + \frac{h}{u}\right)$$
$$= \frac{1}{u} \log \left(1 + \frac{h}{u}\right)^{u/h}$$

Substituting this result in the statement of the derivative, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{1}{u} \log \left(1 + \frac{h}{u}\right)^{u/h}$$
$$= \frac{1}{u} \log \left[\lim_{h \to 0} \left(1 + \frac{h}{u}\right)^{u/h}\right]$$

The application of the limit holds because the log function is a continuous function.

Now we see that the limit of the quantity in the bracket is nothing less than e, which was discussed before. Thus, for

$$y = \log_b u \qquad u > 0$$

we now have
$$\frac{dy}{dx} = \frac{1}{u} \log_b e$$

The base b was reinserted at this point for necessary emphasis. Finally, if we take e as the base for the logarithm, that is, if we employ natural logarithms, and if we remember that $\log_e e = \ln e = 1$, we have the very valuable and simple conclusion that if

$$y = \ln u \qquad u > 0$$

then
$$\frac{dy}{du} = \frac{1}{u}$$

and
$$\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx}$$

We may lift the limitation on u. For $u < 0$, we may write

$$y = \ln |u|$$

then
$$\frac{dy}{dx} = \frac{1}{|u|} \frac{d|u|}{dx} = \frac{1}{-u} \frac{d(-u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

The reader may recall that it was not possible earlier to integrate $1/u$ by the power rule. Now we see that

$$\int_a^b \frac{1}{u}\, du = \ln |u| ;\Big]_a^b = \ln |b| - \ln |a|$$

where a and b have the same sign.

Some illustrations are in order:

a. $y = \ln|x^2 - 2x|;\ y' = \dfrac{1}{x^2 - 2x} \cdot (2x - 2) = \dfrac{2(x-1)}{x^2 - 2x}$

b. $D \ln|\sin x| = \dfrac{1}{\sin x} \cos x = \cot x$

c. $\int \cot x\, dx = \ln|\sin x| + c$

(As u in $\ln u$ takes on various values, it is preferable to use absolute values rather than keep writing $u > 0$.)

The path is now well prepared for the consideration of the exponential functions. If

$$y = a^u \qquad a > 0$$

then

$$\ln y = u \ln a$$

and

$$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx}$$

so

$$\frac{dy}{dx} = y \ln a \frac{du}{dx}$$

or

$$\frac{dy}{dx} = a^u \ln a \frac{du}{dx}$$

Specifically, if

$$y = 2^x$$

then

$$\frac{dy}{dx} = 2^x \ln 2$$

An important special case of this last formula occurs when we take e for the base.

$$y = e^u$$

$$\frac{dy}{dx} = e^u \frac{du}{dx} \qquad (\ln e = 1)$$

If

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

This last fact was referred to earlier. The rate of change of e^x is precisely e^x.

EXERCISES (XII-2)

1. Find dy/dx in each case:

a. $y = \ln|x + 1|$
b. $y = \ln|x^3|$
c. $y = \ln u^2$
d. $y = x \ln|x|$
e. $y = \ln|\cos x|$
f. $y = \ln|\tan x|$
g. $y = \ln \sin^2 x$
h. $y = e^{x^2}$
i. $y = e^{-x}$
j. $y = e^{\cos x}$
k. $y = e^x - e^y$
l. $y = e^{\ln x}$
m. $y = 3^x$
n. $y = 4^{x^2}$
o. $y = \ln|x^2 + 3|$
p. $y = \ln|a^2 - x^3|$

2. Find the value of y in each of the following, where $x > 0$:

a. $\dfrac{dy}{dx} = \dfrac{1}{x}$

b. $\dfrac{dy}{dx} = \dfrac{1}{1 + x}$

c. $\dfrac{dy}{dx} = \dfrac{1}{2x + 3}$

d. $\dfrac{dy}{dx} = -\dfrac{\sin x}{\cos x}$

e. $y = \displaystyle\int_0^x \frac{x}{x^2 + 3}\, dx$

f. $y = \displaystyle\int_0^x \frac{\sin x}{\cos^2 x}\, dx$

g. $y = \displaystyle\int_0^x 2 \sin x \cos x\, dx$

h. $y = \displaystyle\int_0^x e^x \cos e^x\, dx$

(Other restrictions may be placed on x as necessary.)

3. Find the slope of the graph of $y = xe^x$ at $x = 0$.

4. Find the critical value of $y = e^{-x} \sin x$ for $0 \leq x \leq \pi$.

5. Find the area included by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

6. Find the volume of the solid of revolution formed by revolving the area of the preceding exercise around the X-axis.

7. The equation of a curve is given parametrically by $x = \sin 2\theta$, $y = \cos \theta$. Find the slope at $\theta = \pi/3$.

3. INFINITE SERIES

For quite some time now we have been working with derivatives which represented, when they existed, derived functions. But, irrespective of source, a derived function is a function, and so it may itself have a derivative and that in turn another derivative, and so forth. We call the first derived function the first derivative; the next one, the second derivative; then the third, and so on. Of course notations will be necessary and some have been suggested.

First: $\qquad Dy,\ \dfrac{dy}{dx},\ y',\ f'(x),\ \dfrac{df(x)}{dx}$

Second: $\qquad D^2y,\ \dfrac{d^2y}{dx^2},\ y'',\ f''(x),\ \dfrac{d^2f(x)}{dx^2}$, etc.

For

$$\begin{aligned} f(x) &= x^5 - 7x^4 + 3x^2 - 6 \\ f'(x) &= 5x^4 - 28x^3 + 6x \\ f''(x) &= 20x^3 - 84x^2 + 6 \\ f^3(x) &= 60x^2 - 168x \end{aligned}$$

We cannot indulge here in a study of diverse significances of higher derivatives, many of which are very technical. However, this matter was brought up for a particular investigation which will be seen to have considerable interest and merit.

In our contact with the binomial expansion, we have seen that expressions such as $(1 - x)^n$ are expandable for positive integral values of n. Under

certain conditions any value of n will do just as well. For example, we could find by long division that

$$(1 - x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

which is an **infinite power series.** The binomial expansion theorem would yield the same result.

Now what is the meaning of the sum of an infinite number of terms? Surely we cannot actually add that many terms. All that can be said is that, if we take the partial sums

$$s_1, s_2, s_3, \cdots, s_n, \cdots$$

where s_k means the sum of the first k terms (a finite number of terms), and if

$$\lim_{n \to \infty} s_n = N$$

where N is a finite number, then the series **converges** and N is its sum. If the sequence does not have a limit, then the series **diverges** and has no sum.

Now, it may be that a power series converges only for certain values of the variable or for a limited range of the variable. In fact the preceding illustration converges for $|x| < 1$, that is for $-1 < x < 1$. For example, if $x = \frac{1}{2}$, we get

$$\frac{1}{1 - \frac{1}{2}} = 2 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

As more and more terms are added, the partial sums get nearer and nearer to 2, and in fact the limit is 2. On the other hand, for $x = 2$, we get

$$\frac{1}{1 - 2} = -1 = 1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots$$

which is patently preposterous. This is a divergent series and the nth term is getting infinintely large. Or, consider the case where $x = -1$:

$$\frac{1}{1 - (-1)} = \tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

Again this is not a convergent series. The set of values for which the power series converges is called its **region of convergence.**

The illustration may suggest the possibility that other functions are expressible as power series. That is,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

Should such be the case, we expect that the coefficients a_i depend uniquely on $f(x)$. Strangely enough a relationship can be determined by the taking of derivatives successively, assuming that they exist.

To begin with, we can find a_0 through $f(0)$, since all but the first term vanishes when $x = 0$. So, $f(0) = a_0$. Now,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

This is like the previous step. All but the first term of this derived series will vanish for $x = 0$. Thus

$$[f'(x)]_{x=0} = a_1$$

Continuing, $\qquad f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots$

again, by $x = 0$, $\qquad [f''(x)]_{x=0} = 2a_2 \qquad$ so $\qquad a_2 = \frac{1}{2}[f''(x)]_{x=0}$

Similarly

$$[f^{(3)}(x)]_{x=0} = 6a_3 \qquad \text{so} \qquad a_3 = \tfrac{1}{6}[f^{(3)}(x)]_{x=0}$$

$$[f^{(4)}(x)]_{x=0} = 24a_4 \qquad \text{so} \qquad a_4 = \tfrac{1}{24}[f^{(4)}(x)]_{x=0}$$

(Parentheses are used in $f^{(4)}(x)$ to emphasize that 4 is not a power.

We pause for a couple of simplifications. If we trace back the coefficients that appeared in the series, the 24 and 6 denominators, for example, we see that they arose from an ordered, consecutive set of factors. Note that $24 = 4 \cdot 3 \cdot 2 \cdot 1$ and $6 = 3 \cdot 2 \cdot 1$. The next coefficient that will turn up, if we pursue the matter, will be $120 = 5!$ (factorial 5).

Further, the expression $[f^{(4)}(x)]_{x=0}$ is rather clumsy, particularly when many of the same kind will have to be written. So, we introduce the following abbreviation

$$[f^{(n)}(x)]_{x=0} = f^{(n)}(0)$$

where the important thing to note is that the substitution of $x = 0$ is made after $f^{(n)}(x)$ is found and is made in $f^{(n)}(x)$. We followed this procedure earlier. Recapitulating,

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{1}{2!}f''(0),\ a_3 = \frac{1}{3!}f^3(0)$$

$$a_n = \frac{1}{n!}f^{(n)}(0)$$

Substituting these results in the original expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

This is known as Taylor's, and sometimes as Maclaurin's series. Let us try this on $f(x) = e^x$. Since every derivative of e^x is e^x, $f^{(i)}(x) = e^x$ and $f^{(i)}(0) = e^0 = 1$. So,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and this converges for all values of x, that is, for $-\infty < x < \infty$.

By substituting $x = 1$, we get the special case

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$

from which the value of $e = 2.71828\cdots$ can be obtained to any desired degree of accuracy.

Suppose now that $f(x) = \sin x$:

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

By substituting in the Taylor's series, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}(x^{2n-1})}{(2n-1)!} + \cdots$$

which also converges for $-\infty < x < \infty$. The values of x are in radians because the derivatives used are correct only for radian measure. This series may be used as a basis for the construction of a table of sines. Similarly, we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n-1}x^{2n}}{(2n)!} + \cdots$$

A glance at the preceding three series indicates that the first, e^x, except for signs, contains all the terms of the other two put together. The perversity of signs may be overcome as follows:

In e^x substitute ix for x, where i is the imaginary unit. We recall that $i = i$, $i^2 = -1$, $i^3 = -i$, and so forth.

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

This can be written as

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

The first portion of this series represents the series for the $\cos x$ and i is multiplied by the $\sin x$ series. So, we have, by substitution,

$$\mathbf{e^{ix} = \cos x + i \sin x}$$

which is known as Euler's formula, a remarkable and valuable conclusion. We note parenthetically that the imaginary number finally achieved a position as an exponent. It will be recalled that one of our earliest extensions of concepts concerned exponents which started with positive integers.

The field was broadened by definitions to include real numbers. Now, the imaginary number begins to take its place in the field too.

If we substitute $x = \pi$, in the e^{ix} equation, we get

$$e^{\pi i} = -1$$

This is indeed a remarkable phenomenon. A transcendental number raised to a power which is the product of the imaginary unit and π, the other most important transcendental number, yields precisely -1.

EXERCISES (XII-3)

1. a. Find the expansion of e^{-x}.
   b. Check the result by obtaining it from the e^x series.
2. Find the Maclaurin expansion for

a. $\ln(1 - x)$	c. $\sin 2x$
b. $\ln(1 + x)$	d. $e^x + e^{-x}$

(Restrict x as necessary.)

3. Derive exercise 2(b) from (a) by an appropriate substitution.
4. Derive the series in exercise 2(c) and 2(d) from others developed so far.
5. Find the approximate value of $\sin 12° = \sin \pi/15$ by using the first three terms of the expansion. Find the value of the fourth term to determine the degree of accuracy you may count on in the previous computation.
6. Use the first three terms of the $\ln(1 + x)$ expansion to obtain an approximation for $\ln 1.2$. Examine the fourth term for an indication of dependable accuracy in the first three.
7. The expansions may be conveniently stated with the summation symbol. For example,

$$\sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!}$$

   a. Represent the $\cos x$ expansion in this manner.
   b. Do the same for the e^x expansion.
8. Show that $e^{-ix} = \cos x - i \sin x$.
9. Prove that:
   a. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$
   b. $\sin x = \dfrac{1}{2i}(e^{ix} - e^{-ix})$
10. Use the results in the preceding example to show that $\sin^2 x + \cos^2 x = 1$.
11. Show that $e^{i\theta} = \text{cis}\,\theta$.
12. Show that $x + iy = re^{i\theta}$, where r and θ have the usual meanings in the representation of complex numbers.
13. Show that $e^{2i\theta} = \text{cis}\,2\theta$.
14. For integral values of k, show that
   a. $1 + i = \sqrt{2}\, e^{i[(\pi/4)+2k\pi]}$
   b. $\ln(1 + i) = \frac{1}{2}\ln 2 + i\pi(2k + \frac{1}{4})$.

(Note that the logarithm of a complex number is a many valued relation.)

15. Represent the general value of each of the following complex numbers:

a. $\ln (1 - i)$
b. $\ln (\sqrt{3} + i)$
c. $\ln i$
d. $\ln (-1)$
e. $\ln 2$

16. The last two parts of the preceding exercise require a broadening of earlier statements regarding logarithms. What should be pointed out?

XII-3 REVIEW

1. Find the derivatives of each of the following:

a. $\sin 5x$
b. $\cos \frac{3}{2}x$
c. $\sqrt{\sin x}$
d. $\cos^3 x$
e. $\sin \sqrt{x}$

2. Find the angle of intersection of $y = \sin x$ and $y = \cos x$.

3. Find the critical values of $2 \sin x + \sin 2x$ for $0 \leq x \leq 2\pi$.

4. A plane is flying east of an observer at 250 miles per hour and at a constant altitude of 1000 feet. Find the rate of change of the angle of elevation at the observer on the ground when the plane is 4 miles due east of the observer.

5. An 18-foot ladder rests against a wall with its top sliding down the wall at the rate of 1 foot per minute. Find the rate of change of the angle that the ladder makes with the ground when the top of the ladder is 14 feet from the ground.

6. Two sides of a triangle are 10 and 12 inches and include an angle that is increasing at the rate of 0.2 radian per second. Find the rate of change of the area when the angle is 60°.

7. Find the value of the derivative of $x \sin^2 x$ when $x = (5/6)\pi$.

8. Find the derivatives of each of the following: (The letter shown is the independent variable.)

a. $\ln |5x + 3|$
b. $\ln |z^3 - 2|$
c. $\ln |\sin 2x|$
d. e^{4x}
e. $\sin e^x$
f. $e^{\sin x}$
g. $x e^{-x^2}$
h. $e^x \cos x$

9. The motion of a particle is described by the equation $s = 3 \sin t - 3 \cos 2t$. Find the velocity and acceleration of the particle when $t = \pi/4$.

10. Find the slope of the tangent at $t = \pi/6$ for the curve defined parametrically as $x = 3 \sin 2t$ and $y = 2 \cos 2t$.

11. Find the values of each of the following:

a. $\displaystyle\int_a^b \frac{\cos x}{\sin x}\, dx$
b. $\displaystyle\int_0^1 \frac{e^x}{e^x + 1}\, dx$
c. $\displaystyle\int_1^3 \frac{2u + 3}{u^2 + 3u}\, du$

12. Find the Maclaurin expansion for $\sin 2x$, assuming that such exists.

APPENDIX

Common Logarithms*

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

*Richardson: *Fundamentals of Mathematics*, Rev. ed. New York, The Macmillan Company. ©Moses Richardson 1958.

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Trigonometric Functions—Values and Logarithms*

Angle	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
0° 00′	.0000	——	.0000	——	——	——	1.0000	0.0000	**90° 00′**
10	.0029	7.4637	.0029	7.4637	343.77	2.5363	1.0000	0.0000	**89°** 50
20	.0058	7.7648	.0058	7.7648	171.89	2.2352	1.0000	0.0000	40
30	.0087	7.9408	.0087	7.9409	114.59	2.0591	1.0000	0.0000	30
40	.0116	8.0658	.0116	8.0658	85.940	1.9342	.9999	0.0000	20
0° 50	.0145	8.1627	.0145	8.1627	68.750	1.8373	.9999	0.0000	10
1° 00′	.0175	8.2419	.0175	8.2419	57.290	1.7581	.9998	9.9999	**89° 00′**
10	.0204	8.3088	.0204	8.3089	49.104	1.6911	.9998	9.9999	**88°** 50
20	.0233	8.3668	.0233	8.3669	42.964	1.6331	.9997	9.9999	40
30	.0262	8.4179	.0262	8.4181	38.188	1.5819	.9997	9.9999	30
40	.0291	8.4637	.0291	8.4638	34.368	1.5362	.9996	9.9998	20
1° 50	.0320	8.5050	.0320	8.5053	31.242	1.4947	.9995	9.9998	10
2° 00′	.0349	8.5428	.0349	8.5431	28.636	1.4569	.9994	9.9997	**88° 00′**
10	.0378	8.5776	.0378	8.5779	26.432	1.4221	.9993	9.9997	**87°** 50
20	.0407	8.6097	.0407	8.6101	24.542	1.3899	.9992	9.9996	40
30	.0436	8.6397	.0437	8.6401	22.904	1.3599	.9990	9.9996	30
40	.0465	8.6677	.0466	8.6682	21.470	1.3318	.9989	9.9995	20
2° 50	.0494	8.6940	.0495	8.6945	20.206	1.3055	.9988	9.9995	10
3° 00′	.0523	8.7188	.0524	8.7194	19.081	1.2806	.9986	9.9994	**87° 00′**
10	.0552	8.7423	.0553	8.7429	18.075	1.2571	.9985	9.9993	**86°** 50
20	.0581	8.7645	.0582	8.7652	17.169	1.2348	.9983	9.9993	40
30	.0610	8.7857	.0612	8.7865	16.350	1.2135	.9981	9.9992	30
40	.0640	8.8059	.0641	8.8067	15.605	1.1933	.9980	9.9991	20
3° 50	.0669	8.8251	.0670	8.8261	14.924	1.1739	.9978	9.9990	10
4° 00′	.0698	8.8436	.0699	8.8446	14.301	1.1554	.9976	9.9989	**86° 00′**
10	.0727	8.8613	.0729	8.8624	13.727	1.1376	.9974	9.9989	**85°** 50
20	.0756	8.8783	.0758	8.8795	13.197	1.1205	.9971	9.9988	40
30	.0785	8.8946	.0787	8.8960	12.706	1.1040	.9969	9.9987	30
40	.0814	8.9104	.0816	8.9118	12.251	1.0882	.9967	9.9986	20
4° 50	.0843	8.9256	.0846	8.9272	11.826	1.0728	.9964	9.9985	10
5° 00′	.0872	8.9403	.0875	8.9420	11.430	1.0580	.9962	9.9983	**85° 00′**
10	.0901	8.9545	.0904	8.9563	11.059	1.0437	.9959	9.9982	**84°** 50
20	.0929	8.9682	.0934	8.9701	10.712	1.0299	.9957	9.9981	40
30	.0958	8.9816	.0963	8.9836	10.385	1.0164	.9954	9.9980	30
40	.0987	8.9945	.0992	8.9966	10.078	1.0034	.9951	9.9979	20
5° 50	.1016	9.0070	.1022	9.0093	9.7882	0.9907	.9948	9.9977	10
6° 00′	.1045	9.0192	.1051	9.0216	9.5144	0.9784	.9945	9.9976	**84° 00′**
10	.1074	9.0311	.1080	9.0336	9.2553	0.9664	.9942	9.9975	**83°** 50
20	.1103	9.0426	.1110	9.0453	9.0098	0.9547	.9939	9.9973	40
30	.1132	9.0539	.1139	9.0567	8.7769	0.9433	.9936	9.9972	30
40	.1161	9.0648	.1169	9.0678	8.5555	0.9322	.9932	9.9971	20
6° 50	.1190	9.0755	.1198	9.0786	8.3450	0.9214	.9929	9.9969	10
7° 00′	.1219	9.0859	.1228	9.0891	8.1443	0.9109	.9925	9.9968	**83° 00′**
10	.1248	9.0961	.1257	9.0995	7.9530	0.9005	.9922	9.9966	**82°** 50
20	.1276	9.1060	.1287	9.1096	7.7704	0.8904	.9918	9.9964	40
30	.1305	9.1157	.1317	9.1194	7.5958	0.8806	.9914	9.9963	30
40	.1334	9.1252	.1346	9.1291	7.4287	0.8709	.9911	9.9961	20
7° 50	.1363	9.1345	.1376	9.1385	7.2687	0.8615	.9907	9.9959	10
8° 00′	.1392	9.1436	.1405	9.1478	7.1154	0.8522	.9903	9.9958	**82° 00′**
10	.1421	9.1525	.1435	9.1569	6.9682	0.8431	.9899	9.9956	**81°** 50
20	.1449	9.1612	.1465	9.1658	6.8269	0.8342	.9894	9.9954	40
30	.1478	9.1697	.1495	9.1745	6.6912	0.8255	.9890	9.9952	30
40	.1507	9.1781	.1524	9.1831	6.5606	0.8169	.9886	9.9950	20
8° 50	.1536	9.1863	.1554	9.1915	6.4348	0.8085	.9881	9.9948	10
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	0.8003	.9877	9.9946	**81° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Angle
	Cosine		Cotangent		Tangent		Sine		

*Rider: *First Year Mathematics for Colleges.* New York, The Macmillan Company, 1949.

Trigonometric Functions—Values and Logarithms (Continued)

Angle	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	0.8003	.9877	9.9946	**81° 00′**
10	.1593	9.2022	.1614	9.2078	6.1970	0.7922	.9872	9.9944	**80°** 50
20	.1622	9.2100	.1644	9.2158	6.0844	0.7842	.9868	9.9942	40
30	.1650	9.2176	.1673	9.2236	5.9758	0.7764	.9863	9.9940	30
40	.1679	9.2251	.1703	9.2313	5.8708	0.7687	.9858	9.9938	20
9° 50	.1708	9.2324	.1733	9.2389	5.7694	0.7611	.9853	9.9936	10
10° 00′	.1736	9.2397	.1763	9.2463	5.6713	0.7537	.9848	9.9934	**80° 00′**
10	.1765	9.2468	.1793	9.2536	5.5764	0.7464	.9843	9.9931	**79°** 50
20	.1794	9.2538	.1823	9.2609	5.4845	0.7391	.9838	9.9929	40
30	.1822	9.2606	.1853	9.2680	5.3955	0.7320	.9833	9.9927	30
40	.1851	9.2674	.1883	9.2750	5.3093	0.7250	.9827	9.9924	20
10° 50	.1880	9.2740	.1914	9.2819	5.2257	0.7181	.9822	9.9922	10
11° 00′	.1908	9.2806	.1944	9.2887	5.1446	0.7113	.9816	9.9919	**79° 00′**
10	.1937	9.2870	.1974	9.2953	5.0658	0.7047	.9811	9.9917	**78°** 50
20	.1965	9.2934	.2004	9.3020	4.9894	0.6980	.9805	9.9914	40
30	.1994	9.2997	.2035	9.3085	4.9152	0.6915	.9799	9.9912	30
40	.2022	9.3058	.2065	9.3149	4.8430	0.6851	.9793	9.9909	20
11° 50	.2051	9.3119	.2095	9.3212	4.7729	0.6788	.9787	9.9907	10
12° 00′	.2079	9.3179	.2126	9.3275	4.7046	0.6725	.9781	9.9904	**78° 00′**
10	.2108	9.3238	.2156	9.3336	4.6382	0.6664	.9775	9.9901	**77°** 50
20	.2136	9.3296	.2186	9.3397	4.5736	0.6603	.9769	9.9899	40
30	.2164	9.3353	.2217	9.3458	4.5107	0.6542	.9763	9.9896	30
40	.2193	9.3410	.2247	9.3517	4.4494	0.6483	.9757	9.9893	20
12° 50	.2221	9.3466	.2278	9.3576	4.3897	0.6424	.9750	9.9890	10
13° 00′	.2250	9.3521	.2309	9.3634	4.3315	0.6366	.9744	9.9887	**77° 00′**
10	.2278	9.3575	.2339	9.3691	4.2747	0.6309	.9737	9.9884	**76°** 50
20	.2306	9.3629	.2370	9.3748	4.2193	0.6252	.9730	9.9881	40
30	.2334	9.3682	.2401	9.3804	4.1653	0.6196	.9724	9.9878	30
40	.2363	9.3734	.2432	9.3859	4.1126	0.6141	.9717	9.9875	20
13° 50	.2391	9.3786	.2462	9.3914	4.0611	0.6086	.9710	9.9872	10
14° 00′	.2419	9.3837	.2493	9.3968	4.0108	0.6032	.9703	9.9869	**76° 00′**
10	.2447	9.3887	.2524	9.4021	3.9617	0.5979	.9696	9.9866	**75°** 50
20	.2476	9.3937	.2555	9.4074	3.9136	0.5926	.9689	9.9863	40
30	.2504	9.3986	.2586	9.4127	3.8667	0.5873	.9681	9.9859	30
40	.2532	9.4035	.2617	9.4178	3.8208	0.5822	.9674	9.9856	20
14° 50	.2560	9.4083	.2648	9.4230	3.7760	0.5770	.9667	9.9853	10
15° 00′	.2588	9.4130	.2679	9.4281	3.7321	0.5719	.9659	9.9849	**75° 00′**
10	.2616	9.4177	.2711	9.4331	3.6891	0.5669	.9652	9.9846	**74°** 50
20	.2644	9.4223	.2742	9.4381	3.6470	0.5619	.9644	9.9843	40
30	.2672	9.4269	.2773	9.4430	3.6059	0.5570	.9636	9.9839	30
40	.2700	9.4314	.2805	9.4479	3.5656	0.5521	.9628	9.9836	20
15° 50	.2728	9.4359	.2836	9.4527	3.5261	0.5473	.9621	9.9832	10
16° 00′	.2756	9.4403	.2867	9.4575	3.4874	0.5425	.9613	9.9828	**74° 00′**
10	.2784	9.4447	.2899	9.4622	3.4495	0.5378	.9605	9.9825	**73°** 50
20	.2812	9.4491	.2931	9.4669	3.4124	0.5331	.9596	9.9821	40
30	.2840	9.4533	.2962	9.4716	3.3759	0.5284	.9588	9.9817	30
40	.2868	9.4576	.2994	9.4762	3.3402	0.5238	.9580	9.9814	20
16° 50	.2896	9.4618	.3026	9.4808	3.3052	0.5192	.9572	9.9810	10
17° 00′	.2924	9.4659	.3057	9.4853	3.2709	0.5147	.9563	9.9806	**73° 00′**
10	.2952	9.4700	.3089	9.4898	3.2371	0.5102	.9555	9.9802	**72°** 50
20	.2979	9.4741	.3121	9.4943	3.2041	0.5057	.9546	9.9798	40
30	.3007	9.4781	.3153	9.4987	3.1716	0.5013	.9537	9.9794	30
40	.3035	9.4821	.3185	9.5031	3.1397	0.4969	.9528	9.9790	20
17° 50	.3062	9.4861	.3217	9.5075	3.1084	0.4925	.9520	9.9786	10
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	0.4882	.9511	9.9782	**72° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Angle
	Cosine		Cotangent		Tangent		Sine		

Trigonometric Functions—Values and Logarithms (Continued)

Angle	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	0.4882	.9511	9.9782	**72° 00′**
10	.3118	9.4939	.3281	9.5161	3.0475	0.4839	.9502	9.9778	**71°** 50
20	.3145	9.4977	.3314	9.5203	3.0178	0.4797	.9492	9.9774	40
30	.3173	9.5015	.3346	9.5245	2.9887	0.4755	.9483	9.9770	30
40	.3201	9.5052	.3378	9.5287	2.9600	0.4713	.9474	9.9765	20
18° 50	.3228	9.5090	.3411	9.5329	2.9319	0.4671	.9465	9.9761	10
19° 00′	.3256	9.5126	.3443	9.5370	2.9042	0.4630	.9455	9.9757	**71° 00′**
10	.3283	9.5163	.3476	9.5411	2.8770	0.4589	.9446	9.9752	**70°** 50
20	.3311	9.5199	.3508	9.5451	2.8502	0.4549	.9436	9.9748	40
30	.3338	9.5235	.3541	9.5491	2.8239	0.4509	.9426	9.9743	30
40	.3365	9.5270	.3574	9.5531	2.7980	0.4469	.9417	9.9739	20
19° 50	.3393	9.5306	.3607	9.5571	2.7725	0.4429	.9407	9.9734	10
20° 00′	.3420	9.5341	.3640	9.5611	2.7475	0.4389	.9397	9.9730	**70° 00′**
10	.3448	9.5375	.3673	9.5650	2.7228	0.4350	.9387	9.9725	**69°** 50
20	.3475	9.5409	.3706	9.5689	2.6985	0.4311	.9377	9.9721	40
30	.3502	9.5443	.3739	9.5727	2.6746	0.4273	.9367	9.9716	30
40	.3529	9.5477	.3772	9.5766	2.6511	0.4234	.9356	9.9711	20
20° 50	.3557	9.5510	.3805	9.5804	2.6279	0.4196	.9346	9.9706	10
21° 00′	.3584	9.5543	.3839	9.5842	2.6051	0.4158	.9336	9.9702	**69° 00′**
10	.3611	9.5576	.3872	9.5879	2.5826	0.4121	.9325	9.9697	**68°** 50
20	.3638	9.5609	.3906	9.5917	2.5605	0.4083	.9315	9.9692	40
30	.3665	9.5641	.3939	9.5954	2.5386	0.4046	.9304	9.9687	30
40	.3692	9.5673	.3973	9.5991	2.5172	0.4009	.9293	9.9682	20
21° 50	.3719	9.5704	.4006	9.6028	2.4960	0.3972	.9283	9.9677	10
22° 00′	.3746	9.5736	.4040	9.6064	2.4751	0.3936	.9272	9.9672	**68° 00′**
10	.3773	9.5767	.4074	9.6100	2.4545	0.3900	.9261	9.9667	**67°** 50
20	.3800	9.5798	.4108	9.6136	2.4342	0.3864	.9250	9.9661	40
30	.3827	9.5828	.4142	9.6172	2.4142	0.3828	.9239	9.9656	30
40	.3854	9.5859	.4176	9.6208	2.3945	0.3792	.9228	9.9651	20
22° 50	.3881	9.5889	.4210	9.6243	2.3750	0.3757	.9216	9.9646	10
23° 00′	.3907	9.5919	.4245	9.6279	2.3559	0.3721	.9205	9.9640	**67° 00′**
10	.3934	9.5948	.4279	9.6314	2.3369	0.3686	.9194	9.9635	**66°** 50
20	.3961	9.5978	.4314	9.6348	2.3183	0.3652	.9182	9.9629	40
30	.3987	9.6007	.4348	9.6383	2.2998	0.3617	.9171	9.9624	30
40	.4014	9.6036	.4383	9.6417	2.2817	0.3583	.9159	9.9618	20
23° 50	.4041	9.6065	.4417	9.6452	2.2637	0.3548	.9147	9.9613	10
24° 00′	.4067	9.6093	.4452	9.6486	2.2460	0.3514	.9135	9.9607	**66° 00′**
10	.4094	9.6121	.4487	9.6520	2.2286	0.3480	.9124	9.9602	**65°** 50
20	.4120	9.6149	.4522	9.6553	2.2113	0.3447	.9112	9.9596	40
30	.4147	9.6177	.4557	9.6587	2.1943	0.3413	.9100	9.9590	30
40	.4173	9.6205	.4592	9.6620	2.1775	0.3380	.9088	9.9584	20
24° 50	.4200	9.6232	.4628	9.6654	2.1609	0.3346	.9075	9.9579	10
25° 00′	.4226	9.6259	.4663	9.6687	2.1445	0.3313	.9063	9.9573	**65° 00′**
10	.4253	9.6286	.4699	9.6720	2.1283	0.3280	.9051	9.9567	**64°** 50
20	.4279	9.6313	.4734	9.6752	2.1123	0.3248	.9038	9.9561	40
30	.4305	9.6340	.4770	9.6785	2.0965	0.3215	.9026	9.9555	30
40	.4331	9.6366	.4806	9.6817	2.0809	0.3183	.9013	9.9549	20
25° 50	.4358	9.6392	.4841	9.6850	2.0655	0.3150	.9001	9.9543	10
26° 00′	.4384	9.6418	.4877	9.6882	2.0503	0.3118	.8988	9.9537	**64° 00′**
10	.4410	9.6444	.4913	9.6914	2.0353	0.3086	.8975	9.9530	**63°** 50
20	.4436	9.6470	.4950	9.6946	2.0204	0.3054	.8962	9.9524	40
30	.4462	9.6495	.4986	9.6977	2.0057	0.3023	.8949	9.9518	30
40	.4488	9.6521	.5022	9.7009	1.9912	0.2991	.8936	9.9512	20
26° 50	.4514	9.6546	.5059	9.7040	1.9768	0.2960	.8923	9.9505	10
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	0.2928	.8910	9.9499	**63° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Angle
	Cosine		Cotangent		Tangent		Sine		

Trigonometric Functions—Values and Logarithms (Continued)

Angle	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	0.2928	.8910	9.9499	**63° 00′**
10	.4566	9.6595	.5132	9.7103	1.9486	0.2897	.8897	9.9492	**62°** 50
20	.4592	9.6620	.5169	9.7134	1.9347	0.2866	.8884	9.9486	40
30	.4617	9.6644	.5206	9.7165	1.9210	0.2835	.8870	9.9479	30
40	.4643	9.6668	.5243	9.7196	1.9074	0.2804	.8857	9.9473	20
27° 50	.4669	9.6692	.5280	9.7226	1.8940	0.2774	.8843	9.9466	10
28° 00′	.4695	9.6716	.5317	9.7257	1.8807	0.2743	.8829	9.9459	**62° 00′**
10	.4720	9.6740	.5354	9.7287	1.8676	0.2713	.8816	9.9453	**61°** 50
20	.4746	9.6763	.5392	9.7317	1.8546	0.2683	.8802	9.9446	40
30	.4772	9.6787	.5430	9.7348	1.8418	0.2652	.8788	9.9439	30
40	.4797	9.6810	.5467	9.7378	1.8291	0.2622	.8774	9.9432	20
28° 50	.4823	9.6833	.5505	9.7408	1.8165	0.2592	.8760	9.9425	10
29° 00′	.4848	9.6856	.5543	9.7438	1.8040	0.2562	.8746	9.9418	**61° 00′**
10	.4874	9.6878	.5581	9.7467	1.7917	0.2533	.8732	9.9411	**60°** 50
20	.4899	9.6901	.5619	9.7497	1.7796	0.2503	.8718	9.9404	40
30	.4924	9.6923	.5658	9.7526	1.7675	0.2474	.8704	9.9397	30
40	.4950	9.6946	.5696	9.7556	1.7556	0.2444	.8689	9.9390	20
29° 50	.4975	9.6968	.5735	9.7585	1.7437	0.2415	.8675	9.9383	10
30° 00′	.5000	9.6990	.5774	9.7614	1.7321	0.2386	.8660	9.9375	**60° 00′**
10	.5025	9.7012	.5812	9.7644	1.7205	0.2356	.8646	9.9368	**59°** 50
20	.5050	9.7033	.5851	9.7673	1.7090	0.2327	.8631	9.9361	40
30	.5075	9.7055	.5890	9.7701	1.6977	0.2299	.8616	9.9353	30
40	.5100	9.7076	.5930	9.7730	1.6864	0.2270	.8601	9.9346	20
30° 50	.5125	9.7097	.5969	9.7759	1.6753	0.2241	.8587	9.9338	10
31° 00′	.5150	9.7118	.6009	9.7788	1.6643	0.2212	.8572	9.9331	**59° 00′**
10	.5175	9.7139	.6048	9.7816	1.6534	0.2184	.8557	9.9323	**58°** 50
20	.5200	9.7160	.6088	9.7845	1.6426	0.2155	.8542	9.9315	40
30	.5225	9.7181	.6128	9.7873	1.6319	0.2127	.8526	9.9308	30
40	.5250	9.7201	.6168	9.7902	1.6212	0.2098	.8511	9.9300	20
31° 50	.5275	9.7222	.6208	9.7930	1.6107	0.2070	.8496	9.9292	10
32° 00′	.5299	9.7242	.6249	9.7958	1.6003	0.2042	.8480	9.9284	**58° 00′**
10	.5324	9.7262	.6289	9.7986	1.5900	0.2014	.8465	9.9276	**57°** 50
20	.5348	9.7282	.6330	9.8014	1.5798	0.1986	.8450	9.9268	40
30	.5373	9.7302	.6371	9.8042	1.5679	0.1958	.8434	9.9260	30
40	.5398	9.7322	.6412	9.8070	1.5597	0.1930	.8418	9.9252	20
32° 50	.5422	9.7342	.6453	9.8097	1.5497	0.1903	.8403	9.9244	10
33° 00′	.5446	9.7361	.6494	9.8125	1.5399	0.1875	.8387	9.9236	**57° 00′**
10	.5471	9.7380	.6536	9.8153	1.5301	0.1847	.8371	9.9228	**56°** 50
20	.5495	9.7400	.6577	9.8180	1.5204	0.1820	.8355	9.9219	40
30	.5519	9.7419	.6619	9.8208	1.5108	0.1792	.8339	9.9211	30
40	.5544	9.7438	.6661	9.8235	1.5013	0.1765	.8323	9.9203	20
33° 50	.5568	9.7457	.6703	9.8263	1.4919	0.1737	.8307	9.9194	10
34° 00′	.5592	9.7476	.6745	9.8290	1.4826	0.1710	.8290	9.9186	**56° 00′**
10	.5616	9.7494	.6787	9.8317	1.4733	0.1683	.8274	9.9177	**55°** 50
20	.5640	9.7513	.6830	9.8344	1.4641	0.1656	.8258	9.9169	40
30	.5664	9.7531	.6873	9.8371	1.4550	0.1629	.8241	9.9160	30
40	.5688	9.7550	.6916	9.8398	1.4460	0.1602	.8225	9.9151	20
34° 50	.5712	9.7568	.6959	9.8425	1.4370	0.1575	.8208	9.9142	10
35° 00′	.5736	9.7586	.7002	9.8452	1.4281	0.1548	.8192	9.9134	**55° 00′**
10	.5760	9.7604	.7046	9.8479	1.4193	0.1521	.8175	9.9125	**54°** 50
20	.5783	9.7622	.7089	9.8506	1.4106	0.1494	.8158	9.9116	40
30	.5807	9.7640	.7133	9.8533	1.4019	0.1467	.8141	9.9107	30
40	.5831	9.7657	.7177	9.8559	1.3934	0.1441	.8124	9.9098	20
35° 50	.5854	9.7675	.7221	9.8586	1.3848	0.1414	.8107	9.9089	10
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	0.1387	.8090	9.9080	**54° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Angle
	Cosine		Cotangent		Tangent		Sine		

Trigonometric Functions—Values and Logarithms (Continued)

Angle	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	0.1387	.8090	9.9080	**54° 00′**
10	.5901	9.7710	.7310	9.8639	1.3680	0.1361	.8073	9.9070	**53°** 50
20	.5925	9.7727	.7355	9.8666	1.3597	0.1334	.8056	9.9061	40
30	.5948	9.7744	.7400	9.8692	1.3514	0.1308	.8039	9.9052	30
40	.5972	9.7761	.7445	9.8718	1.3432	0.1282	.8021	9.9042	20
36° 50	.5995	9.7778	.7490	9.8745	1.3351	0.1255	.8004	9.9033	10
37° 00′	.6018	9.7795	.7536	9.8771	1.3270	0.1229	.7986	9.9023	**53° 00′**
10	.6041	9.7811	.7581	9.8797	1.3190	0.1203	.7969	9.9014	**52°** 50
20	.6065	9.7828	.7627	9.8824	1.3111	0.1176	.7951	9.9004	40
30	.6088	9.7844	.7673	9.8850	1.3032	0.1150	.7934	9.8995	30
40	.6111	9.7861	.7720	9.8876	1.2954	0.1124	.7916	9.8985	20
37° 50	.6134	9.7877	.7766	9.8902	1.2876	0.1098	.7898	9.8975	10
38° 00′	.6157	9.7893	.7813	9.8928	1.2799	0.1072	.7880	9.8965	**52° 00′**
10	.6180	9.7910	.7860	9.8954	1.2723	0.1046	.7862	9.8955	**51°** 50
20	.6202	9.7926	.7907	9.8980	1.2647	0.1020	.7844	9.8945	40
30	.6225	9.7941	.7954	9.9006	1.2572	0.0994	.7826	9.8935	30
40	.6248	9.7957	.8002	9.9032	1.2497	0.0968	.7808	9.8925	20
38° 50	.6271	9.7973	.8050	9.9058	1.2423	0.0942	.7790	9.8915	10
39° 00′	.6293	9.7989	.8098	9.9084	1.2349	0.0916	.7771	9.8905	**51° 00′**
10	.6316	9.8004	.8146	9.9110	1.2276	0.0890	.7753	9.8895	**50°** 50
20	.6338	9.8020	.8195	9.9135	1.2203	0.0865	.7735	9.8884	40
30	.6361	9.8035	.8243	9.9161	1.2131	0.0839	.7716	9.8874	30
40	.6383	9.8050	.8292	9.9187	1.2059	0.0813	.7698	9.8864	20
39° 50	.6406	9.8066	.8342	9.9212	1.1988	0.0788	.7679	9.8853	10
40° 00′	.6428	9.8081	.8391	9.9238	1.1918	0.0762	.7660	9.8843	**50° 00′**
10	.6450	9.8096	.8441	9.9264	1.1847	0.0736	.7642	9.8832	**49°** 50
20	.6472	9.8111	.8491	9.9289	1.1778	0.0711	.7623	9.8821	40
30	.6494	9.8125	.8541	9.9315	1.1708	0.0685	.7604	9.8810	30
40	.6517	9.8140	.8591	9.9341	1.1640	0.0659	.7585	9.8800	20
40° 50	.6539	9.8155	.8642	9.9366	1.1571	0.0634	.7566	9.8789	10
41° 00′	.6561	9.8169	.8693	9.9392	1.1504	0.0608	.7547	9.8778	**49° 00′**
10	.6583	9.8184	.8744	9.9417	1.1436	0.0583	.7528	9.8767	**48°** 50
20	.6604	9.8198	.8796	9.9443	1.1369	0.0557	.7509	9.8756	40
30	.6626	9.8213	.8847	9.9468	1.1303	0.0532	.7490	9.8745	30
40	.6648	9.8227	.8899	9.9494	1.1237	0.0506	.7470	9.8733	20
41° 50	.6670	9.8241	.8952	9.9519	1.1171	0.0481	.7451	9.8722	10
42° 00′	.6691	9.8255	.9004	9.9544	1.1106	0.0456	.7431	9.8711	**48° 00′**
10	.6713	9.8269	.9057	9.9570	1.1041	0.0430	.7412	9.8699	**47°** 50
20	.6734	9.8283	.9110	9.9595	1.0977	0.0405	.7392	9.8688	40
30	.6756	9.8297	.9163	9.9621	1.0913	0.0379	.7373	9.8676	30
40	.6777	9.8311	.9217	9.9646	1.0850	0.0354	.7353	9.8665	20
42° 50	.6799	9.8324	.9271	9.9671	1.0786	0.0329	.7333	9.8653	10
43° 00′	.6820	9.8338	.9325	9.9697	1.0724	0.0303	.7314	9.8641	**47° 00′**
10	.6841	9.8351	.9380	9.9722	1.0661	0.0278	.7294	9.8629	**46°** 50
20	.6862	9.8365	.9435	9.9747	1.0599	0.0253	.7274	9.8618	40
30	.6884	9.8378	.9490	9.9772	1.0538	0.0228	.7254	9.8606	30
40	.6905	9.8391	.9545	9.9798	1.0477	0.0202	.7234	9.8594	20
43° 50	.6926	9.8405	.9601	9.9823	1.0416	0.0177	.7214	9.8582	10
44° 00′	.6947	9.8418	.9657	9.9848	1.0355	0.0153	.7193	9.8569	**46° 00′**
10	.6967	9.8431	.9713	9.9874	1.0295	0.0126	.7173	9.8557	**45°** 50
20	.6988	9.8444	.9770	9.9899	1.0235	0.0101	.7153	9.8545	40
30	.7009	9.8457	.9827	9.9924	1.0176	0.0076	.7133	9.8532	30
40	.7030	9.8469	.9884	9.9949	1.0117	0.0051	.7112	9.8520	20
44° 50	.7050	9.8482	.9942	9.9975	1.0058	0.0025	.7092	9.8507	10
45° 00′	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	.7071	9.8495	**45° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Angle
	Cosine		Cotangent		Tangent		Sine		

ANSWERS

To Selected Odd-numbered Exercises

I–1

1a.

14	8
28	4
56	2
112	1

$14 \times 8 = 112$

b.

15	7
30	3
60	1

$15 \times 7 = 15 + 30 + 60 = 105$

c.

28	21
56	10
112	5
224	2
448	1

$28 \times 21 = 28 + 112 + 448 = 588$

d.

104	17
208	8
416	4
832	2
1664	1

$104 \times 17 = 104 + 1664$
$= 1768$

3a. V + (I + II) = VIII
b. (V + I)II = X + II = XII
c. V + (III + II) = X
d. (V + III)II = X + VI = XVI

5a.

1	6
2	12
4	24
8	48

$8 \times 6 = 48$

b.

1	15
2	30
4	60
8	120

$13 \times 15 = 120 + 60 + 15 = 195$

c.

1	26
2	52
4	104
8	208
16	416

$23 \times 26 = 416 + 104 + 52 + 26 = 598$

7. כ ח

I–4

1a. $7(10) + 3$
b. $8(10)^2 + 4(10)$
c. $6(10)^3 + 5(10)^2 + 7$
d. $6(10)^3 + 5(10) + 7$

I–5

1a. 19
b. 19
c. 25
d. 41
e. 550

3a. 242
b. 266
c. 27t
d. 1,010,111,110,010

5a. 130
b. 3,031
c. 1,000,010
d. 10,001
e. 200
f. 4,052
g. t4
h. 2,243

I–8

1a. Sum of two numbers is a number; substitution; association; commutation
b. Product of two numbers is a number; substitution; distribution

3a. $mp + mq; mq + mp$
$pm + mq;\ qm + mp; \cdots$
b. $mpq,\ pmq,\ qpm, \cdots$

5a. $ac + ad + bc + bd$
b. $ab + 3a + 2b + 6$
c. $x^2 + 5x + 6$

I–9

1a. $30 + 12a$ substitution, distribution, association
b. $10m + 8p$ commutation, similar terms
c. $x(y + z)$ distribution
d. $12ab + 18ac$ distribution, commutation, association, substitution
e. $12mk$ commutation, association, substitution
f. $30abcd$ commutation, association, substitution
g. $6a^2 + 11a + 3$ commutation, association, distribution, substitution

I–9 REVIEW

1a. Sum of two numbers is a number.
b. Product of two numbers is a number.
c. Commutation, association
d. Commutation, association, substitution
e. Distribution, substitution
f. Association
g. Distribution

3. 23

5a.

13	24
26	12
52	6
104	3
208	1

$104 + 208 = 312$

b.

1	24
2	48
4	96
8	192

$24 + 96 + 192 = 312$

7. ||∩∩
||

9. 1,001,000
10,010,111,111

11. $xz + xw + yz + yw$

I–10

1a. $20x^3$
b. $18a^3b^3$
c. $20x^5y$
d. $3x^3 + 6x^2$
e. $5m^5 + 10m^4$
f. $12x^2 + 11x + 2$
g. $x^4 + 5x^2 + 6$
h. 10^9
i. 5^5
j. 1.8×10^{20}
k. $(cd)^8$
l. 1.3×10^{36}
m. 7^7
n. a^6
o. n^{x+y}

3. The exponent refers only to its indicated base.

5a. 15
b. 16
c. $3a + 2b$

I–11

1. 2.4×10^{46}
3. 10^{16}

4a. 1.4×10^{17}
b. 8.1×10^{33}

I–12

1a. 6
b. 1
c. 9
d. x
e. y
f. 5
g. x
h. 1
i. 1

I–12 REVIEW

1a. $30a^3b^2$
b. $30t^5$
c. $40a^5b^6$
d. 10^a
e. abh^{m+t}
f. 5^6

3a. 76
b. 250
c. 340
d. 1

5. 10, 100, 1000, 10,000

I–13

1a. 1, 2, 3, 5, 6, 10, 15, 30
b. 1, 2, 5, 7, 10, 14, 35, 70
c. 1, 3, 5, 7, 15, 21, 35, 105
d. 1, 2, 3, 4, 6, 8, 12, 24

3a. Meaningless
b. 0
c. Indeterminate

5. A natural number has unique prime factors.

I–14

1. Consider $1 \cdot 2 = 2$ and $8 \cdot 1 = 8$

3. 1

5a. p/m, $m \neq 0$, p and m are natural numbers.
b. $\frac{c}{a + b}$, $a + b \neq 0$,; a, b, and c are natural numbers.
c. $\frac{Q}{M + N}$, $M + N \neq 0$; M, N, and Q are natural numbers.

7. 11 and 13, 29 and 31, 47 and 49
9. p is larger than $x, y, z, \cdots$, or w and is not divisible by any of them.
11. $17 = 4^2 + 1^2$, $29 = 5^2 + 2^2$

I–15

1a. Equal numbers multiplied by equal numbers result in equals.
b. The squares of equal numbers are equal.

3a. $b \neq 0$, $d \neq 0$
b. $bd \neq 0$

5. $x + y = \frac{pa + mc}{pc}$

I–17

1a. $\frac{4}{5}$
b. $\frac{x}{2y}$
c. $\frac{1}{x}$
d. $\frac{2a}{7b}$
e. $\frac{2}{3}$

3a. $6(m + 1)$
b. $x(y + z)$
c. $x^2(1 + x)$
d. $ab(a + b)$
e. $6p(m + 2p)$
f. $4(3p + 8)$
g. $4(k + 1)(3k + 7)$
h. $6(m^2 + 2m + 3)$
i. $\frac{1}{2}rh(r + h)$

5. It is 3, 6, or 12.

7a. $\frac{2(3)}{2(4)} = \frac{3(3)}{3(4)}$
b. $\frac{4(4)}{4(5)} = \frac{7(4)}{7(5)}$
c. $\frac{3(4)}{3(9)} \neq \frac{4(2)}{4(3)}$
d. $\frac{3a}{2p} \neq \frac{3p(3a)}{3p(2)}$ unless $p = 1$ and/or $a = 0$

I–18

1a. $\frac{6}{35}$
b. $\frac{18}{49}$
c. $\frac{4}{3a}$
d. ax
e. 1
f. $\frac{6}{7}$
g. $\frac{1 + x}{y^2}$

3. $5x = 4$, $3y = 2$

$15xy = 8$, $xy = \frac{8}{15}$

I–19

1a. $\frac{x}{2}$

b. $\frac{b}{a}$

c. $\frac{a}{b}$

d. $\frac{8}{25}$

e. $2M$

f. $\frac{p^3}{K}$

g. $\frac{4y + 5}{5}$

h. $\frac{1 + T}{T}$

i. $6m + 2$

j. $k + 1$

k. 30

l. 7

m. $6m$

3a. 48 cu in.

b. 3 in.

c. 5

d. 120 lb

e. $\frac{x}{y}$ in.

f. 27,000 ft

g. KQ lb

h. 100 gm

5a. Division by any number is equivalent to multiplication by its reciprocal.

b. Division by zero is meaningless.

I–19 REVIEW

1a. $2 \cdot 3 \cdot 7$

b. $2 \cdot 7 \cdot 13$

c. $3 \cdot 5 \cdot 11$

d. $a(b + c)$

e. $\pi(x + y + z)$

f. $5d^2(1 + d)$

3. a. $\frac{43}{36}$

b. $\frac{ac + cb}{bc}$

c. c/b

d. a^3b^3/c^4

e. $\frac{2}{6}$

f. $a + 2$

g. $9/a$

h. 1.6×10^7

i. $b + a$

j. $\frac{1}{9uv}$

k. $\frac{19}{7}$

l. $5w/v$

m. $\frac{3ab}{5}$

5a. $2^4 \cdot 3^2$

b. $2^3 \cdot 3^4$

c. $2^3 \cdot 3^2 \cdot 5^2$

7a. "1" is not prime by definition.

b. Commutative.

I–20

1a. $5 < 7, 7 > 5$
b. $12 < 13, 13 > 12$
c. $\frac{1}{2} > \frac{1}{3}, \frac{1}{3} < \frac{1}{2}$

3a. 4
b. 5
c. 8, 9, 10, 11
d. None

5. Descriptively as "2° below zero"; red ink in ledger.

I–21

1a. 3
b. -3
c. $+2$ or -2
d. Impossible
e. $-P$

I–22

1a. -15
b. -2
c. $8x^2 - 9x$
d. $2ab$
e. $2m + 3m^2$
f. $-7(a + b)$
g. $3mn^2 - 4m^2n$
h. $-\frac{1}{6}$
i. $-\frac{5}{12}$

3a. 5
b. -2
c. 7
d. 11
e. 3
f. $2\frac{1}{3}$
g. $4\frac{1}{2}$
h. $6\frac{2}{3}$

I–23

1a. 38
b. -7
c. 16
d. $12x^2$
e. $-8ab$
f. $-3y(z + 2)$
g. $\frac{1}{12}$

3a. $7a - 3$
b. $2m^2 + 6m$
c. $5x - 3$

5a. Subtraction of a quantity is equivalent to the addition of its negative. The sum of 0 and any number is that number.
b. Same as (a). Start with $0 - (+n)$.

I–25

1a. 30
b. -140
c. $-6a - 3$
d. $-10t + 15t^2$
e. $-a^2 + 3a$
f. $5 - b$
g. $-t - 3$
h. $-18m^3n^3$
i. -3
j. -3
k. $\frac{3}{4}$
l. $-a$
m. $-5a^2$
n. $a^2 - a - 6$
o. $6a^2 + 13a - 5$
p. $x^2 - y^2$
q. $6a^3 - 4a^2 + 2a$
r. $-c$
s. -2
t. $15a^2 + 7ab - 2b^2$

3a. $4 \cdot 54 = 216$
b. $1 \cdot 31 = 31$
c. $2 \cdot 74 = 148$
d. $2 \cdot 280 = 560$

5a. Since $(-1)(-x) = (-1)a$, then $x = -a$.
b. Since $\frac{-x}{-1} = \frac{a}{-1}$, then $x = -a$.
c. Add $-a + x$ to both members or subtract $-x = a$ from $0 = 0$.

I–25 REVIEW

1a. -5
b. 2
c. $-\frac{9}{40}$
d. $1\frac{1}{3}$
e. $2\frac{1}{2}$
f. 18
g. -4
h. -12

3a. $n < 1, n \neq 0$
b. $4 \leq n \leq 7$

5a. $30a^6$
b. $a^2 + 4a$
c. $\frac{4a^2}{3b}$
d. $h^2 - 9$
e. $12t - 3$
f. -9
g. 1
h. 1.6×10^{12}
i. $10w^2 - 11w + 3$
j. $-\frac{7}{16}$
k. $9h^2 - 12h + 4$
l. $a^2 - a$
m. $-h - 17$
n. -4
o. $\frac{3h}{5}$
p. $\frac{19h - 2}{15}$
q. $\frac{5 + x}{5x}$
r. $\frac{11a - 11}{6}$
s. $\frac{2a}{(a - b)(a + b)}$

7. $\frac{2}{3}$, $\frac{2}{3}$, $2 - 3x$, $b - a$

I–27

1a. 1
b. 6
c. m
d. $h^5 + h$
e. $-2k + 2$
f. -2
g. 19
h. 27
i. 16
j. 1

3a. $\frac{8}{x^2}$
b. $\frac{1}{64x^3}$
c. $\frac{v^2}{u^2}$
d. $\frac{6y^3}{x^2}$
e. $-\frac{8}{m^2n^2}$
f. $\frac{6}{(s + t)^3}$
g. $\frac{2.7}{10^{14}}$
h. $\frac{a}{b} - \frac{b}{a}$
i. $24x^2$
j. $\frac{72}{v^5}$
k. $2a^4$

5a. 6.2×10^{-6}
b. 3.7×10^{-6}
c. 2.9×10^{-10}

I–28

1a. 2
b. $-\frac{1}{2}$
c. 8
d. Impossible
e. 10^5
f. 7^3
g. $\frac{1}{2}$
h. 9
i. $4\frac{15}{16}$
j. 64
k. 2
l. 6
m. a^p
n. 16

3a. Non-negative
b. All values
c. No values

5. Start with $2^{1/3} = p/q$ and cube both members, in manner of text with $2^{1/2}$.

7a. $3\sqrt{2}$
b. $|x|\sqrt{x}$
c. $2\sqrt[3]{2}$
d. $-2\sqrt[5]{2}$
e. Impossible as yet
f. $a^2b^2\sqrt{a}$
g. $13\sqrt{2}$
h. $4\sqrt[3]{3}$
i. $|a| - a\sqrt[5]{a}$
j. $5\sqrt{3}$

9. Let $N = u \,.\, a_1a_2\cdots a_n$, u an integer, and $a_i = 0, 1, 2, \cdots 9$. Then,

$$N = u + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

N is then a sum of rational numbers which is itself a rational number.

11. $$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

I–28 REVIEW

1a. -5
b. $\frac{1}{8}$
c. 2.0×10^{-6}
d. 5.9×10^{13}
e. $-\sqrt[3]{2}$
f. $31\sqrt{2}$

3a. $\sqrt{5}$
b. $\dfrac{\sqrt{x}}{x^2}$
c. $\sqrt[3]{4}$
d. $\dfrac{\sqrt{10}}{4}$
e. $\dfrac{\sqrt[3]{h^2}}{h^2}$

5. $m > 0$, $p > 0$ for all values of m, $\sqrt[m]{p} = +\sqrt[m]{p}$

7b. $(2n + 1)^2 = 2(2n^2 + 2n) + 1$. By (a), the right-hand member is odd.

I–29

1a. $3i$
b. i
c. $2i\sqrt{2}$
d. $17i\sqrt{2}$
e. $|a|i$
f. $i|m|\sqrt{m}$
g. $20\sqrt{3} + 3i\sqrt{3}$
h. $\frac{7}{6}i$
i. $-|a|i$
j. i

3a. $(m - ni)(m + ni) = m^2 + n^2$. Since m and n are real, their squares and sum of squares are real, positive.

5a. 13
b. $4i$
c. 2
d. $5 + 14i$
e. $a^2 + b^2$
f. $a^2 - b^2 + 2abi$

7. $\sqrt{-1} \neq \dfrac{\sqrt{1}}{\sqrt{-1}}$, as may be seen from $\sqrt{a}\sqrt{b} = -\sqrt{ab}$ when a and b are both negative.

I–32 REVIEW

3a. Origin

b. $(a, b) - (c, d) = 0$
$(a - c, b - d) = 0$
$\therefore a - c = 0 \quad b - d = 0$
$a = c \qquad b = d$

c. Two complex numbers are equal if and only if the real and imaginary parts are equal, respectively.

I–33

1. $p \le \frac{1}{2}(p + q) \le q$. $\frac{1}{2}(p+q)$ is a number. The average of this with p or q may be taken, yielding a new average, and so forth.

3a. I_1: 1, 2
I_2: 1.7, 1.8
I_3: 1.73, 1.74
I_4: 1.732, 1.733

b. $(I_4 + I_4')$: 1.414 + 1.732, 1.415 + 1.733
: 3.146, 3.148
$3.146 < \sqrt{2} + \sqrt{3} < 3.148$

c. $I_n \cdot I_n'$: a_na_n', b_nb_n'
If $n = 4$, $I_4 \cdot I_4'$: 2.449, 2.452
$(1.414 \times 1.732 = 2.449)$
$(1.415 \times 1.733 = 2.452)$
$2.449 < \sqrt{2}\sqrt{3} < 2.452$

I–34

1. 0, 1, −1, 2, −2, 3, −3, ⋯
1, 2, 3, 4, 5, 6, 7, ⋯

3. $\left(\frac{0}{1}\right), \left(\frac{1}{1}\right), \left(\frac{2}{1}, \frac{1}{2}\right), \left(\frac{3}{1}, \frac{1}{3}\right), \left(\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}\right), \cdots$

5. $A \cup B$ is the set of reals

7. $\aleph_0$

9. $A \cap B = B$

11a. Every number between 0 and 1, expressed in binary system, is of form $0.a_1a_2a_3 \cdots$ Zeros can be added so that all numbers have an infinite number of places. Construct new binary $0.b_1b_2b_3 \cdots$ where $b_1 \ne a_1$, $b_2 \ne a_2$, and so forth, and b_i is 0 or 1.

13. Set A contains rationals.

15. $A \cap B = 0$

II–1

1a. x is any point on X-axis between m and n, inclusive.

b. x is the half-line beginning with m and extending to the right.

c. All points on X-axis greater than n, exclusive of n.

d. All points on X-axis between m and n, including m but not n.

e. All points on X-axis between m and n with neither m nor n.

f. All points on X-axis less than or equal to n.

g. All points on X-axis between m and n, with n but not m included.

h. All points on X-axis.

II–2

1. If x is an acute angle,

   $0 < x < 90°$
   $0 < x < 100$ grades
   $0 < x < 1600$ mils

   If x is a right angle,

   $x = 90°$, $x = 100$ grades
   $x = 1600$ mils

   If x is obtuse,

   $90° < x < 180°$
   100 grades $< x < 200$ grades
   1600 mils $< x < 3200$ mils

3a. $65° \, 39'$

b. $73° \, 43'$

II–3

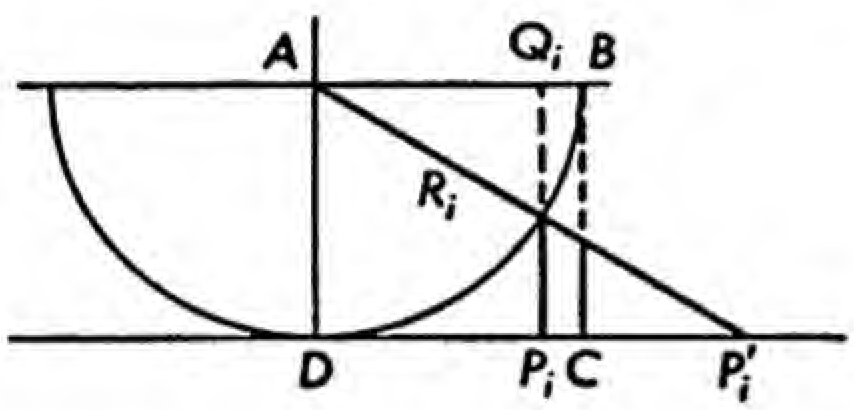

Fig. II–3–1

1a. Segment AB is put into 1 to 1 correspondence with DC via perpendiculars P_iQ_i. By means of AR_i extended, P_i is put into 1 to 1 with P_i'.

b. A similar procedure may be used here.

3. Since a set contains at least one element not in a proper subset of itself, the two sets cannot be put into 1 to 1 correspondence if either is finite.

II–5

1. Mass-produced parts are $\cong$. Blueprints and scale drawings utilize similarity.

3a. The distance between the second pair of points is k times the former distance.

b. Distances are the same.

5. (a) and (c) are $\cong$. (a) or (c) is $\backsim$ to (b); constant of proportionality = 2:1.

7. 9, 25, 49

9. (a) is valid by definition.

II–6

1a. Any side corresponds to any other. So, any $\angle$ corresponds to any other. Therefore all $\angle$s are =.

b. No, since b corresponds only to b, and so, only $\angle B = \angle B$.

3. Corresponding $\triangle$s are $\backsim$ by *sas*.

5. Draw diagonals. Use fact concerning segment joining midpoints of 2 sides of a $\triangle$.

7a. $180n°$

b. $180(n - 2)°$

c. $180n - 180(n - 2) = 360°$

11. Right and vertical $\angle$s make $\triangle$s $\backsim$.

13. $\angle A + \angle ACB = 90°$
    $\angle ECD + \angle ACB = 90°$
    $\therefore \angle A = \angle ECD$

II–6 REVIEW

1. Should lines intersect, an impossible △ would result.

3a. ∠s A and ADE are supplementary; also ∠s C and CED.

b. The line segment that joins the midpoints of 2 sides of a △ is one-half the third side, and if extended, will not intersect it.

5a. 72 b. 60 c. 36

7. 2:5

9. The = half-segments and vertical ∠s yield 2 pairs of ≅ △s.

11. The Y-axis is the perpendicular bisector of PP'. The ordinates of the two points are equal, while the abscissas are the negatives of each other.

13. $A'(-1, 2)$, $B'(-6, 4)$, $C'(-3, 7)$
$A''(1, -2,)$, $B''(6, -4)$, $C''(3, -7)$

II–7

1. Equal supplements of ∠s 1 and 2, together with vertical ∠s, give ∼ △s.

3a. The ∠ bisector leads to *sas*.

b. Use *saa*.

5a. Use ≅ △s.

b. The 3 altitudes are =.

7. If an ∠ of one (vertex or base) equals a corresponding ∠ of other (vertex or base).

9. Prove △s ≅ by *sas*.

II–8

1a. $4\frac{1}{2}$
b. 10
c. $1\frac{1}{4}$
d. $\frac{14}{27}$
e. $1\frac{5}{23}$
f. $2\sqrt{10}$

3a. Use right ∠s and = corresponding ∠s.

b. Use halves of = ∠s instead of the right ∠s of (a).

5. $\angle A = \angle CDB$ via supplements, $\angle C = \angle C$, △s are ∼, and so, $BC:BD$ = CE:AE

II–9

1a. $\angle C = \angle C$ plus given.
b. Does not; otherwise, △s ≅
c. Corresponding sides of ∼ △s.
d. $8\sqrt{3}$
e. $10\frac{3}{4}$

3a. ± 6
b. $\pm 2\sqrt{7}$
c. $\pm \frac{\sqrt{3}}{3}$ or $\pm \frac{1}{\sqrt{3}}$
d. $\pm \frac{1}{5}\sqrt{210}$

II–10

1. $a = 6,\ b = 12\sqrt{2}$
 $\sqrt{a^2 + b^2} = \sqrt{324} = 18 = c$

3a. 9, 40, 41 for $p = 5,\ q = 4$
 20, 21, 29 for $p = 5,\ q = 2$

3b. $(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$
 $p^4 + 2p^2q^2 + q^4 = p^4 + 2p^2q^2 + q^4$

5. $\sqrt{89}$

7a. 10, 15
 b. $2\sqrt{41},\ 3\sqrt{41},\ 5\sqrt{41} \approx 32.0$
 c. Taking $AR = 6$, $PR + QR = 10 + \sqrt{505} \approx 32.5$

9. $\dfrac{d}{\sqrt{2}}$

11. 17

13. Prove rt. $\triangle$s $\sim$.

II–11

1a. If $3 > 2$ and $15 > 7$, then $18 > 9$.
 If $2 > -5$ and $-2 > -3$, then $0 > -8$.
 If $-1 > -4$ and $-1 > -6$, then $-2 > -10$.
 b. If $a < b$ and $c < d$, then $a + c < b + d$.

3. $c = d + k, \quad k > 0$
 $c + b = (d + b) + k$
 $\therefore c + b > d + b$

5a. $b = a + k \quad k > 0$
 $bc = ac + ck \quad c < 0,\ ck < 0$
 $bc < ac$

 b. If both members of an inequality are multiplied by -1, the sense of the inequality is reversed.

7a. $x > 2$
 b. $x > 2$
 c. $4 < x < 8$
 d. $1 \leq x \leq 5$
 e. $x < 10$
 f. $3 < x < -3$
 g. $x > \frac{3}{5}$
 h. $x > 6$
 i. $-1\frac{2}{3} < x < 1$
 j. $x < 6\frac{2}{3}$
 k. $x < -5$
 l. $x < 4\frac{1}{2}$
 m. $3 < x < -3$

9. Draw a diagonal; use inequalities; sum of 2 sides of a $\triangle$ is greater than the third.

11. In $\triangle ABC$, $a + b > c$; so, $a > c - b$.

13. If $c/c' = a/a' = k$, then $c^2 = c'^2k^2$ and $a^2 = a'^2k^2$. Use these with Pythagorean theorem to show $b = b'k$ and so $b/b' = k$. $\triangle$s are $\sim$ by *sss*.

15. Draw altitude to base and use conclusion of exercise 12.

17. (b) and (c)

II–11 REVIEW

1. Prove $\triangle BFE \sim \triangle DFC$

3a. ± 10

b. ± 6

c. $\pm 2\sqrt{2}$

d. $\pm\sqrt{10}$

5a. $d^2 = l^2 + w^2$; $d = \sqrt{l^2 + w^2}$

b. $D^2 = d^2 + h^2 = l^2 + w^2 + h^2$

$D = \sqrt{l^2 + w^2 + h^2}$

7. $d = \sqrt{1025} = 32'$

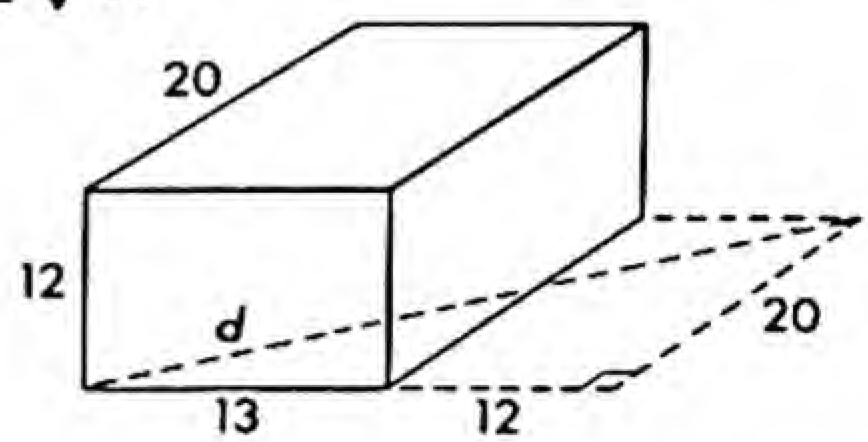

Fig. II–11 Rev.–7

9. $\angle B = \angle B$, $\angle BAD = \angle C$

11. $(x - y)^2 = x^2 - 2xy + y^2 \geq 0$

$x^2 + 2xy + y^2 = (x + y)^2 \geq 4xy$, etc.

13a. 1.5, 1.6, 1.7

b. $\sqrt{0.50}$, $\sqrt{0.51}$, $\sqrt{0.52}$

II–12

1. Show base $\angle$s =.

3. $\cong \triangle$s by saa or asa.

5. Use two pairs of equal alt. int. $\angle$s.

7a. Draw diagonals and prove $\triangle$s $\cong$.

b. Same as (a).

c. Use consecutive supplementary $\angle$s.

d. Prove $\triangle$s $\cong$; use (a) or (b)

e. See (c)

9. Show 2 adjacent sides =.

11. Use = base $\angle$s then $\cong\triangle$s.

II–13

1a. 32 sq in.

b. Area of each $\triangle$ formed by a diagonal $= \frac{1}{4}d^2$.

3. The 2$\triangle$s have = bases and a common altitude.

5. Use diagonal, parallel, or perpendiculars.

9. See II–11 Review exercise 8.

11a. $AE \parallel BD$ by right $\angle$s.

$ED \parallel AB$ through midpoints.

b. Two sets of $\cong$ right $\triangle$s and the addition of a common trapezoid.

d. $\angle EAC + \angle DBC = \angle C$ from $\cong \triangle$s. Add to both members of this = $\angle$s CAB and CBA.

II–13 REVIEW

1. Each side is one-half of one of the equal diagonals.

3. Bisectors may be proved $\parallel$ via corresponding $\angle$s; $\cong \triangle$s available too.

5. $6k^2$

7a. $h = \dfrac{2A}{b + B}$

b. $b = \dfrac{2A}{h} - B$

c. $B = \dfrac{2A}{h} - b$

9a. $16\sqrt{3}$

b. Note $h = \frac{s}{2}\sqrt{3}$

11a. 2:3

b. 2:3

c. 4:9

d. $56\frac{1}{4}$ sq cm

13. 98 sq ft

III–1

1. Altitude via pressure; temperature via expansion.

3a. $QR = \tan P$, $PR = \sec P$
b. $PQ = \cot P$, $PR = \csc P$

5. $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{1}{\sqrt{2}}, \sqrt{2}, 1, 1$

7a. Lowest prod = 18.8.
Usual prod = 19.2.
Highest prod = 19.7.
The usual is about average between the other two.
b. 0.00034, 0.00037, 0.00040
c. Preferred prod. in (a), 19; in (b), 0.0004.
9. 48.9 ft
11. 35°
13a. 110 ft
b. 53.4
15. 71 mi/hr, 10 mi/hr

III–2

1a. Only one extra significant figure needed in computation.
b. Given side contained only two significant figures.

3a. 59
b. 38
5. If $a = b$, $A = B$ via Law of Sines; ($A = 180 - B$ is impossible.)
7. $m + CB = AB \tan y$; $CB = AB \tan x$; then, $m = AB(\tan y - \tan x)$.

9a. $\frac{m}{p} = \frac{q}{r}, \frac{m}{q} = \frac{p}{r}$
$\frac{m}{q} + 1 = \frac{p}{r} + 1; \frac{m+q}{q} = \frac{p+r}{r}$
$\frac{m+q}{p+r} = \frac{q}{r}$

b. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
$\frac{a+b+c}{\sin A + \sin B + \sin C} = \frac{a}{\sin A}$
and so forth

III–3

1. 163 yd ≈ 160 yd.
3. $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

III–3 REVIEW

1. 94 mi/hr, 13 mi/hr
3. 54 ft
5. 69 mi/hr, 92 mi/hr
7. 173 mi
9. 31 ft

11a. $\cos A$
b. $\tan A$
c. $\tan A$
d. $\cot A$

13b. $a^2 + b^2 = r^2$
$r^2(\sin^2 \theta + \cos^2 \theta) = r^2$
$\sin^2 \theta + \cos^2 \theta = 1$

III–4

1a. sin 12
b. tan 67
c. −sec 27
d. cos 50
e. −tan 76
f. −cot 65
g. cos 42
h. tan 16
i. −cot 25
j. csc 48

3a. $-\frac{1}{2}$
b. 0
c. $-\sqrt{2}$
d. 0
e. $-\frac{1}{2}$
f. −1
g. −2
h. ∞
i. $\frac{1}{\sqrt{2}}$

5. $c^2 = h^2 + (x + b)^2$ and use $x = -a \cos C$

7. 129°

9a. $\sqrt{13}$, 124°
b. $\sqrt{41}$, 321°

III–5

1. $\cot R = \dfrac{-\cos R}{-\sin R} = \cot R$

3a. cot 23
b. sin 6
c. cos 18
d. −csc 17
e. −sec 19
f. tan 36

5a. $\sin \theta$
b. $\cos x$
c. $\dfrac{\sin^4 y}{\cos^2 y}$
d. $\cos y$
e. $\sin t + \cos t$
f. $\dfrac{1}{\sin^2 M \cos^2 M}$

7a. 45°, 225°
b. 210°, 330°
c. 45°, 135°, 225°, 315°
d. 90°, 270°
e. 27°, 207°
f. 66°, 114°, 246°, 294°

III–6

1d. See Fig. III–6–1(d).
e. See Fig. III–6–1(e).
f. See Fig. III–6–1(f).

3a. 1, 90°
b. $\frac{1}{2}$, 120°
c. 3, 720°
d. 4, 180°
e. $\frac{1}{2}$, 720°
f. 0.2, 1.8°

III–7

1a. $\frac{1}{3}\pi, \frac{2}{3}\pi, \frac{\pi}{4}, \frac{\pi}{6}, \frac{5}{6}\pi$

b. $\frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{9}, \frac{7\pi}{24}$

3a. 6″, $3\frac{3}{4}$″

b. $s = r\theta$

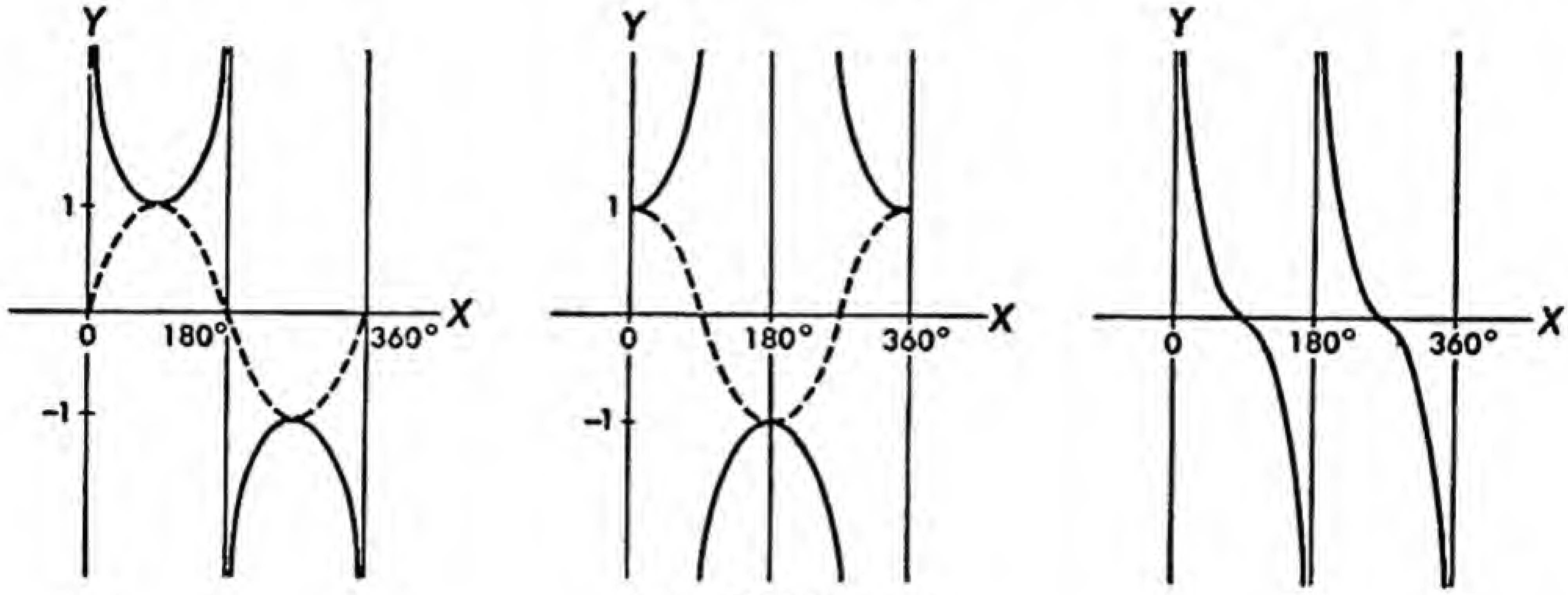

Fig. III–6–1def

5a. $\frac{\sqrt{3}}{2}$

b. 0

c. 1

d. 0.7265

III–8

1a. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

b. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

3a. $3 \sin x \cos^2 x - \sin^3 x$

b. $\cos^3 x - 3 \sin^2 x \cos x$

5. 14″; insufficient data.

7a. $\frac{\sin y}{\cos y} + \frac{\cos y}{\sin y} = \frac{1}{\sin y \cos y}$

$\frac{1}{\sin y \cos y} = \frac{1}{\sin y \cos y}$

b. $\frac{1}{\sin \theta \cos \theta} = \text{same}$

c. $\tan x = \frac{1 - (\cos^2 x - \sin^2 x)}{2 \sin x \cos x}$

$= \frac{2 \sin^2 x}{2 \sin x \cos x}$

$= \tan x$

d. $\tan^2 x = \frac{1 - (\cos^2 x - \sin^2 x)}{1 + \cos^2 x - \sin^2 x}$

$\tan^2 x = \tan^2 x$

e. $-\sqrt{2}\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x\right) = \cos x\left(\frac{\sin x}{\cos x} - 1\right)$

$\sin x - \cos x = \sin x - \cos x$

f. $\cos^2 m - \sin^2 m = \cos^2 m - \sin^2 m$

9a. $\frac{\pi}{2}, \frac{11\pi}{6}$

b. $\frac{3\pi}{4}, \frac{7\pi}{4}$

c. $\frac{\pi}{3}, \frac{5\pi}{3}$

d. $\frac{\pi}{3}, \frac{5\pi}{3}$

e. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

III–9

1a. $\text{cis}\,\frac{5\pi}{12}$

b. $2\,\text{cis}\,0$

c. $6\,\text{cis}\,\frac{5\pi}{6}$

d. $\text{cis}\,\frac{\pi}{2}$

e. $\text{cis}\,\frac{\pi}{12}$

f. $\text{cis}\,\frac{5\pi}{6}$

g. $\text{cis}\,\frac{11\pi}{6}$

3a. $\frac{1}{\sqrt{2}}(1 + i)$

b. -1

c. $-1 - \sqrt{3}i$

d. $\frac{3}{2}(1 + \sqrt{3}i)$

e. $\frac{-5}{\sqrt{2}}(1 + i)$

f. 6

g. $\frac{1}{\sqrt{2}}(1 + i)$

h. $\frac{3}{2}(\sqrt{3} + i)$

7. $r^{1/m}\,\text{cis}\left(\frac{\theta + 2k\pi}{m}\right)$ for integral values of k.

III–9 REVIEW

1. Amplitude 2, *pd* $\frac{\pi}{4}$; for both.

3a. $\frac{1}{\tan\theta} - \tan\theta = \frac{2}{\tan 2\theta}$

$$\frac{1 - \tan^2\theta}{\tan\theta} = \frac{1 - \tan^2\theta}{\tan\theta}$$

b. $\frac{\tan x}{1 - \tan^2 x} = \frac{\sin x \cos x}{\cos^2 x - \sin^2 x}$

$$= \frac{\sin x \cos x}{1 - 2\sin^2 x}$$

5. $\left\{\text{cis}\left(\frac{3\pi}{2} + 2k\pi\right)\right\}^{1/4} = \text{cis}\left(\frac{3\pi}{8} + \frac{k\pi}{2}\right)$

7. *b* and *c*

9a. $\sqrt{13}$ b. $\sqrt{13}$

11. $|\cot x| = \frac{|\cos x|}{\sqrt{1 - \cos^2 x}}$

13a. $-\sqrt{3} + i$

b. $\frac{3}{\sqrt{2}}(1 - i)$

c. $-5i$

d. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

17. $\frac{h(\tan\theta' - \tan\theta)}{\tan\theta}$ or $\frac{h(\cot\theta - \cot\theta')}{\cot\theta'}$

19. 48 sq in.

IV–3

1. Use birectangular triangles.
3. Yes, a trirectangular $\triangle$.
5. Prove $\triangle$s $\cong$ by *sas*. Same as Euclidean proof.

7a. Use exercise 6(a).
b. Use exercise 6(b).
c. The vertex $\angle$ is a pole and the original $\triangle$ is birectangular.

9a. Extend = sides to pole and drop perpendicular to summit. Base $\angle$s in upper triangle may be shown acute.

9b. Nonexistent.

11. Measure off quadrant's distance on each line from point of intersection. Segment joining the two points will provide the two right $\angle$s.

IV–6

1b. All lines through P and within $\angle BPA$ are intersectors; all others, excepting the parallels, are nonintersectors.
c. Take any other point P' and draw $P'A$. This will be a Lobatchevskian right parallel.

3b. Extend AP and BP and see Lobatchevskian figures in text.
c. Lobatchevskian parallels meet at ∞.
d. $P < \pi$; otherwise, APB is one geodesic arc. Sum is less than π.

IV–7

1. Proposition: Equal perpendiculars have = $\angle$s of parallelism. Contrapositive: Unequal $\angle$s of parallelism have unequal perpendiculars.

5b. If perpendiculars are not equal, then $\angle$s of parallelism are not equal.

7a. Draw BD as indicated in Fig. IV–22. $\angle CDB > \angle P$ by valid Euclidean theorem. As a model for a Lobatchevskian figure, this consistently shows that larger $\angle$ of parallelism goes with smaller perpendicular.
b. In this case $\angle CPB$ would be the exterior $\angle$ and greater than CDB.

IV–8

1a. $\angle 1$, as an $\angle$ of parallelism, is acute; $\angle 2$ is obtuse.
b. $\angle GFC$ is acute.
c. Perpendiculars between parallels at equal distances from foot of same perpendicular are equal.
d. Whole segment greater than part.
e. Perpendiculars between parallels grow shorter in direction of parallelism.

3. Since $\angle A < \angle 1$, and all other angles are the same, the sum of the $\angle$s of $ABDE$ is less than of $FCDE$.

IV–9

1. Summit $\angle$s = in both: acute in Lobatchevskian and obtuse in Riemannian; sum of birectangular quadrilateral $\angle$s greater than 2π in Riemannian and less than 2π in Lobatchevskian.
3. $\angle 1 = \angle 2$ in Euclid.
   $\angle 1 < \angle 2$ in Lobatchevskian.
   $\angle 1 > \angle 2$ in Riemannian.
5. Since summit angles are acute, a summit DY can be drawn within $\angle D$. Thus $CB > YB = AD$. Fig. IV–9–5. Similar procedure for $DC > AB$. Compare with corresponding cases for Riemannian, IV–3, exercise 10.
7. See Fig. IV–35, in which $GE = \frac{1}{2}HD$. By exercise 6, $HD < AB$ (in Lobatchevskian.) So, $GE < \frac{1}{2}AB$. (In Riemannian, $HD > AB$.)

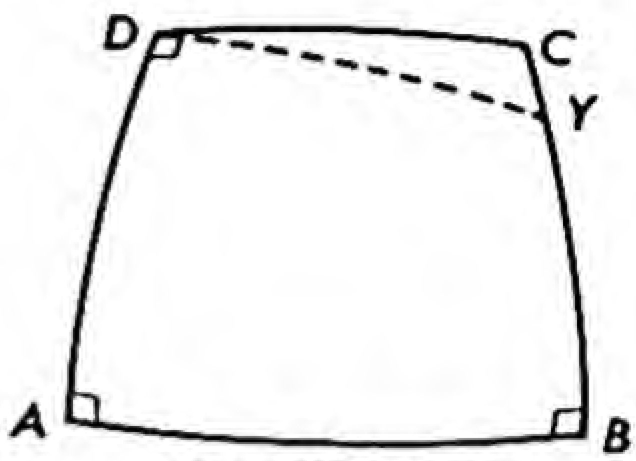

Fig. IV–9–5

9. Recall that the summit $\angle$s are obtuse in Riemannian figures.
11. The corresponding figure with the same lettering leads to an almost identical proof as in text. Rather than add the 4 sets of = $\angle$s, add the last 3 sets and subtract the first. This is the only actual change in proof. (Note that III is $\triangle ECF$, $\angle 3$ is $\angle FCE$ and $\angle 5 = \angle CAB$.)

IV–10 REVIEW

1. Drop a perpendicular from B to m and show that it coincides with the $\angle$ bisector.
3. Through midpoint of AB draw a parallel to one of the lines which becomes parallel to other too. By exercise 2, new line forms = $\angle$s with AB, and so is perpendicular to AB.
5. Assume sides $\neq$. Extend shorter to = longer and join ends to form an isosceles birectangular quadrilateral. Contradiction develops.
7. Suppose m and m' are parallels and p is a perpendicular between them. For any other parallels, s and s', a perpendicular $q = p$ can be found. The corresponding $\angle$s of parallelism will be = and the figures will be $\cong$.

V–2

1a.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

b.

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

c.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

d.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

e. No.

3a. Exercise 1(a): The roots of the equation are irrational and equal.
Exercise 1(b): The roots of the equation are rational and $\neq$.
Exercise 1(c): It is not true that the roots of the equation are rational and equal.
Exercise 2(a): The roots of the equation are not rational or $=$.
Exercise 2(b): The roots of the equation are rational or they are $\neq$.
Exercise 2(d): It is not true that the roots of the equation are rational or $=$.

3b, c, d, e, f
In same manner as for exercise 3(a).

5a. The opposite sides of a figure are either not parallel or not equal.
b. The $\angle$s are either not supplementary or $\neq$.
c. The geometry is not Lobatchevskian and not Riemannian.
d. The angle is either not acute or the lines are not parallel.
e. The man is not a dupe or not a liar.

V–3

1a.

q	r	$\sim r$	$q \rightarrow \sim r$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

b.

q	r	$\sim q$	$\sim q \rightarrow r$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

3. If $\triangle ABC$ is isosceles, then its vertex $\angle$ is acute. If $\triangle ABC$ is isosceles, then its vertex $\angle$ is not acute. If $\triangle ABC$ is not isosceles, then its vertex $\angle$ is acute. If the vertex $\angle$ of $\triangle ABC$ is not acute, then the $\triangle$ is not isosceles.

V–4

1a. If $\angle$s are vertical, then they are formed by intersecting lines.
b. A binary number system has only 2 distinct symbols.
c. If a figure is a square, then its area is the square of any one side.
d. $a^n = \dfrac{1}{a^{-n}} \qquad a \neq 0$
e. A school is an institution dedicated to formal instruction.

3a. $x = \dfrac{b}{a} \qquad a \neq 0$
b. $x = b - a$
c. $m = 0$
d. The line joins a vertex to midpoint of opposite side.
e. They lie in a (Euclidean) plane and do not intersect, or they lie in a Lobatchevskian plane, do not meet, and are not nonintersectors.
f. $b > a$ or $a - b$ is imaginary.
g. p

5a. $x^2 > 0$ if x is real and $|x| > 0$.
b. An equilateral $\triangle$ is equiangular.
c. Each side of a trirectangular $\triangle$ is a quadrant.
d. A trirectangular quadrilateral contains the largest number of right $\angle$s that could be contained in the non-Euclidean geometries we studied.

V–6

1. Two propositions are equivalent if they have the same truth values. A tautology has only T as its truth value.

V–7

1. Not valid. Take out "not" in each of last two sentences.
3. Valid. We saw earlier that $\backsim(\backsim r) \leftrightarrow r$.
5. Not valid. Take converse of first sentence.
7. Not valid. Take converse of first sentence.
9. Not valid.

V–8

1. If yesterday was not Tuesday, then today is not Wednesday.
3. If $m^2 \nless n^2$, then $|m| \nless n$.
5. If one will be a good business man, then one is not indifferent to success.
7. Valid.

V–9

1. $[(p \vee q) \wedge \backsim p] \rightarrow q$

p	q	$\backsim p$	$p \vee q$	$(p \vee q) \wedge \backsim p$	$[(p \vee q) \wedge \backsim p] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

3. Opposite sides of $ABCD$ are $\neq$.
5. x is not in quadrant IV.
7. It suffices for ABC to be a $\triangle$ to know that the sum of the $\angle$s $= 180°$. The sum of 180° is a necessary condition for triangularity.
9. Non-Euclideanism is a necessary but not sufficient condition. The sum of the $\angle$s greater than 360° is a sufficient condition for non-Euclideanism.
11. $y \neq 1 - i\sqrt{3}$ is a necessary but not sufficient condition. $y^3 = 8$ is a sufficient condition.
13. Sufficient.
15. The conjunction "$b = 1$ and a is an integer" is the sufficient condition.
17. "$\angle C$ is a right $\angle$ in a Euclidean plane" is a necessary and sufficient condition.
19. Only a necessary condition.

V–9 REVIEW

1a. $\backsim(p \wedge \backsim q)$
b. $\backsim p \vee q$

3.

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

5. *a*, *d*

7a. If two $\angle$s are either vertical or right $\angle$s and they are not vertical, then they are right $\angle$s.

b. $x^2 = 4$ implies $x = 2$ and $x = -2$, and it not being true that $x = 2$ and $x = -2$, then $x^2 \neq 4$.

c. It is not true that $2 = 6/3$ and $3 = 5$, but since $2 = 6/3$, then $3 \neq 5$.

d. If a man wears gaudy clothes, he is ostentatious, and if he is ostentatious, he is conceited; then a man who wears gaudy clothes is conceited.

VI–1

1. Area of square = square of a side; independent variable ÷ length of side, dependent variable ÷ area of square; domain:non-negative numbers; range: non-negative numbers; (1, 1), (2, 4), (3, 9)

3a. Domain: $-\infty < x < \infty$
range: $-\infty < y < \infty$

b. $-\infty < x < +\infty$
$-1 \leq y \leq +1$

c. d. $-\infty < x < +\infty$
$|y| \geq 1$

e. $|x| < \infty$
$|y| \leq 2$

f. $|x| < \infty$, $|y| \leq 1$

g. $-\infty < x < \infty$
$-\infty < y < \infty$

h. $|x| < \infty$, $y = 2$

5a. 0

b. 1

c. $\dfrac{1}{\sqrt{3}}$

d. $\pm\infty$

7a. $(f + g)$: $\{x, \cos x + \sin x\}$

b. $(f - g)$: $\{x, \cos x - \sin x\}$

c. $\dfrac{1}{f}$:$\{x, \sec x\}$

d. fg: $\{x, \cos x \sin x\}$ and $\cos x \sin x = \frac{1}{2} \sin 2x$

e. $\dfrac{f}{g}$:$\left\{x, \dfrac{\cos x}{\sin x}\right\}$

and $\dfrac{\cos x}{\sin x} = \cot x \qquad (\sin x \neq 0)$

VI–2 REVIEW

1a. The set of 30°, 60°, 90° $\triangle$s in Riemannian geometry. The set of real roots of $x^2 + 1 = 0$.

b. The set of positive roots of $x^2 - 4 = 0$. The set of decimal numbers equal to the binary number 10.

3. (0, 0)

5. $2x$, x^2, and $\sqrt{x}$, respectively.

7a. 1 and -1.

b. From $-\sqrt{2}$ to $\sqrt{2}$ inclusive.

9. The domain needs to be indicated. Thus the domain, the permitted values of x, may be the real numbers, the complex numbers, the irrationals, the primes, the positive even integers, and so forth. In any such case the range will be determined by $x + 2$, and the relation will be single-valued.

11a. 9
b. 1
c. 3
d. $\sqrt{x}$
e. $\frac{2}{a}$
f. $2x + x^2$
g. $\{x, 2x + x^2\}$
h. $x^2 - 2x$
i. $\{x, x^2 - 2x\}$

13a. $f^{-1}: \{x, \frac{1}{3}x\}$
b. $g^{-1}:\{x, \sqrt{x}\} \veebar g^{-1}:\{x, -\sqrt{x}\}$ for $x \geq 0$.
c. $h^{-1}: \{x, x^{1/3}\}$
d. $k^{-1}: \{x, x\}$

VI-3

1a. $y = x + 2$
b. $y = \frac{1}{3}x$
c. $y = \frac{1}{x}$
d. $y = \frac{\sqrt{x}}{2}$, $x > 0$

5a. $x^2 + 4 = 0$
b. $9x^2 + 4 = 0$
c. $x^2 - 6x + 25 = 0$
d. $x^2 - 2x + 2 = 0$

7a. The smallest absolute values of the a's and n are each 1.
b. $x = 0$, defining 0.
c. $x - 1 = 0$ (1)
$-x - 1 = 0$ (-1)
$x^2 = 0$ (0)
$x - 2 = 0$ (2)
$x + 2 = 0$ (-2)
$2x + 1 = 0$ $(-\frac{1}{2})$
$2x - 1 = 0$ $(\frac{1}{2})$
$3x = 0$ (0)
$x^2 - 1 = 0$ (± 1)
$-x^2 - 1 = 0$ $(\pm i)$
$x^3 = 0$ (0)
d. Sixth.
e. Multiply both members of the equation by an appropriate integer. No effect on roots.
f. The quantity i may be factored from all the pure imaginary terms. $k(x) = 0$ and $m(x) = 0$.

9a. By substitution, $(3^{\sqrt{2}})^{\sqrt{2}} - 9 = 3^2 - 9 = 0$
b. The equation is not polynomial by definition because of nonpermissible exponent.

11a. $y = x$
b. $y = \sqrt{\frac{x}{2}}$ $x \geq 0$
c. $y = \frac{\sqrt{3x}}{2}$ $x \geq 0$
d. $y = \sqrt[3]{x}$
e. $y = \sqrt[4]{x}$ $x \geq 0$
$y = -\sqrt[4]{x}$ $x \geq 0$
f. $y = \frac{x - 3}{2}$
g. $y = \frac{2x - 6}{3}$

VI–4

1a. $(x + 3)(x + 1)$
b. $(a - 4)(a + 1)$
c. $(2m + 5)(m - 1)$
d. $(h - 12)(h + 1)$
e. Prime in field of polynomials with integer coefficient.
f. $(k + 3)(k - 3)$
g. $(2t - 5)(2t + 5)$
h. $(x + a)^2$
i. $(3a + b)(a - 2b)$
j. $(5 - 2t)(3 + t)$
k. $(2x + 3y)^2$
l. $k(a - 4)(a + 1)$

3a. $x = -1, x = 2\frac{1}{2}$
b. $x = +3, x = +3$
c. $x = -1\frac{1}{2}, x = 1\frac{1}{3}$
d. $x = 5, x = -2$

5. $4\frac{1}{2}$ in., $11\frac{1}{2}$ in.

7a. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
b. $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
c. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
d. 45°, 225°, 63°, 243°
e. 74°, 254°
f. 0°, 60°, 300°, 360°
g. $\frac{2\pi}{3}, \frac{4\pi}{3}$
h. 72°, 153°, 252°, 333°

9a. $1, -\frac{5}{3}$
b. $\frac{-1 \pm \sqrt{13}}{2}$
c. $\frac{1 \pm \sqrt{13}}{3}$
d. $\frac{-3 \pm \sqrt{29}}{10}$
e. $\pm i$
f. $3 \pm 4i$
g. $1 \pm i\sqrt{5}$
h. 201°, 339°
i. 58°, 238°, 77°, 257°
j. 65°, 295°
k. 28°, 332°, 70°, 290°

11.
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}$$
$$r_1r_2 = \frac{4ac}{4a^2} = \frac{c}{a}$$

Or by comparison of coefficients of $ax^2 + bx + c = 0$ and equation in exercise 10(h).

VI–5

3a. 6
b. 274
c. 666

5a. $x^3 - 2x^2 - 5x + 6 = 0$
b. $3x^3 - 10x^2 - 9x + 4 = 0$
c. $x^3 - 6x^2 + 13x - 10 = 0$

7a. -4
b. 1

VI–5 REVIEW

1a. $x^3 - 2x^2 - x + 2 = 0$
b. $y^3 - 4y^2 + 6y - 4 = 0$
c. $u^3 - 19u + 30 = 0$
d. $m^3 - 8 = 0$

3. $2i, -i$

5a. $-2 + 2i$
b. $(1 + i)(-i) - 1 + i = 0$

7a. $R:\left\{x, \dfrac{x^3 - 3x^2 + 1}{(x - 2)(x + 3)}\right\} \quad x \neq 2 \quad x \neq -3$
b. $R(x) = \dfrac{x^3 - 3x^2 + 1}{(x - 2)(x + 3)} \quad x \neq 2 \quad x \neq -3$

11a. $y = 1, y = \dfrac{1}{b}, b \neq 0$
b. 4
c. $\dfrac{\sqrt{2} \pm \sqrt{10}}{2}$

13. $i = \text{cis}\left(\dfrac{\pi}{2} + 2k\pi\right)$

$\sqrt{i} = \text{cis}\left(\dfrac{\pi}{4} + k\pi\right)$

$x_1 = \sqrt{i} = \dfrac{\sqrt{2}}{2}(1 + i)$

$x_2 = -\sqrt{i} = -\dfrac{\sqrt{2}}{2}(1 + i)$

VI–6

1a. $x^4 + 8x^3 + 24x^2 + 32x + 16$
b. $x^5 + 10x^4y + 40x^3y^2 + \cdots + 32y^5$
c. $x^6 - 6x^5 + 15x^4 - \cdots + 1$
d. $\dfrac{m^7}{128} + \dfrac{7m^6}{16} + \dfrac{21m^5}{2} + \cdots + 16{,}384$
e. $a^n - na^{n-1}b + \dfrac{n(n - 1)}{2}a^{n-2}b^2 + \cdots + (-1)^{N+1}\, b^n$
f. $1 + 3t + 3t^2 + t^3$
g. $a^4 + 4ia^3b - 6a^2b^2 - 4iab^3 + b^4$
h. $\tan^3 x + 3 \tan x + 3 \cot x + \cot^3 x$

3a. $\dfrac{12!}{4!8!}a^4b^8 = 495a^4b^8$
b. $-\dfrac{14!}{9!5!}x^9(3)^5 = -486{,}486x^9$
c. $\dfrac{4845}{8}y^{14}$

5a. 28 b. 66 c. 66
d. $C_{n,r} = C_{n,n-r}$

VI–7

	Domain	Range	Asymptotes
1.	$\lvert x\rvert < \infty$ $x \neq 2$	$\lvert y\rvert < \infty$ $y \neq 1$	$x = 2$ $y = 1$
3.	$\lvert x\rvert < \infty$ $x \neq 0, x \neq 2$	$0 < y \leq -1$	$x = 0$ $x = 2$ $y = 0$

	Domain	*Range*	*Asymptotes*
5.	$\lvert x\rvert < \infty$ $x \neq 0$	$\lvert y\rvert < \infty$ $y \neq 2$	$x = 0$ $y = 2$
7.	$\lvert x\rvert < \infty$ $x \neq -1$ $x \neq -3$	$(2 - \sqrt{3}) \geq y \geq (2 + \sqrt{3})$	$x = -1$ $x = -3$ $y = 0$
11.	$\lvert x\rvert < \infty$	$y > 0$	$y = 0$
13.	$\lvert x\rvert < \infty$	$0 \leq y \leq 1$	

15. exs. 2, 6, 13, 14

VI–8

1. To keep x the independent variable and (thereby) the frame of reference the same.

3a. x
b. y
c. b^{x+y}
d. $x + y$
e. b^{x-y}
f. $x - y$
g. b^{kx}
h. kx

5a. 1.80
b. 0.450
c. 457
d. 0.000264
e. 6.0
f. 1.3
g. 0.919
h. 200

7a. 100°
b. 109°
c. $B = 45°$, $a = 522$
d. $A = 64°$

9. 19.9 cu in.

11. 2.60×10^{11} cu mi

13. $158.50

VI–8 REVIEW

1. $10x^4$

3. $1 - \cos^2 x$

5a. $x = 0, y = -1$
b. $x = -1, y = 0$

7. 90,720 a^4b^4

9a. 1.77
b. 7.08
c. 0.792

11a. 1.3010
b. 2.9957
c. 3.6888
d. 5.2983

VI–9

1a. $x = 0, y = 0$
b. $y = 0, x = -1$
c. $y = 5, x = 1, x = -1$
d. $x = 0$
e. $x = 2, x = -2, y = 0$

VI–10

1a. 1
b. 0
c. 1
d. 0
e. ∞, div.
f. 0
g. div.
h. $\frac{1}{3}$
i. 0
j. 0 if $b < c$
div. if $b > c$
k. 0
l. c
m. 0

3. Take, for instance, $a_n = \dfrac{n}{n+1} + \dfrac{n^2+n}{3n^2-n}$ from exercises 1(a) and 1(h) and show that $\lim a_n = 4/3$. Similarly, take product, and so forth.

5. For positive integral values of n,

$$n^n \geq n$$

So,
$$\frac{1}{n^n} \leq \frac{1}{n}$$

and
$$\frac{1}{n} - \frac{1}{n^n} \geq 0$$

$$\lim\left(\frac{1}{n} - \frac{1}{n^n}\right) = \lim\frac{1}{n} - \lim\frac{1}{n^n}$$
$$= -\lim\frac{1}{n^n}$$

But,
$$\lim\left(\frac{1}{n} - \frac{1}{n^n}\right) \geq 0$$

So,
$$\lim\frac{1}{n^n} = 0$$

VI–11

1. Take $\{x\}: 1, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, \cdots, 1 + \dfrac{n-1}{n}, \cdots$

so $\{y\}: 3, 3\frac{1}{2}, 3\frac{2}{3}, 3\frac{3}{4}, \cdots, 3 + \dfrac{n-1}{n}$

Note: $3 + \dfrac{n-1}{n} = 4 - \dfrac{1}{n}$

$\delta\left(L, 4 - \dfrac{1}{n}\right) = \left| L - 4 + \dfrac{1}{n} \right|$

$\delta\left(L, 4 - \dfrac{1}{n}\right) \to 0$ as $n \to \infty$ if $L = 4$.

VI–12

1a. Conditions for continuity are met for all nonintegral values of x.

b. Right-hand limits at integral values do not agree with their corresponding function values.

3. Take $y = x$ and $y = x^2$ which are each continuous at $x = 2$, as may be shown or assumed. Then definition of continuity can be shown to hold for $y = x^2 + x$, $y = x^2 - x$, $y = x^3$, and $y = x^2/x$, $x \neq 0$.

5a. $x = -2, y = 1$

b. $x = 2, x = -2, y = -1$

c. $x = -6, x = 1, y = 0$

7. Take
$$y = \begin{cases} \dfrac{3x^2 - 12}{2x + 4}, & x \neq -2 \\ -6 \quad , & x = -2 \end{cases}$$

9. (1) $\left(a, a \sin \dfrac{1}{a}\right)$ is defined for every real value in f.

(2) $\lim\limits_{x \to a} y = a \sin \dfrac{1}{a}$

e.g., $\lim\limits_{n \to \infty} \left(a \pm \dfrac{1}{n}\right) \sin \dfrac{1}{a \pm 1/n} = a \sin \dfrac{1}{a}$

(3) $f(a) = a \sin \dfrac{1}{a}$

(4) $\lim\limits_{x \to 0} \left(x \sin \dfrac{1}{x}\right) = \lim x \lim \sin \dfrac{1}{x} = 0$

So, (0, 0) completes definition.

11. See Figs. VI–12–11(a)(b).

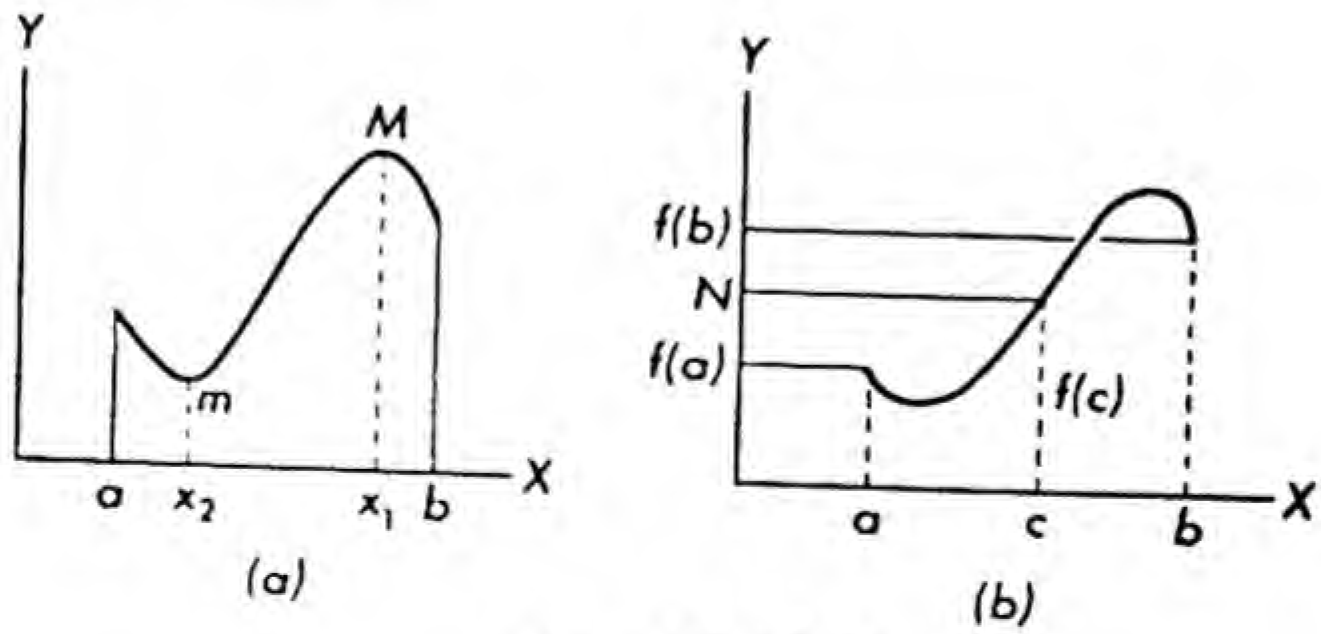

Fig. VI–12–11

VI–13 REVIEW

1a. ∞, divergent	b. 1	c. 1	d. 2
5a. 1	b. 1	c. 1	d. 0

7. See Fig. VI–13, Review VI–7.

VII–1

3a. $-\frac{1}{4}$, 10, $-\frac{13}{27}$

b. 166°, 84°, 120°

5. $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}$$
$$x_1 \neq x_2$$

7a. $y - 5 = \frac{1}{3}(x - 2)$

b. $y + 4 = -\frac{4}{5}(x + 3)$

c. $\dfrac{y + 3}{x - 2} = -1$

d. $\dfrac{y - 4}{x + 5} = \dfrac{4}{7}$

e. $y = -2x + 3$

f. $y = \frac{4}{5}x - 4$

g. $\dfrac{x}{3} - \dfrac{y}{5} = 1$

h. $\dfrac{y}{2} - \dfrac{x}{4} = 1$

i. $x = -2$

j. $y = 4$

9a. $(0\ \ -3)$, $(1, -2\frac{1}{3})$

b. $(4, 0)$, $(0, 2\frac{2}{3})$

c. $(0, 6)$, $(-1, 8\frac{1}{2})$

d. $(0, 0)$, $(2, -5)$

e. $(0, 1\frac{3}{5})$, $(7, -1\frac{3}{5})$

f. $(14, 0)$, $(0, 35)$

11. $$\frac{(y_1 + mh) - y_1}{(x_1 + h) - x_1} = \frac{mh}{h} = m$$
$$h \neq 0$$

VII–2

1a. 13

b. 17

c. $2\sqrt{5}$

d. $\sqrt{89}$

3. It would be necessary that $\sqrt{(y - 7)^2 + 4} = \sqrt{(y - 3)^2 + 4}$ or $(y - 7)^2 = (y - 3)^2$. This is possible for $y = 5$. But $(3, 5)$ is on line AB.

5. By Pythagorean theorem:

a. $(\sqrt{20})^2 + (\sqrt{45})^2 = (\sqrt{65})^2$

b. $(\sqrt{32})^2 + (\sqrt{18})^2 = (\sqrt{50})^2$

c. $(\sqrt{5})^2 + (\sqrt{45})^2 = (\sqrt{50})^2$

7. Sides are each $\sqrt{20}$ and both diagonals are $\sqrt{40}$ so that the figure is in fact a square.

9. $d_1 = d_2 = \sqrt{(b + a)^2 + c^2}$

11. $$\begin{aligned} x_m - x_1 &= x_2 - x_m \\ 2x_m &= x_1 + x_2 \\ x_m &= \tfrac{1}{2}(x_1 + x_2) \end{aligned}$$

Similarly for y_m. The abscissa (ordinate) of the midpoint of a line segment is the algebraic average of the abscissas (ordinates) of the end points.

13a. $(3, 5)$, $(2, 6)$, $(4, 7)$

b. $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$

c. Each is one-half of opposite side.

15. $(3, 3 + \sqrt{15})$, $(3, 3 - \sqrt{15})$

17. Take $A(0, 0)$, $B(a, 0)$ and $C(b, c)$.
For P: $x_m = \frac{1}{2}b$, $y_m = \frac{1}{2}c$
For Q: $x_m = \frac{1}{2}(a + b)$, $y_m = \frac{1}{2}c$
$$PQ = \sqrt{\frac{a^2}{4}} = \frac{a}{2}$$
$AB = a$

19. Use $(0, 0)$, $(2a, 0)$, $(0, 2b)$
$x = y = z = \sqrt{a^2 + b^2}$

VII–3

1a. $3x + y = 7$
c. $(2, 1)$

3a. $y = x$
b. $3y - 2x = 0$
c. $ny - mx = 0$

VII–3 REVIEW

1a. $2y - 3x = 4$
b. $2y - 3x = 6$

3. $-1\frac{8}{11}$

5. $(0, 5)$

7. $\sqrt{a^2 + b^2}$

9. $(8, 7)$

13a. $\frac{\pi}{6}$

b. $\frac{\pi}{3}$

c. $\frac{2\pi}{3}$

d. $\frac{3\pi}{4}$

15. $12x + 2y = 13$

VII–4

1a. $x = 5, y = 6$
b. $x = 1, y = -3$
c. $x = -\frac{1}{2}, y = 1\frac{1}{2}$
d. $x = 5, y = 4$
e. $x = \dfrac{km + 2k}{m^2 + 1}, y = \dfrac{2mk - k}{m^2 + 1}$
f. $x = 5\frac{7}{19}, y = -3\frac{18}{19}$
g. $x = 2, y = 1$

3. $a_1m + b_1m - c_1 = 0$
$a_2m + b_2m - c_2 = 0$
and $0 + k \cdot 0 = 0$

VII–5

1a. -10

b. 13

c. 12

d. 0

e. $\frac{1}{x} - \frac{1}{y}$

3a. $mad - mbc = m(ad - bc)$
b. $mad - mbc = m(ad - bc)$
c. $kmad - kmbc = km(ad - bc)$

5. The value of the new determinant is (-1) multiplied by the original value.

7a. $\begin{vmatrix} 5-3 & 3 \\ 6-2 & 2 \end{vmatrix}$

b. $\begin{vmatrix} 3+1 & 4-2 \\ 1 & -2 \end{vmatrix}$

c. $\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$

d. Factor 2 from the second row.
e. Factor $\frac{1}{6}$ from first row and 1/12 from second row.

VII–6

1. d and g are parallel; a and h are coincident. The following pairs are perpendicular: b, i; c, e; a, d; d, h; g, h; a, g.

3a. $y - 4 = \frac{2}{3}(x - 1)$
 b. $y + 2 = \frac{1}{2}(x + 3)$

5. Take as vertices: $(0, 0)$, $(a, 0)$, $(b + a, c)$, (b, c).

$$a = \sqrt{b^2 + c^2} \qquad\qquad m_1 m_2 = \frac{c^2}{b^2 - a^2} = -1$$
$$a^2 = b^2 + c^2$$
$$m_1 = \frac{c}{b + a},\ m_2 = \frac{c}{b - a}$$

7. Take $(0, 0)$, $(2\,a, 0)$, $(2d, 2c)$, $(2b, 2c)$ and call bases B and B', median M. Slopes of lines in question are all zero. $B = 2a$, $B' = 2d - 2b$

$$M = a + d - b = \tfrac{1}{2}(B + B')$$

9. $x = 2\frac{3}{5}$, $y = \frac{1}{5}$

11. $\dfrac{6}{\sqrt{13}}$

13a. $3x - y - 5 + k(x + 2y - 4) = 0$
 b. $3x - y = 5$
 $4x + 3y - 11 = 0$
 $y - 1 = 0$, $x - 2 = 0$

17a. 138°
 b. 101°
 c. 90°

19. Show opposite sides of new figure are equal to each other and a pair of adjacent sides are perpendicular.

21a. $\dfrac{7}{\sqrt{13}}$
 b. $\dfrac{1}{\sqrt{26}}$
 c. $\dfrac{10}{\sqrt{34}}$

25. $\dfrac{27}{\sqrt{10}}$

VII–6 REVIEW

1. $x = 1\frac{7}{11}$, $y = 1\frac{8}{11}$

3. 13°, 51°, 117°

5. 1

7. $y - 3 = -\frac{3}{2}(x - 5)$

9. $\pm 2\sqrt{3}$

11. 7

13. $X = x\cos\theta + y\sin\theta$
 $Y = y\cos\theta - x\sin\theta$

15. No.

VII–7

1. $(x - 3)^2 + (y + 4)^2 = 36$
3. $(x - 5)^2 + (y + 6)^2 = 36$
5. $(x + 4)^2 + (y + 5)^2 = 106$
7. $(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 = \frac{25}{2}$

9a. $-9 \leq x \leq 3$; $-2 \leq y \leq 10$
  b. Within.
  c. $4 \pm \sqrt{11}$

11. ± 3

13. (3, 4) and (5, 0)

15a. (3, 4), (3, 4). Line is tangent to circle.
  b. (1, −3), (1, −3). Line is tangent to circle.
  c. $\left(\dfrac{-6 \pm i\sqrt{34}}{2}, \dfrac{-6 \pm i\sqrt{34}}{2}\right)$

No intersection in the real plane.

VII–8

1a. $(2, -3)$, 3
  b. $(-4, -5)$, 7
  c. $(3, 0)$, 3
  d. $(\frac{3}{2}, \frac{1}{2})$, $2\sqrt{2}$
  e. $(0, -\frac{3}{2})$, $2\frac{1}{2}$
  f. $(\frac{5}{6}, -1)$, $\frac{1}{6}\sqrt{97}$

3. $d = -2h$, $e = -2k$, $f = h^2 + k^2 - r^2$

5a. Take $d = 2$, $e = 4$, $f = 6$
  $(x + 1)^2 + (y + 2)^2 = -1$
  $r^2 = -1$, $r = i$

  The circle is not in the real plane.

  b. Take $d = 2$, $e = 4$, $f = 5$
  $(x + 1)^2 + (y + 2)^2 = 0$

  $r = 0$. The graph is the center point, or point circle.

7. $x^2 + y^2 + x + y + 3 = 0$ (Imaginary radius)

9a. A straight line.
  b. A circle passing through the points of intersection of the first two circles. (The points of intersection may even be complex numbers.)

VII–9

1a. $x = \frac{4}{3}$, $y = \frac{1}{3}$, $z = -2$
  b. $a = -1$, $b = -1$, $c = \frac{1}{2}$

3a. (0, 2, 0), (0, 0, −4), (2, 3, 4)
  b. (2, 0, −2), (−1, 4, 3), (5, −4, −7)
  c. $3x + y - z = 8$, (2, 0, −2)

VII–10

1a. −31 b. −31 c. −355

3c. If each element of a row or column is 0, the determinant is 0.
d. The sign of the value is changed.
e. The determinant is 0.
f. No change.
g. No change.

5. Consistency requires that a counterclockwise circuit of the vertices be defined as enclosing a positive area; negative the other way.

7. Consider the three points as forming a $\triangle$ with area $= 0$; see exercise 5.

VII–11

1a. $(2, 3)$
b. $(-3, -4)$
c. $(7, -1)$
d. $(7, 10)$
e. $(-3, 13)$
3a. $X^2 + Y^2 = 25$ [Fig. VII–11–3a]
b. $X^2 + Y^2 = 10$
c. $X^2 + Y^2 = 16$

5. Translation leaves the slope unaffected.

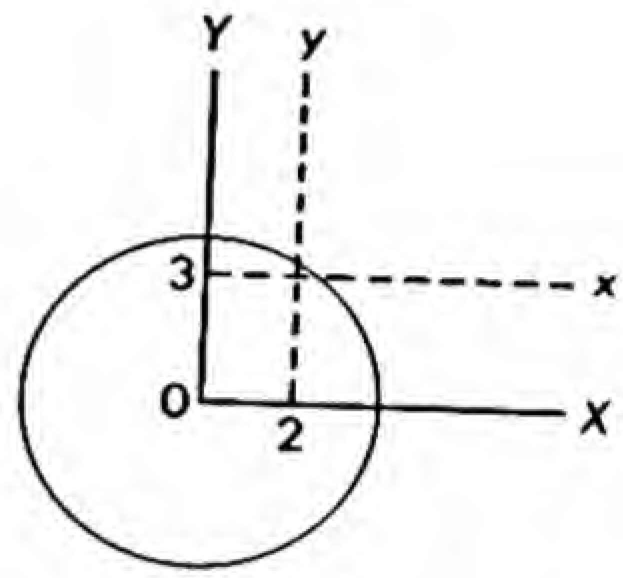

Fig. VII–11–3a

VII–11 REVIEW

1. $\dfrac{y-4}{x-3} = -\dfrac{3}{4}$

3a. $x = 2, y = 4; x = 5, y = 1$

5. Take $x^2 + y^2 = r^2$ and $y = mx + b$. The abscissas of the common solution will be given by $x^2 + (mx + b)^2 = r^2$ which, as a second-degree equation, can have at most 2 real solutions.

7a. Circle, center $(-3, 2)$, $r = 6$.
b. Circle, center $(1, 2\frac{1}{2})$, $r = 5$.

9a. $x^2 + y^2 - 6x + 8y = 0$
b. $x^2 + y^2 + 2x + 4y - 20 = 0$

11. $\begin{vmatrix} 1 & 4 & 1 \\ 3 & 9 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 0$

13. $x = \frac{1}{2}, y = 1, z = -\frac{1}{2}$

15a. $3x' - 5y' = 0$, $(x = 5, y = 0)$
b. $3x'' - 5y'' = 0$ $(x = 0, -3)$

VII–12

1a. $(0, 0), (0, 3), y = -3$
b. $(0, 0), (0, -2), y = 2$
c. $(0, 0), (4, 0), x = -4$
d. $(0, 0), (1, 0), x = -1$
e. $(0, 0), (0, \frac{1}{2}), y = -\frac{1}{2}$
f. $(2, 3), (2, 7), y = -1$
g. $(4, -3), (9, -3), x = -1$
h. $(-3, -4), (-3, -3\frac{3}{4}), y = -4\frac{1}{4}$

3a. Vertex $(-1, 4)$; focal point $(-1, 3\frac{3}{4})$; axis $x = -1$
b. Vertex $(-\frac{5}{16}, \frac{3}{2})$; focal point $(\frac{11}{16}, \frac{3}{2})$; principal axis $y = \frac{3}{2}$.
c. $(-\frac{5}{4}, -\frac{17}{48})$; $(-\frac{5}{4}, \frac{19}{48})$; $x = -\frac{5}{4}$

5a. $18x^2 - 68x - 129y + 179 = 0$
b. $7y^2 + 12x - 18y - 28 = 0$

7. $7x^2 = 9000y$

9. $(x - 2)^2 = 2(y - 3)$
Use $(x - h)^2 = 4p(y - k)$. We are given h and k and so needed only the $(4, 5)$ to find p. Actually the point $(0, 5)$ is implied by the given data, and so, in effect, we have 3 given points and the orientation of the curve which may be used with the other general equation.

11. 20 ft

VII–13

1. $(x^a y^b)^c = x^{ac} y^{bc}$

$\left(\dfrac{x^a}{y^b}\right)^c = \dfrac{x^{ac}}{y^{bc}}$

3. Use $a^2 = b^2 + c^2$

7. $\dfrac{2b^2}{|a|}$

9. $\dfrac{y^2}{36} + \dfrac{(x - 2)^2}{20} = 1$

11. $4x^2 + 3y^2 - 22y + 7 = 0$

13a. $(2, 3)$, $4\sqrt{17}$, $2\sqrt{17}$
b. $(2, -3)$, 8, 6

VII–14

3. $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

5a. $y = \frac{2}{3}x$, $y = -\frac{2}{3}x$
b. $y = x$, $y = -x$
c. $y = \dfrac{4x}{3}$, $y = -\dfrac{4x}{3}$
d. $y = \dfrac{x}{5}$, $y = -\dfrac{x}{5}$
e. $y = x$, $y = -x$
f. $y = \dfrac{x}{2}$, $y = -\dfrac{x}{2}$

7. $x^2 - 2y = 7$

VII–15

1. $4X^2 + 4Y^2 - 10XY - 1 = 0$

3. $(2, 1)$, $(-\frac{5}{7}, -4\frac{3}{7})$

7a. $(4, 3)$ $(4, -3)$ $(-4, 3)$ $(-4, -3)$
b. $(3, 5)$ $(-3, 5)$
c. $(3, 4)$ $(-3, -4)$ $(4, 3)$ $(-4, -3)$

VII–16 REVIEW

1. $0 \le x < \infty$, $0 \le y < \infty$

5. $x^2 - 4y - 2x + 5 = 0$

7. New equation $4x^2 + y^2 = 25$

11a. $(\pm 4, 0)$, 10, 6, $(\pm 5, 0)$, $(0, \pm 3)$
b. $(0, \pm\sqrt{2})$, $2\sqrt{2}$, 4, $(\pm\sqrt{2}, 0)$, $(0, \pm 2)$
c. $(3 \pm \sqrt{3}, -1)$, 4, 2, $(5, -1)$, $(1, -1)$
$(3, 0)$, $(3, -2)$
d. $(0, \pm\sqrt{5})$, 6, 4, $(\pm 2, 0)$, $(0, \pm 3)$

13. $(\pm 4, 3), (\pm 4, -3)$

17. $X^2 - Y^2 = 8$

19a. $3X^2 + 9Y^2 = 212$
b. $2X^2 + Y^2 = -2$

VIII–1

1. Take sides of $\angle$s 1 and 2 parallel to each other. Let $\angle 3$ be formed by extending a side of $\angle 2$. Continue that line to meet a side of $\angle 1$ extended, forming $\angle 4$. $\angle 1 = \angle 2$ (via $\angle 4$)
$\angle 1$ supp. $\angle 3$ (via $\angle 2$)

3. Assume intersections are not parallel.

5. $FB \perp AB$ and CB. Thus, $FB \perp$ plane $DABC$ and $FB \perp DB$, a line through its foot.

7a. By exercise 6(b), $d^2 = 3e^2$
b. Use $\sin \frac{\beta}{2} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

$\sin \frac{\alpha}{2} = \frac{1}{\sqrt{3}}$

9. Pass plane through perpendicular and use exercise 3.

11. Use indirect proof.

VIII–2

7. $x = 2, y = 1, z = 4$

9. A plane passing through line of intersection of first two. If first two are parallel, the new equation is but a third parallel plane.

VIII–3

1a. $\sqrt{30}$
b. $\sqrt{105}$
c. $4\sqrt{5}$
d. $\sqrt{a^2 + b^2 + c^2}$

3. $3x + 4y - 3z = 14$

5. $y^2 - 2x - 4y + 4z + 9 = 0$

VIII–4

3. $(x - 3)^2 + (y - 5)^2 + (z - 4)^2 = 16$

5. $\frac{13}{\sqrt{14}}$

7a. Sphere at origin, $r = 6$.
 b. Paraboloid of revolution, vertex at origin, axis on Z-axis.
 c. Circular ellipsoid, Y-axis the major axis. Center at origin.
 d. Circular cone around Z-axis, vertex at origin; xy-traces are circles.
 e. Elliptic paraboloid, axis on Z-axis.
 f. Hyperbolic paraboloid.
 g. Circular hyperboloid of one sheet.
 h. Elliptic hyperboloid of two sheets.
 i. Circular ellipsoid.
 j. Ellipsoid.
 k. Circular hyperboloid of one sheet.
 l. Circular hyperboloid of two sheets.
 m. Hyperbolic paraboloid.

VIII–5 REVIEW

3. arc tan 1, arc tan $\dfrac{4}{\sqrt{34}}$, arc tan $\dfrac{3}{\sqrt{41}}$

5. $\dfrac{3}{6} = \dfrac{-2}{-4} = \dfrac{1}{2}$

   $3(1) + (-2)(2) + (1)(1) = 0$

7. $3x + 2y = 19$

9. $x + y + z = 15$

11. Use Pythagorean theorem.

13. $\dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

IX–1

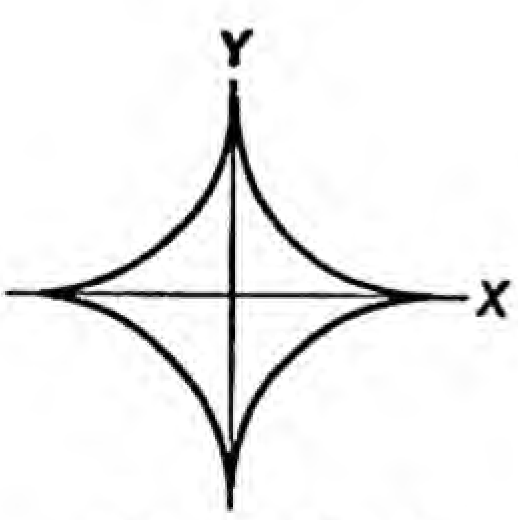

Fig. IX–1–1f

1f. See Fig. IX–1–1f

3a. $xy = 12$
 b. $y = \frac{1}{4}x^{3/2}$
 c. $y = (x + 1)(x + 3)$
 d. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
 e. $y^2 = \dfrac{x^3}{2 - x}$
 f. $x^{2/3} + y^{2/3} = 1$

5. $x^2 + y^2 = r^2$

7a. $x = a \cos \theta,\ y = b \sin \theta$
 c. Repeat determination of other P's.

9. $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

11. $(1, 1, 1), (-1, 1, -1), (-2, 4, -8)$

IX–2

1. Substitute for $\tan\theta$ and $\cot\theta$.

3a. $(-2, -140°)$ $(2, 400°)$
 b. $(3, -30°)$ $(3, 330°)$
 c. $\left(-2, \frac{11\pi}{5}\right)\left(2, -\frac{4\pi}{5}\right)$
 d. $(k, \alpha + 2\pi)$ $(-k, \alpha - \pi)$

5. Drop perpendicular from P to axis.

7b. See Fig. IX–2–7(b).

9a. $r = \dfrac{k}{\theta}$
 b. See Fig. IX–2–9(b).

11a. $r = a \sec\theta - b$
 b. See Fig. IX–2–11(b).

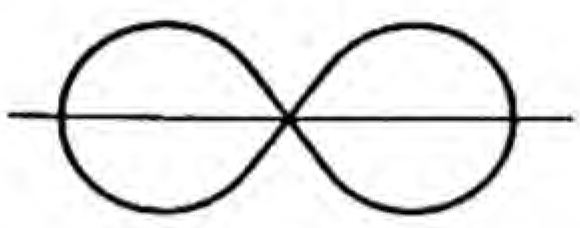

Fig. IX–2–7b

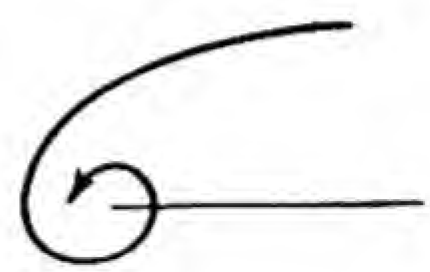

Fig. IX–2–9b

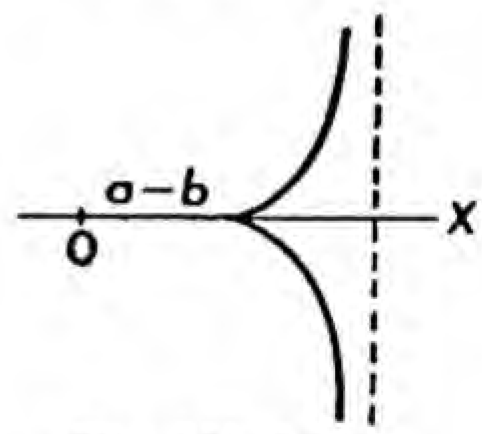

Fig IX–2–11b

13a. $r = 6\cos\theta$
 b. $r^2 = \cos 2\theta$
 c. $\theta = \dfrac{\pi}{4}$
 d. $r = 0$
 e. $r = 8\tan\theta\sin\theta$

IX–3

1b. $\tan\theta = \dfrac{y}{x}$, $x^2 + y^2 = r^2$

3a. Circle, $r = 3$, intersection of sphere and plane.
 b. Helix-wound on cylinder whose radius is 3.
 c. Line of intersection of plane $z = 3$ and plane $\theta = \pi/4$.

5a. $r^2 = 2pz$
 $\rho\tan\phi\sin\phi = 2p$
 b. $r^2 + 9z^2 = 16$
 $\rho^2\sin^2\phi + 9\rho^2\cos^2\phi = 16$
 c. $z^2 - 4(x^2 + y^2) = 16$
 $\rho^2\cos^2\phi - 4\rho^2\sin^2\phi = 16$
 d. $x^2 + y^2 = -4y$
 $\rho\sin\phi = 4\sin\theta$
 e. $r^2 = 4z$
 $x^2 + y^2 = 4z$
 f. $z^2 - r^2 = 9$
 $z^2 - (x^2 + y^2) = 9$

IX–3 REVIEW

1. For integral values of x and y only.

3. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, hyperbola.

5. See Fig. IX–1–1(f).

7. 6

11. $x^2 - y^2 = 2$

13a. $r\cos\theta' = -4$
 b. $r\sin\theta' = 2$
 c. $r = 4\cos 2\theta'$

15. $(\sqrt{6}, \sqrt{6}, 2)$

X–1

5b. $-\frac{1}{4}$

7. Limit of sum of continuous functions is sum of limits.

X–2

1a. $5x^4 + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$
b. $3(2xh + h^2)$
c. $\sqrt{x+h} - \sqrt{x}$
d. $h + 2xh + h^2$
e. $(mx^{m-1}h + \cdots + h^m) + (nx^{n-1}h + \cdots + h^n)$

3a. $15x^2$
b. $-\dfrac{1}{x^2}$
c. -1
d. $-\frac{1}{4}$
e. $\frac{1}{4}$
f. $-\frac{1}{16}$

5. $30x^4$, $120x^3$

X–3

1a. $20x^3$
b. $-12x^2$
c. $12t$
d. $48w^2$
e. 16

X–4

1a. 5
b. 1
c. $-\frac{2}{27}$
d. $\frac{1}{2}\sqrt{2}$

3. $\frac{1}{4}$

5a. $\dfrac{dw}{dt}$ b. $\dfrac{d\theta}{dt}$ c. $\dfrac{d\omega}{dt}$

X–5

1a. $\dfrac{ds}{dt} = 5$
b. $\dfrac{ds}{dt} = 12t - 10$
c. $\dfrac{ds}{dt} = -\dfrac{1}{t^2}$
d. $\dfrac{ds}{dt} = -7$

3a. $\dfrac{ds}{dt} = v = 48 - 32t$
b. $v = 16, -16$
c. Away from, toward, the ground.
d. $t = 1\frac{1}{2}$, body is at rest, neither rising nor falling; a change in direction takes place.
e. $v = -48$

5a. $v = v_0 \sin\theta - gt$
b. 560 ft/sec
c. $18\frac{3}{4}$ sec
d. $37\frac{1}{2}$ sec
e. Highest point reached in one-half the total time of flight.
f. -600 ft/sec

7. $(4, -12), (-1, 113)$

9a. $3x^2$
b. $32x^3$
c. $15x^2 - 6x + 7$
d. $32m - 2$
e. $3T^2 - 8T$
f. $3x^2 - 10x$
g. $-\frac{1}{x^2} - 1$
h. $-\frac{1}{2\sqrt{x^3}} - \frac{1}{2\sqrt{x}}$
i. $3t^2 - 6t$
j. $-\frac{10}{m^3} - \frac{1}{m^2}$

X–5 REVIEW

1a. $12x^2 - 6x + 1$
b. $4t$
c. $36v(4v^4 - 12v^2 + 9)$
d. $\frac{3}{2}(v^2 + 2v + 1)$
e. $4\pi r^2$
f. $\frac{1}{2}(1 + 2x)$
g. $8\pi r$

3a. $16y - 8x + 17 = 0$
b. $4y + 8x = 7$

5. 98

X–6

1a. $6(3x + 1)$
b. $2(2x - 1)(x^2 - x)$
c. $3(x - 2)^2$
d. $30x(2 + 5x^2)^2$

X–7

1a. t
b. $9t^2$
c. $2t^2$
d. $1\frac{2}{3}$

3. $(1, -6)$

X–8

1a. $\frac{5}{2y}, \frac{2y}{5}$
b. $\frac{1 - 2x}{3y^2}, \frac{3y^2}{1 - 2x}$
c. $\frac{1}{2y}, 2y$
d. $-\frac{x}{y}, -\frac{y}{x}$
e. $\frac{3}{1 + 2y}, \frac{1 + 2y}{3}$
f. $\frac{px^{p-1}}{qy^{q-1}}, \frac{qy^{q-1}}{px^{p-1}}$
g. $2(x - 1), \frac{1}{2\sqrt{y}}$
h. $\frac{x}{2y\sqrt{x^2 - 3}}, \frac{2y^3}{x}$
i. $\frac{3x^2}{2y}, \frac{2y}{3x^2}$

3. $y - \sqrt{3} = -\frac{1}{3\sqrt{3}}(x - 3)$

5. $\frac{4v - 3}{6u + 2}, \frac{6u + 2}{4v - 3}$

X-8 REVIEW

1. $6x,\ y = 9t^2 + 30t + 31$

   $\dfrac{dy}{dt} = 18t + 30 = 6x$

3a. $4x(x^2 - 5)$

b. $4x(x^2 - 5)\dfrac{dx}{dt}$

c. $6x^5(5 + 6x)(5 + 4x)^2$

d. $54z^2(6z^3 + 9)^2$

e. $5y^{34}(9y^2 + 7)(y^2 + 1)^4$

f. $2u^3(2 - 3u)(1 - u)\dfrac{du}{dv}$

5a. $6x^2(x^3 + 1)$

b. $59\frac{1}{16}$

7. $2\frac{1}{2}$

9a. $g[F] + h[F]$

b. $Dg \cdot DF + Dh \cdot DF$

c. $Df[F]$

d. $\dfrac{df}{dv}$

11. $\dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$

13. $y - 4 = \frac{1}{3}(x - 8)$

15. $y - 3\sqrt{3} = -\dfrac{1}{\sqrt{3}}(x - 3)$

    $y + 3\sqrt{3} = \dfrac{1}{\sqrt{3}}(x - 3)$

17. $y - 1 = -\frac{3}{2}(x + 1)$

X-9

1. 2.074
3. 2.53
5. 2.93

X-10

1. 28°
3. 152°

X-11

1. $1\frac{3}{5}$
3. $2\frac{5}{8}$ ft/min
5. 32 sq in./sec
7. $-\dfrac{45}{\sqrt{73}}, \dfrac{45}{\sqrt{265}}$ knots
9. $-\dfrac{2}{9\pi}$ cm/sec

11a. $V = \frac{3}{4}\pi h^3$

b. $\dfrac{4}{5\pi}$ ft/min

X-12

1a. $x^2(5x^2 + 3)$

b. $\dfrac{3\sqrt{x}}{2}$

c. $\dfrac{3\sqrt{x}}{2}$

d. $\dfrac{1}{6y^5}$

e. $\dfrac{3}{\sqrt{x}} + \dfrac{8}{3\sqrt[3]{x}}$

f. $\dfrac{2x^2 + x + 1}{\sqrt{x^2 + 1}}$

g. $\dfrac{1}{8y}$

h. $-\sqrt{\dfrac{y}{x}}$

i. $\dfrac{t\sqrt{t}}{(t - 1)(\sqrt{t} + 1)}$

j. 1

k. $\dfrac{2x - x^2}{(x + 1)^4}$

l. $3x(x^2 + 1)^{1/2}$

m. $\dfrac{-1}{(x + 1)^2}$

3. 104°

X–12 REVIEW

3. 8°

5a. $\frac{dy}{dt} = (3x^2 - 3)\frac{dx}{dt}$

b. $\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$

c. $\frac{du}{dt} = 6v(v^2 - 1)^2\frac{dv}{dt}$

d. $\frac{dy}{dt} = -\frac{1}{2y(x-1)^2}\frac{dx}{dt}$

e. $-\frac{3}{w^4}\frac{dw}{dt} = 2z\frac{dz}{dt}$

f. $(x + 2y)\frac{dy}{dt} + (1 + y)\frac{dx}{dt} = 0$

7. Increasing $\frac{2}{15}$
9. 10 in./min
11. $-\sqrt{3}$
13. $-\frac{2\sqrt{t}}{(t-1)^2}$

X–13

1a. $(2\frac{1}{2}, -12\frac{1}{4})$ min.
b. $(\frac{1}{3}, \frac{4}{27})$ max., (1,0) min.
c. (1, 0) inflection point, horizontal tangent.
d. $(-2, 9)$ max.
e. $(-3, 16)$ max., $(1, -16)$ min.
f. $(-1, 1)$ min., (0, 2) max., (1, 1) min.
g. $(1, -\sqrt[3]{9})$ min.
h. (0, 0) max., $(\frac{2}{3}, -\frac{4}{27})$ min.
i. $(2, -8)$ min., $(-2, 24)$ max.
j. (0, 0) min., vertical tangent.
k. (0, 0) min.
l. (0, 0) inflection point, vertical tangent.
m. (1, 1) inflection point, horizontal tangent
n. (1, 2) min., $(-1, -2)$ max.
o. (0, 2) min., no tangent.
p. (0, 0) min., no tangent.
q. (0, 1) max., $(-1, 0)$ and (1, 0) minima, no tangents.

3a. $5\sqrt{2}$ in. $\times$ $5\sqrt{2}$ in.
b. $5\sqrt{2}$ in. $\times$ $5\sqrt{2}$ in.

5a. $x = 3, y = 12$
b. $y = 12, x = 3$

7. (17, 36)

9. $\left(4\sqrt{5} \times 3\sqrt{5} \times \frac{12}{\sqrt{5}}\right)$ in.

11. 4 in. diam., 4 in. high.

13a. $\frac{x^2}{4} + \frac{y^2}{2} = 1$

b. $Y = \sqrt{\frac{4 - X^2}{2}}$

c. $4|X|\sqrt{\frac{4 - X^2}{2}}$

d. $2 \times 2\sqrt{2}$

X–14

1a. $\frac{2}{5}x^5 + x^3 - x^2 + k$
 b. $\frac{3}{2}x^2 - \frac{2}{3}x^3 + k$
 c. $\frac{5}{4}x^4 - 7x + k$
 d. $6x + k$
 e. $6x^{3/2} + k$
 f. $6x^{4/3} + k$
 g. $\frac{2}{3}x^{3/2} + 2x^{1/2} + k$
 h. $\dfrac{1}{3}x^3 - \dfrac{1}{x} + k$
 i. $\dfrac{y^4}{2} + \dfrac{2}{y^2} + k$
 j. $\frac{1}{4}(x^2 + 1)^4 + k$
 k. $4\sqrt{1 - x^3} + k$

5. $y = kx + b$ (b a constant).

7a. $v = 32t$
 b. $v = 32t + 60$
 c. $v = 32t + 60$, $s = 16t^2 + 60t$

9. $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$

11. $x^2 + y^2 = k$

13. $\dfrac{dy}{dx} = m$ and $m = \dfrac{y}{x}$

X–15

1a. $y = -\frac{1}{8}(4 - x^2)^4 + k$
 b. $y = -\dfrac{5}{2(x^2 - 1)} + k$
 c. $\frac{1}{28}(3x^4 + 1)^7 + k$
 d. $\frac{1}{3}(x^2 + 1)^{3/2} + k$
 e. $-\frac{2}{3}(x^3 + 1)^{-1/2} + k$
 f. $y^2 + x^2 = c$
 g. $5y^2 = 2x^3 + c$
 h. $3y^2 + 8x^3 = c$

3a. $\frac{1}{3}x^3$, $\frac{1}{3}x^3 + 1$, $\frac{1}{3}x^3 - 1$
 $-\infty < x < \infty$

X–15 REVIEW

1. $(0, 0)$ min., $(3, 0)$ min., $(\frac{3}{2}, \frac{81}{16})$ max.

5. \$4.07

7. $\dfrac{m}{2}, \dfrac{m}{2}$

9a. $2x^3 - \frac{1}{2}x^2 + 3x + k$
 b. $\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + k$
 c. $-(x - 1)^{-1} + k$
 d. $\frac{5}{9}(x^3 + 1)^3 + k$
 e. $\sqrt{x^2 - 1} + k$

11a. $\frac{1}{4}x^4 - \frac{5}{3}x^3 + x + k$
 b. $-\dfrac{3}{2(z^2 - 1)^2} + k$
 c. $-\frac{1}{3}(1 - u^2)^{3/2} + k$
 d. $\frac{1}{4}(v^2 + 1)^4 + k$

13. $g(x) = \dfrac{7}{6} - \dfrac{5}{2(x^2 - 1)}$

XI–1

1a. 30
 b. 84
 c. 39
 c. Meaningless.
 d. $40\frac{1}{2}$

3a. 4
 b. $13\frac{1}{6}$

5. $10\frac{2}{3}$

7. $15\frac{1}{3}$

11. $-\frac{1}{6}$

XI–2

1. $1{,}555\ \frac{1}{5}\pi$
5. $34\frac{2}{15}\pi$
7. $31\frac{1}{3}\pi$

9a. 8π
 b. $5\frac{1}{3}\pi$
 c. $\frac{16}{15}\pi$

11. 144

XI–3

1. $\frac{2}{15}k$
3. 63π ft-tons

7a. 600 in.-lb.
 b. 1200 in.-lb.

XI–4

1. $4\sqrt{10}$

3. 9.1

XI–5

1. $16\pi\sqrt{2}$
3. $2\pi\int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

5a. $S = \pi s(R + r)$

XI–5 REVIEW

5. $21\frac{1}{12}$
7. $2\frac{2}{3}$
9. $13\frac{1}{3}\pi$

11. $\frac{6075}{64}\pi$ ft-tons
13. $2\sqrt{5}$
15. 52.1
17. $\frac{\pi}{6}(17\sqrt{17} - 1)$

XII–1

3a. $-\csc^2 x$
 b. $\sec x \tan x$
 c. $-\csc x \cot x$

5. $D \sin x° = D \sin \frac{\pi}{180}x$
$= \frac{\pi}{180} \cos \frac{\pi}{180}x = \frac{\pi}{180} \cos x°$

7a. $\frac{\pi}{4}, \frac{5\pi}{4}$
 b. $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$

9. Graphs intersect where $\cos x = \tan x$; that is, where $\cos^2 x = \sin x$. The product of the derivatives is $-\frac{\sin x}{\cos^2 x}$ which is -1 at points of intersection.
11. 2
13. $\frac{\pi}{2}$

XII-2

1a. $\dfrac{1}{x+1}$

b. $\dfrac{3}{x}$

c. $\dfrac{2}{u}\dfrac{du}{dx}$

d. $1 + \ln|x|$

e. $-\tan x$

f. $\dfrac{1}{\sin x \cos x}$

g. $2 \cot x$

h. $3x^2 e^{x^3}$

i. $-e^{-x}$

j. $-\sin x\ (e^{\cos x})$

k. $e^x - e^y \dfrac{dy}{dx}$

l. 1

m. $3^x \ln 3$

n. $2x(4^{x^2}) \ln 4$

o. $\dfrac{2x}{x^2+3}$

p. $\dfrac{-3x^2}{a^2 - x^3}$

3. 1

5. 1.72

7. $\dfrac{\sqrt{3}}{2}$

XII-3

1. $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots + \dfrac{(-1)^n x^n}{n!} + \cdots,\ n = 0, 1, 2, \cdots$

7a. $\cos x = \sum_{k=1}^{\infty} \dfrac{(-1)^{k-1} x^{2k-2}}{(2k-2)!}$

b. $e^x = \sum_{k=1}^{\infty} \dfrac{x^{k-1}}{(k-1)!}$

e. $\ln 2 + 2k\pi i$

15a. $\frac{1}{2} \ln 2 + i\pi(\frac{7}{4} + 2k)$

b. $\ln 2 + i\pi(\frac{1}{6} + 2k)$

c. $i\pi(2k + \frac{1}{2})$

d. $i\pi(2k + 1)$

XII-3 REVIEW

1a. $5 \cos 5x$

b. $-\frac{3}{2} \sin \frac{3}{2}x$

c. $\dfrac{\cos x}{2\sqrt{\sin x}}$

d. $-3 \cos^2 x \sin x$

e. $\dfrac{\cos \sqrt{x}}{2\sqrt{x}}$

3. Max. at $\pi/3$; min. at $5\pi/3$; inflection point at π.

5. $\dfrac{-\sqrt{2}}{16}$ radians/min

7. $\dfrac{1}{4} - \dfrac{5\pi\sqrt{3}}{12}$

9. $\dfrac{3\sqrt{2}}{2} + 6;\ -\dfrac{3\sqrt{2}}{2}$

11a. $\ln \left| \dfrac{\sin b}{\sin a} \right|$

b. $\ln \frac{1}{2}(e + 1)$

c. $\ln \frac{9}{2}$

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XII–2

1a. $\dfrac{1}{x+1}$
b. $\dfrac{3}{x}$
c. $\dfrac{2}{u}\dfrac{du}{dx}$
d. $1 + \ln|x|$
e. $-\tan x$
f. $\dfrac{1}{\sin x \cos x}$
g. $2 \cot x$
h. $3x^2 e^{x^3}$
i. $-e^{-x}$
j. $-\sin x \, (e^{\cos x})$
k. $e^x - e^y \dfrac{dy}{dx}$
l. 1
m. $3^x \ln 3$
n. $2x(4^{x^2}) \ln 4$
o. $\dfrac{2x}{x^2+3}$
p. $\dfrac{-3x^2}{a^2 - x^3}$

3. 1
5. 1.72
7. $\dfrac{\sqrt{3}}{2}$

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